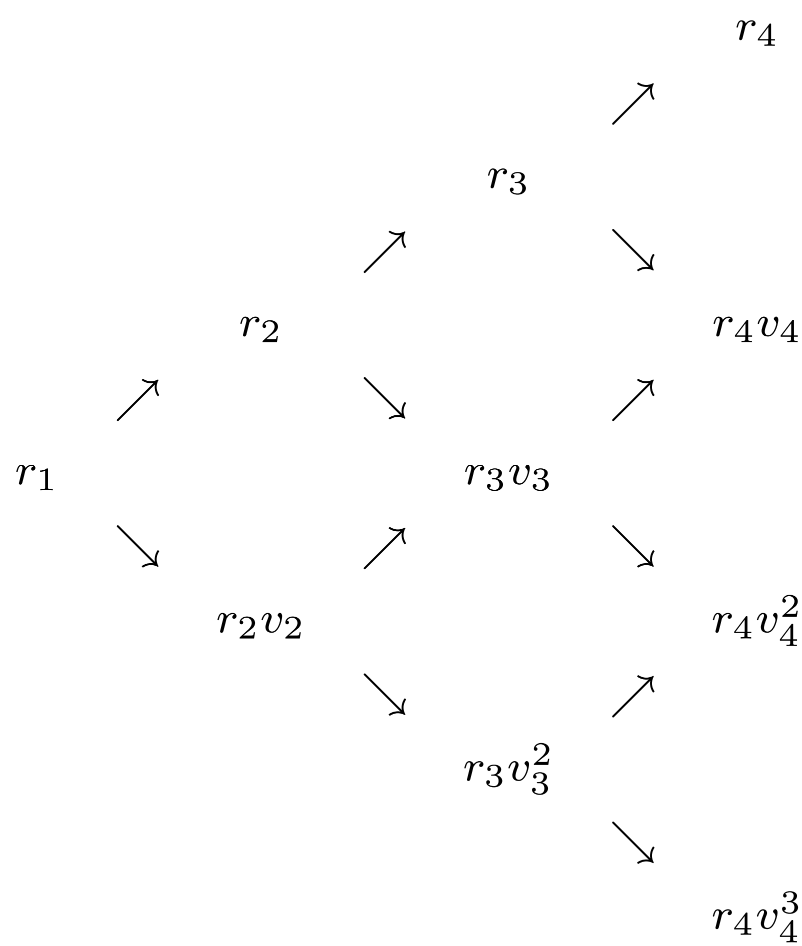
The Black-Derman-Toy Model[a]

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 820ff (repeated on next page).
- The volatility structure is given by the market.
- From it, the short rate volatilities (thus v_i) are determined together with r_i.

[a]Black, Derman, and Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).



r_4
r_3
r_2
$r_4 v_4$
r_1
$r_3 v_3$
$r_2 v_2$
$r_4 v_4^2$
$r_3 v_3^2$
$r_4 v_4^3$

The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes v_i are given a priori.
  - A related model of Salomon Brothers takes v_i to be a given constant.[a]
- Lognormal models preclude negative short rates.

[a] Tuckman (2002).

The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the i-period zero-coupon bond be denoted by κ_i.
- P_{u} is the price of the i-period zero-coupon bond one period from now if the short rate makes an up move.
- P_{d} is the price of the i-period zero-coupon bond one period from now if the short rate makes a down move.

The BDT Model: Volatility Structure (concluded)

- Corresponding to these two prices are the following yields to maturity,

$$y_{\mathrm{u}} \equiv P_{\mathrm{u}}^{-1/(i-1)} - 1,$$
$$y_{\mathrm{d}} \equiv P_{\mathrm{d}}^{-1/(i-1)} - 1.$$

- The yield volatility is defined as

$$\kappa_i \equiv \frac{\ln(y_{\mathrm{u}}/y_{\mathrm{d}})}{2}.$$

The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

$$(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1}).$$

  - They define the binomial tree up to period $i - 1$.
- We now proceed to calculate r_i and v_i to extend the tree to period i.

The BDT Model: Calibration (continued)

- Assume the price of the i-period zero can move to P_{u} or P_{d} one period from now.
- Let y denote the current i-period spot rate, which is known.
- In a risk-neutral economy,

$$\frac{P_{\mathrm{u}} + P_{\mathrm{d}}}{2(1 + r_1)} = \frac{1}{(1 + y)^i}. \tag{117}$$

- Obviously, P_{u} and P_{d} are functions of the unknown r_i and v_i.

The BDT Model: Calibration (continued)

- Viewed from now, the future $(i-1)$-period spot rate at time 1 is uncertain.
- Recall that y_{u} and y_{d} represent the spot rates at the up node and the down node, respectively (p. 984).
- With κ^2 denoting their variance, we have

$$\kappa_i = \frac{1}{2} \ln \left(\frac{{P_{\mathrm{u}}}^{-1/(i-1)} - 1}{{P_{\mathrm{d}}}^{-1/(i-1)} - 1} \right). \tag{118}$$

The BDT Model: Calibration (continued)

- We will employ forward induction to derive a quadratic-time calibration algorithm.[a]
- Recall that forward induction inductively figures out, by moving *forward* in time, how much \$1 at a node contributes to the price (review p. 846(a)).
- This number is called the state price and is the price of the claim that pays \$1 at that node and zero elsewhere.

[a] Chen (`R84526007`) and Lyuu (1997); Lyuu (1999).

The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period i be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i-1$ be $P_1, P_2, \ldots, P_i$, corresponding to rates $r, rv, \ldots, rv^{i-1}$, respectively.
- One dollar at time i has a present value of

$$f(r,v) \equiv \frac{P_1}{1+r} + \frac{P_2}{1+rv} + \frac{P_3}{1+rv^2} + \cdots + \frac{P_i}{1+rv^{i-1}}.$$

The BDT Model: Calibration (continued)

- The yield volatility is

$$g(r,v) \equiv \frac{1}{2} \ln \left(\frac{\left(\frac{P_{\mathrm{u},1}}{1+rv} + \frac{P_{\mathrm{u},2}}{1+rv^2} + \cdots + \frac{P_{\mathrm{u},i-1}}{1+rv^{i-1}} \right)^{-1/(i-1)} - 1}{\left(\frac{P_{\mathrm{d},1}}{1+r} + \frac{P_{\mathrm{d},2}}{1+rv} + \cdots + \frac{P_{\mathrm{d},i-1}}{1+rv^{i-2}} \right)^{-1/(i-1)} - 1} \right).$$

- Above, $P_{\mathrm{u},1}, P_{\mathrm{u},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the up node (like $r_2 v_2$ on p. 981).
- And $P_{\mathrm{d},1}, P_{\mathrm{d},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the down node (like r_2 on p. 981).

The BDT Model: Calibration (concluded)

- Note that every node maintains 3 state prices.
- Now solve

$$\begin{aligned} f(r, v) &= \frac{1}{(1+y)^i}, \\ g(r, v) &= \kappa_i, \end{aligned}$$

for $r = r_i$ and $v = v_i$.

- This $O(n^2)$-time algorithm appears in the text.

The BDT Model: Continuous-Time Limit

- The continuous-time limit of the BDT model is

$$d \ln r = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t) \, dW.$$

- The short rate volatility clearly should be a declining function of time for the model to display mean reversion.
  - That makes $\sigma'(t) < 0$.
- In particular, constant volatility will not attain mean reversion.

The Black-Karasinski Model[a]

- The BK model stipulates that the short rate follows

$$d\ln r = \kappa(t)(\theta(t) - \ln r)\,dt + \sigma(t)\,dW.$$

- This explicitly mean-reverting model depends on time through $\kappa(\cdot)$, $\theta(\cdot)$, and $\sigma(\cdot)$.
- The BK model hence has one more degree of freedom than the BDT model.
- The speed of mean reversion $\kappa(t)$ and the short rate volatility $\sigma(t)$ are independent.

---

[a]Black and Karasinski (1991).

The Black-Karasinski Model: Discrete Time

- The discrete-time version of the BK model has the same representation as the BDT model.
- To maintain a combining binomial tree, however, requires some manipulations.
- The next plot illustrates the ideas in which

$$\begin{aligned} t_2 &\equiv t_1 + \Delta t_1, \\ t_3 &\equiv t_2 + \Delta t_2. \end{aligned}$$

$$
\begin{array}{ccccc}
 & & & \nearrow & \\
 & & \ln r_{\mathrm{d}}(t_2) & & \\
 & \nearrow & & \searrow & \\
\ln r(t_1) & & & & \ln r_{\mathrm{du}}(t_3) = \ln r_{\mathrm{ud}}(t_3) \\
 & \searrow & & \nearrow & \\
 & & \ln r_{\mathrm{u}}(t_2) & & \\
 & & & \searrow &
\end{array}
$$

The Black-Karasinski Model: Discrete Time (continued)

- Note that

$$\begin{aligned}\ln r_{\mathrm{d}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 - \sigma(t_1)\sqrt{\Delta t_1}\,,\\ \ln r_{\mathrm{u}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 + \sigma(t_1)\sqrt{\Delta t_1}\,.\end{aligned}$$

- To ensure that an up move followed by a down move coincides with a down move followed by an up move, impose

$$\begin{aligned}& \ln r_{\mathrm{d}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{d}}(t_2))\,\Delta t_2 + \sigma(t_2)\sqrt{\Delta t_2}\,,\\ = \;& \ln r_{\mathrm{u}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{u}}(t_2))\,\Delta t_2 - \sigma(t_2)\sqrt{\Delta t_2}\,.\end{aligned}$$

The Black-Karasinski Model: Discrete Time (concluded)

- They imply

$$\kappa(t_2) = \frac{1 - (\sigma(t_2)/\sigma(t_1))\sqrt{\Delta t_2/\Delta t_1}}{\Delta t_2}. \tag{119}$$

- So from Δt_1, we can calculate the Δt_2 that satisfies the combining condition and then iterate.
  - $t_0 \to \Delta t_0 \to t_1 \to \Delta t_1 \to t_2 \to \Delta t_2 \to \cdots \to T$ (roughly).[a]

[a] As $\kappa(t), \theta(t), \sigma(t)$ are independent of r, the Δt_is will not depend on r.

Problems with Lognormal Models in General

- Lognormal models such as BDT and BK share the problem that $E^{\pi}[\,M(t)\,] = \infty$ for any finite t if they the continuously compounded rate.
- Hence periodic compounding should be used.
- Another issue is computational.
- Lognormal models usually do not give analytical solutions to even basic fixed-income securities.
- As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.

Problems with Lognormal Models in General (concluded)

- This problem can be somewhat mitigated by adopting different time steps: Use a fine time step up to the maturity of the short-dated derivative and a coarse time step beyond the maturity.[a]
- A down side of this procedure is that it has to be tailor-made for each derivative.
- Finally, empirically, interest rates do not follow the lognormal distribution.

[a]Hull and White (1993).

The Extended Vasicek Model[a]

- Hull and White proposed models that extend the Vasicek model and the CIR model.
- They are called the extended Vasicek model and the extended CIR model.
- The extended Vasicek model adds time dependence to the original Vasicek model,

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\, dW.$$

- Like the Ho-Lee model, this is a normal model, and the inclusion of $\theta(t)$ allows for an exact fit to the current spot rate curve.

[a]Hull and White (1990).

The Extended Vasicek Model (concluded)

- Function $\sigma(t)$ defines the short rate volatility, and $a(t)$ determines the shape of the volatility structure.
- Under this model, many European-style securities can be evaluated analytically, and efficient numerical procedures can be developed for American-style securities.

The Hull-White Model

- The Hull-White model is the following special case,

$$dr = (\theta(t) - ar)\, dt + \sigma\, dW.$$

- When the current term structure is matched,[a]

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

[a] Hull and White (1993).

The Extended CIR Model

- In the extended CIR model the short rate follows

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t) \sqrt{r}\, dW.$$

- The functions $\theta(t)$, $a(t)$, and $\sigma(t)$ are implied from market observables.
- With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.

The Hull-White Model: Calibration[a]

- We describe a trinomial forward induction scheme to calibrate the Hull-White model given a and σ.
- As with the Ho-Lee model, the set of achievable short rates is evenly spaced.
- Let r_0 be the annualized, continuously compounded short rate at time zero.
- Every short rate on the tree takes on a value

$$r_0 + j\Delta r$$

for some integer j.

[a] Hull and White (1993).

The Hull-White Model: Calibration (continued)

- Time increments on the tree are also equally spaced at Δt apart.
- Hence nodes are located at times $i\Delta t$ for $i = 0, 1, 2, \ldots$.
- We shall refer to the node on the tree with

$$\begin{aligned} t_i &\equiv i\Delta t, \\ r_j &\equiv r_0 + j\Delta r, \end{aligned}$$

as the (i, j) node.

- The short rate at node (i, j), which equals r_j, is effective for the time period $[\,t_i, t_{i+1})$.

The Hull-White Model: Calibration (continued)

- Use

$$\mu_{i,j} \equiv \theta(t_i) - a r_j \tag{120}$$

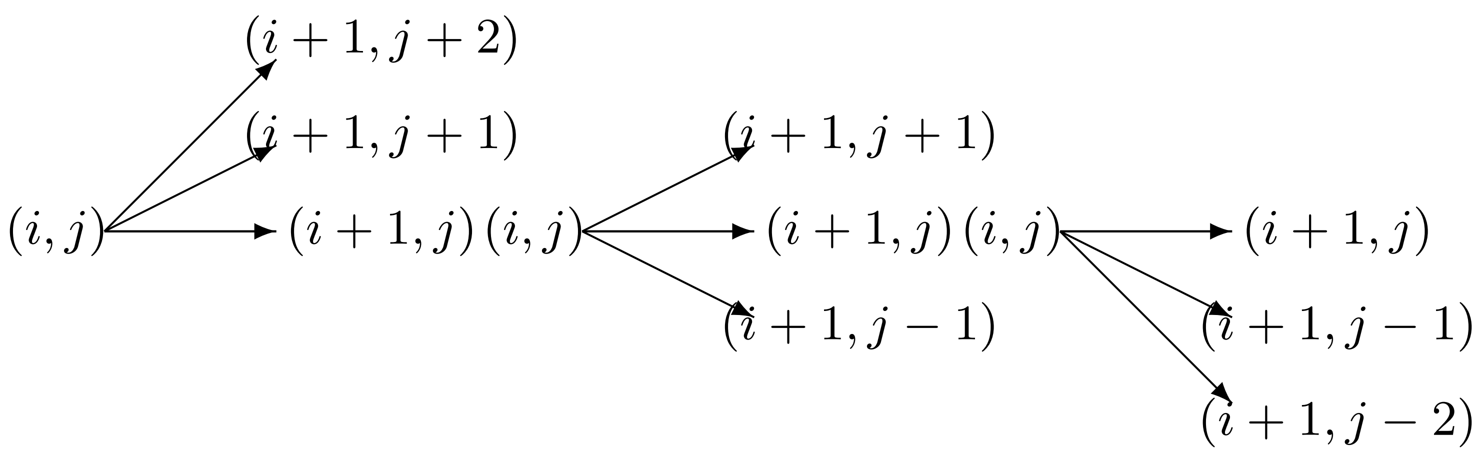
to denote the drift rate, or the expected change, of the short rate as seen from node (i, j).

- The three distinct possibilities for node (i, j) with three branches incident from it are displayed on p. 1007.[a]

- The interest rate movement described by the middle branch may be an increase of Δr, no change, or a decrease of Δr.

[a] A predecessor to Lyuu and Wu's (R90723065) (2003, 2005) mean-tracking idea.

The Hull-White Model: Calibration (continued)



The Hull-White Model: Calibration (continued)

- The upper and the lower branches bracket the middle branch.
- Define

$$
\begin{aligned}
p_1(i,j) &\equiv \text{the probability of following the upper branch from node } (i,j) \\
p_2(i,j) &\equiv \text{the probability of following the middle branch from node } (i,j) \\
p_3(i,j) &\equiv \text{the probability of following the lower branch from node } (i,j)
\end{aligned}
$$

- The root of the tree is set to the current short rate r_0.
- Inductively, the drift $\mu_{i,j}$ at node (i,j) is a function of $\theta(t_i)$.

The Hull-White Model: Calibration (continued)

- Once $\theta(t_i)$ is available, $\mu_{i,j}$ can be derived via Eq. (120) on p. 1006.
- This in turn determines the branching scheme at every node (i, j) for each j, as we will see shortly.
- The value of $\theta(t_i)$ must thus be made consistent with the spot rate $r(0, t_{i+2})$.

The Hull-White Model: Calibration (continued)

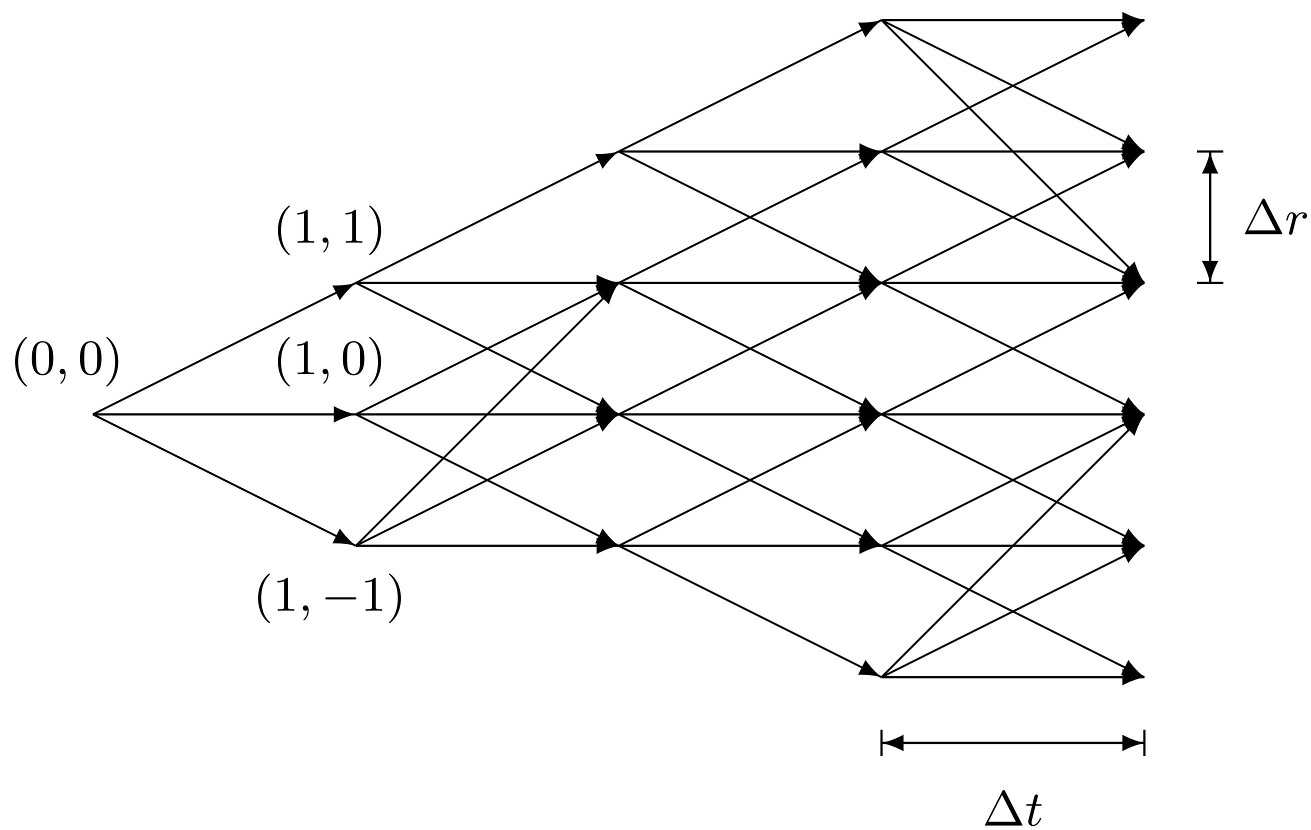
- The branches emanating from node (i, j) with their accompanying probabilities[a] must be chosen to be consistent with $\mu_{i,j}$ and σ.
- This is accomplished by letting the middle node be as close as possible to the current value of the short rate plus the drift.
- Let k be the number among $\{ j-1, j, j+1 \}$ that makes the short rate reached by the middle branch, r_k, closest to

$$r_j + \mu_{i,j} \Delta t.$$

[a] $p_1(i, j)$, $p_2(i, j)$, and $p_3(i, j)$.

The Hull-White Model: Calibration (continued)

- Then the three nodes following node (i, j) are nodes $(i + 1, k + 1)$, $(i + 1, k)$, and $(i + 1, k - 1)$.
- The resulting tree may have the geometry depicted on p. 1012.
- The resulting tree combines because of the constant jump sizes to reach k.



(0, 0)
(1, 1)
(1, 0)
(1, −1)
Δr
Δt

The Hull-White Model: Calibration (continued)

- The probabilities for moving along these branches are functions of $\mu_{i,j}$, σ, j, and k:

$$p_1(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} + \frac{\eta}{2\Delta r} \tag{121}$$

$$p_2(i,j) = 1 - \frac{\sigma^2 \Delta t + \eta^2}{(\Delta r)^2} \tag{121$'$}$$

$$p_3(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} - \frac{\eta}{2\Delta r} \tag{121$''$}$$

where

$$\eta \equiv \mu_{i,j} \Delta t + (j-k)\, \Delta r.$$

The Hull-White Model: Calibration (continued)

- As trinomial tree algorithms are but explicit methods in disguise, certain relations must hold for Δr and Δt to guarantee stability.
- It can be shown that their values must satisfy

$$\frac{\sigma\sqrt{3\Delta t}}{2} \leq \Delta r \leq 2\sigma\sqrt{\Delta t}$$

for the probabilities to lie between zero and one.

  - For example, Δr can be set to $\sigma\sqrt{3\Delta t}$.[a]

[a] Hull and White (1988).

The Hull-White Model: Calibration (continued)

- Now it only remains to determine $\theta(t_i)$.
- At this point at time t_i, $r(0, t_1)$, $r(0, t_2)$, ..., $r(0, t_{i+1})$ have already been matched.
- Let $Q(i, j)$ denote the value of the state contingent claim that pays one dollar at node (i, j) and zero otherwise.
- By construction, the state prices $Q(i, j)$ for all j are known by now.
- We begin with state price $Q(0, 0) = 1$.

The Hull-White Model: Calibration (continued)

- Let $\hat{r}(i)$ refer to the short rate value at time t_i.
- The value at time zero of a zero-coupon bond maturing at time t_{i+2} is then

$$
\begin{aligned}
& e^{-r(0,t_{i+2})(i+2)\,\Delta t} \\
= & \sum_j Q(i,j)\, e^{-r_j \Delta t}\, E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right]. \quad (122)
\end{aligned}
$$

- The right-hand side represents the value of \$1 obtained by holding a zero-coupon bond until time t_{i+1} and then reinvesting the proceeds at that time at the prevailing short rate $\hat{r}(i+1)$, which is stochastic.

The Hull-White Model: Calibration (continued)

- The expectation (122) can be approximated by

$$
\begin{aligned}
& E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \\
\approx \quad & e^{-r_j\,\Delta t}\left(1 - \mu_{i,j}(\Delta t)^2 + \frac{\sigma^2(\Delta t)^3}{2}\right). \quad (123)
\end{aligned}
$$

- Substitute Eq. (123) into Eq. (122) and replace $\mu_{i,j}$ with $\theta(t_i) - ar_j$ to obtain

$$
\theta(t_i) \approx \frac{\sum_j Q(i,j)\, e^{-2r_j\,\Delta t}\left(1 + ar_j(\Delta t)^2 + \sigma^2(\Delta t)^3/2\right) - e^{-r(0,t_{i+2})(i+2)\,\Delta t}}{(\Delta t)^2 \sum_j Q(i,j)\, e^{-2r_j\,\Delta t}}.
$$

The Hull-White Model: Calibration (continued)

- For the Hull-White model, the expectation in Eq. (123) on p. 1017 is actually known analytically by Eq. (19) on p. 152:

$$E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] = e^{-r_j\,\Delta t + (-\theta(t_i) + a r_j + \sigma^2 \Delta t/2)(\Delta t)^2}.$$

- Therefore, alternatively,

$$\theta(t_i) = \frac{r(0, t_{i+2})(i+2)}{\Delta t} + \frac{\sigma^2 \Delta t}{2} + \frac{\ln \sum_j Q(i,j)\, e^{-2r_j\,\Delta t + a r_j\,(\Delta t)^2}}{(\Delta t)^2}.$$

The Hull-White Model: Calibration (concluded)

- With $\theta(t_i)$ in hand, we can compute $\mu_{i,j}$, the probabilities, and finally the state prices at time t_{i+1}:

$$Q(i+1, j) = \sum_{(i, j^*) \text{ is connected to } (i+1, j) \text{ with probability } p_{j^*}} p_{j^*} e^{-r_{j^*} \Delta t} Q(i, j^*).$$

- There are at most 5 choices for j^* (why?).
- The total running time is $O(n^2)$.
- The space requirement is $O(n)$ (why?).

Comments on the Hull-White Model

- One can try different values of a and σ for each option or have an a value common to all options but use a different σ value for each option.
- Either approach can match all the option prices exactly.
- If the demand is for a single set of parameters that replicate all option prices, the Hull-White model can be calibrated to all the observed option prices by choosing a and σ that minimize the mean-squared pricing error.[a]

[a] Hull and White (1995).

The Hull-White Model: Calibration with Irregular Trinomial Trees

- The previous calibration algorithm is quite general.
- For example, it can be modified to apply to cases where the diffusion term has the form σr^b.
- But it has at least two shortcomings.
- First, the resulting trinomial tree is irregular (p. 1012).
  - So it is harder to program.
- The second shortcoming is again a consequence of the tree's irregular shape.

The Hull-White Model: Calibration with Irregular Trinomial Trees (concluded)

- Recall that the algorithm figured out $\theta(t_i)$ that matches the spot rate $r(0, t_{i+2})$ in order to determine the branching schemes for the nodes at time t_i.
- But without those branches, the tree was not specified, and backward induction on the tree was not possible.
- To avoid this dilemma, the algorithm turned to the continuous-time model to evaluate Eq. (122) on p. 1016 that helps derive $\theta(t_i)$ later.
- The resulting $\theta(t_i)$ hence might not yield a tree that matches the spot rates exactly.

The Hull-White Model: Calibration with Regular Trinomial Trees[a]

- The next, simpler algorithm exploits the fact that the Hull-White model has a constant diffusion term σ.
- The resulting trinomial tree will be regular.
- All the $\theta(t_i)$ terms can be chosen by backward induction to match the spot rates exactly.
- The tree is constructed in two phases.

[a]Hull and White (1994).

The Hull-White Model: Calibration with Regular Trinomial Trees (continued)

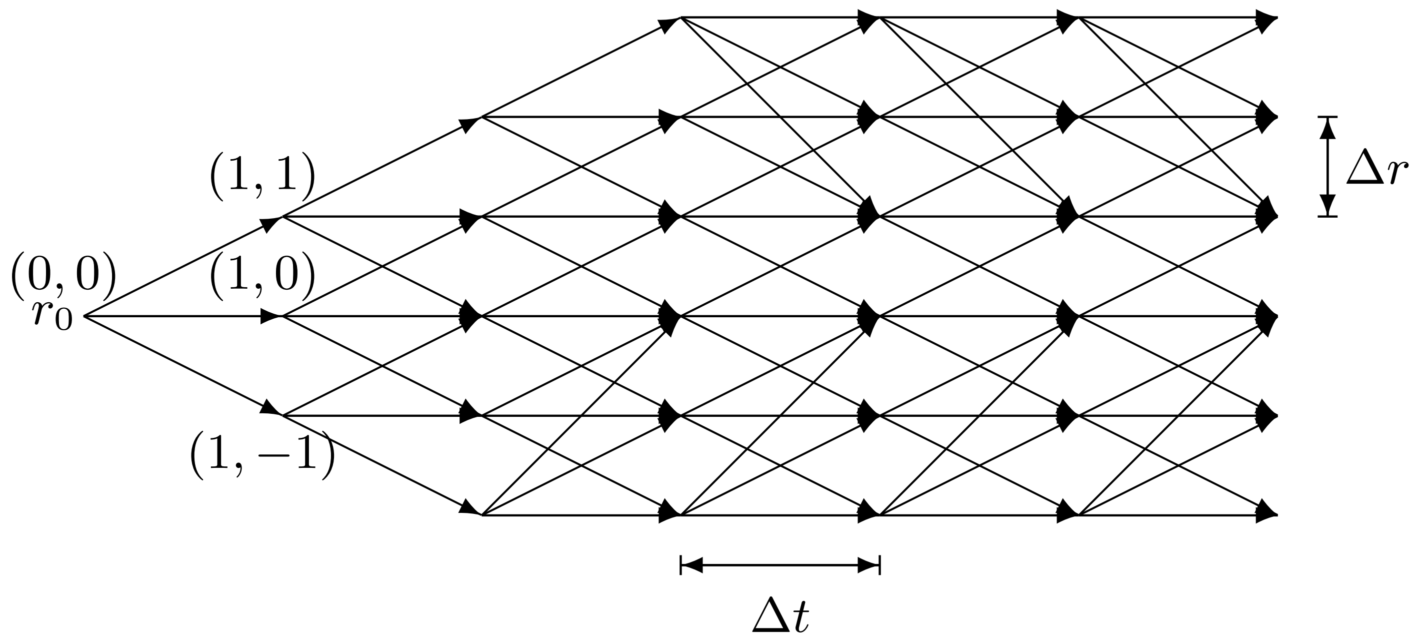
- In the first phase, a tree is built for the $\theta(t) = 0$ case, which is an Ornstein-Uhlenbeck process:

$$dr = -ar\,dt + \sigma\,dW, \quad r(0) = 0.$$

  - The tree is dagger-shaped (p. 1026).
  - The number of nodes above the r_0-line, $j_{\max}$, and that below the line, $j_{\min}$, will be picked so that the probabilities (121) on p. 1013 are positive for all nodes.
  - The tree's branches and probabilities are in place.

The Hull-White Model: Calibration with Regular Trinomial Trees (concluded)

- Phase two fits the term structure.
  - Backward induction is applied to calculate the β_i to add to the short rates on the tree at time t_i so that the spot rate $r(0, t_{i+1})$ is matched.



The short rate at node (0, 0) equals $r_0 = 0$; here $j_{\max} = 3$ and $j_{\min} = 2$.

The Hull-White Model: Calibration

- Set $\Delta r = \sigma\sqrt{3\Delta t}$ and assume that $a > 0$.
- Node (i, j) is a top node if $j = j_{\max}$ and a bottom node if $j = -j_{\min}$.
- Because the root of the tree has a short rate of $r_0 = 0$, phase one adopts $r_j = j\Delta r$.
- Hence the probabilities in Eqs. (121) on p. 1013 use

$$\eta \equiv -aj\Delta r\Delta t + (j - k)\,\Delta r.$$

The Hull-White Model: Calibration (continued)

- The probabilities become

$$p_1(i,j) = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 - aj\Delta t + (j-k)}{2}, \quad (124)$$

$$p_2(i,j) = \frac{2}{3} - \left[a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 \right], \quad (125)$$

$$p_3(i,j) = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 + aj\Delta t - (j-k)}{2}. \quad (126)$$

The Hull-White Model: Calibration (continued)

- The dagger shape dictates this:
  - Let $k = j - 1$ if node (i, j) is a top node.
  - Let $k = j + 1$ if node (i, j) is a bottom node.
  - Let $k = j$ for the rest of the nodes.
- Note that the probabilities are identical for nodes (i, j) with the same j.
- Furthermore, $p_1(i, j) = p_3(i, -j)$.

The Hull-White Model: Calibration (continued)

- The inequalities

$$\frac{3-\sqrt{6}}{3} < ja\Delta t < \sqrt{\frac{2}{3}} \tag{127}$$

ensure that all the branching probabilities are positive in the upper half of the tree, that is, $j > 0$ (verify this).

- Similarly, the inequalities

$$-\sqrt{\frac{2}{3}} < ja\Delta t < -\frac{3-\sqrt{6}}{3}$$

ensure that the probabilities are positive in the lower half of the tree, that is, $j < 0$.

The Hull-White Model: Calibration (continued)

- To further make the tree symmetric across the r_0-line, we let $j_{\min} = j_{\max}$.
- As $\frac{3-\sqrt{6}}{3} \approx 0.184$, a good choice is

$$j_{\max} = \lceil 0.184/(a\Delta t) \rceil.$$

- Phase two computes the β_is to fit the spot rates.
- We begin with state price $Q(0, 0) = 1$.
- Inductively, suppose that spot rates $r(0, t_1)$, $r(0, t_2)$, $\ldots$, $r(0, t_i)$ have already been matched at time t_i.

The Hull-White Model: Calibration (continued)

- By construction, the state prices $Q(i,j)$ for all j are known by now.
- The value of a zero-coupon bond maturing at time t_{i+1} equals

$$e^{-r(0,t_{i+1})(i+1)\,\Delta t} = \sum_j Q(i,j)\, e^{-(\beta_i + r_j)\Delta t}$$

by risk-neutral valuation.

- Hence

$$\beta_i = \frac{r(0,t_{i+1})(i+1)\,\Delta t + \ln \sum_j Q(i,j)\, e^{-r_j\,\Delta t}}{\Delta t},$$

and the short rate at node (i,j) equals $\beta_i + r_j$.

The Hull-White Model: Calibration (concluded)

- The state prices at time t_{i+1},

  $$Q(i+1, j), \quad -\min(i+1, j_{\max}) \leq j \leq \min(i+1, j_{\max}),$$

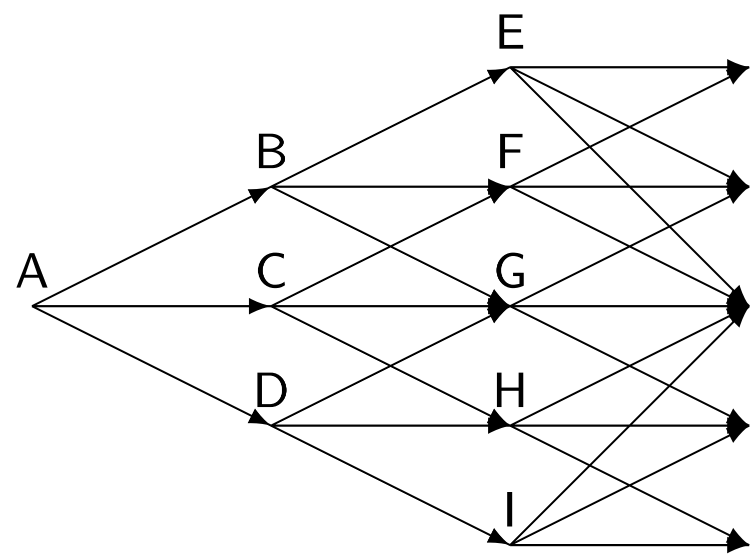
  can now be calculated as before.

- The total running time is $O(nj_{\max})$.

- The space requirement is $O(n)$.

A Numerical Example

- Assume $a = 0.1$, $\sigma = 0.01$, and $\Delta t = 1$ (year).
- Immediately, $\Delta r = 0.0173205$ and $j_{\max} = 2$.
- The plot on p. 1035 illustrates the 3-period trinomial tree after phase one.
- For example, the branching probabilities for node E are calculated by Eqs. (124)–(126) on p. 1028 with $j = 2$ and $k = 1$.



Node	A, C, G	B, F	E	D, H	I
r (%)	0.00000	1.73205	3.46410	-1.73205	-3.46410
p_1	0.16667	0.12167	0.88667	0.22167	0.08667
p_2	0.66667	0.65667	0.02667	0.65667	0.02667
p_3	0.16667	0.22167	0.08667	0.12167	0.88667

A Numerical Example (continued)

- Suppose that phase two is to fit the spot rate curve

$$0.08 - 0.05 \times e^{-0.18 \times t}.$$

- The annualized continuously compounded spot rates are

$$r(0,1) = 3.82365\%, r(0,2) = 4.51162\%, r(0,3) = 5.08626\%.$$

- Start with state price $Q(0,0) = 1$ at node A.

A Numerical Example (continued)

- Now,

$$\beta_0 = r(0,1) + \ln Q(0,0)\, e^{-r_0} = r(0,1) = 3.82365\%.$$

- Hence the short rate at node A equals

$$\beta_0 + r_0 = 3.82365\%.$$

- The state prices at year one are calculated as

$$\begin{aligned} Q(1,1) &= p_1(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414, \\ Q(1,0) &= p_2(0,0)\, e^{-(\beta_0 + r_0)} = 0.641657, \\ Q(1,-1) &= p_3(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414. \end{aligned}$$

A Numerical Example (continued)

- The 2-year rate spot rate $r(0,2)$ is matched by picking

$$\beta_1 = r(0,2) \times 2 + \ln \left[Q(1,1)\, e^{-\Delta r} + Q(1,0) + Q(1,-1)\, e^{\Delta r} \right] = 5.20459\%.$$

- Hence the short rates at nodes B, C, and D equal

$$\beta_1 + r_j,$$

  where $j = 1, 0, -1$, respectively.

- They are found to be 6.93664%, 5.20459%, and 3.47254%.

A Numerical Example (continued)

- The state prices at year two are calculated as

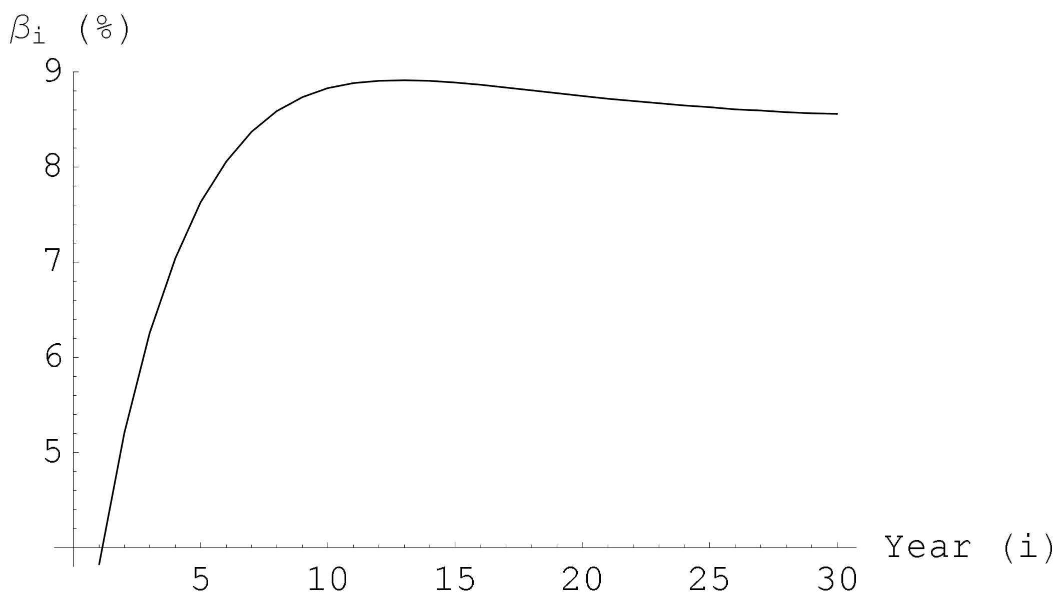
$$
\begin{aligned}
Q(2,2) &= p_1(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) = 0.018209, \\
Q(2,1) &= p_2(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) + p_1(1,0)\, e^{-(\beta_1+r_0)} Q(1,0) \\
&= 0.199799, \\
Q(2,0) &= p_3(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) + p_2(1,0)\, e^{-(\beta_1+r_0)} Q(1,0) \\
&\quad + p_1(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) = 0.473597, \\
Q(2,-1) &= p_3(1,0)\, e^{-(\beta_1+r_0)} Q(1,0) + p_2(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) \\
&= 0.203263, \\
Q(2,-2) &= p_3(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) = 0.018851.
\end{aligned}
$$

A Numerical Example (concluded)

- The 3-year rate spot rate $r(0,3)$ is matched by picking

$$\begin{aligned}\beta_2 &= r(0,3) \times 3 + \ln \Big[Q(2,2)\, e^{-2 \times \Delta r} + Q(2,1)\, e^{-\Delta r} + Q(2,0) \\ &\quad + Q(2,-1)\, e^{\Delta r} + Q(2,-2)\, e^{2 \times \Delta r} \Big] = 6.25359\%.\end{aligned}$$

- Hence the short rates at nodes E, F, G, H, and I equal $\beta_2 + r_j$, where $j = 2, 1, 0, -1, -2$, respectively.
- They are found to be 9.71769%, 7.98564%, 6.25359%, 4.52154%, and 2.78949%.
- The figure on p. 1041 plots β_i for $i = 0, 1, \ldots, 29$.


β_i (%)
9
8
7
6
5
Year (i)
5
10
15
20
25
30

The (Whole) Yield Curve Approach

- We have seen several Markovian short rate models.
- The Markovian approach is computationally efficient.
- But it is difficult to model the behavior of yields and bond prices of different maturities.
- The alternative yield curve approach regards the whole term structure as the state of a process and directly specifies how it evolves.

The Heath-Jarrow-Morton Model[a]

- This influential model is a forward rate model.
- It is also a popular model.
- The HJM model specifies the initial forward rate curve and the forward rate volatility structure, which describes the volatility of each forward rate for a given maturity date.
- Like the Black-Scholes option pricing model, neither risk preference assumptions nor the drifts of forward rates are needed.

[a] Heath, Jarrow, and Morton (HJM) (1992).

Introduction to Mortgage-Backed Securities

Anyone stupid enough to promise to be
responsible for a stranger's debts
deserves to have his own property
held to guarantee payment.
— Proverbs 27:13

Mortgages

- A mortgage is a loan secured by the collateral of real estate property.
- The lender — the mortgagee — can foreclose the loan by seizing the property if the borrower — the mortgagor — defaults, that is, fails to make the contractual payments.

Mortgage-Backed Securities

- A mortgage-backed security (MBS) is a bond backed by an undivided interest in a pool of mortgages.
- MBSs traditionally enjoy high returns, wide ranges of products, high credit quality, and liquidity.
- The mortgage market has witnessed tremendous innovations in product design.

Mortgage-Backed Securities (concluded)

- The complexity of the products and the prepayment option require advanced models and software techniques.
  - In fact, the mortgage market probably could not have operated efficiently without them.[a]
- They also consume lots of computing power.
- Our focus will be on residential mortgages.
- But the underlying principles are applicable to other types of assets.

[a]Merton (1994).

Types of MBSs

- An MBS is issued with pools of mortgage loans as the collateral.
- The cash flows of the mortgages making up the pool naturally reflect upon those of the MBS.
- There are three basic types of MBSs:
  1. Mortgage pass-through security (MPTS).
  2. Collateralized mortgage obligation (CMO).
  3. Stripped mortgage-backed security (SMBS).

Problems Investing in Mortgages

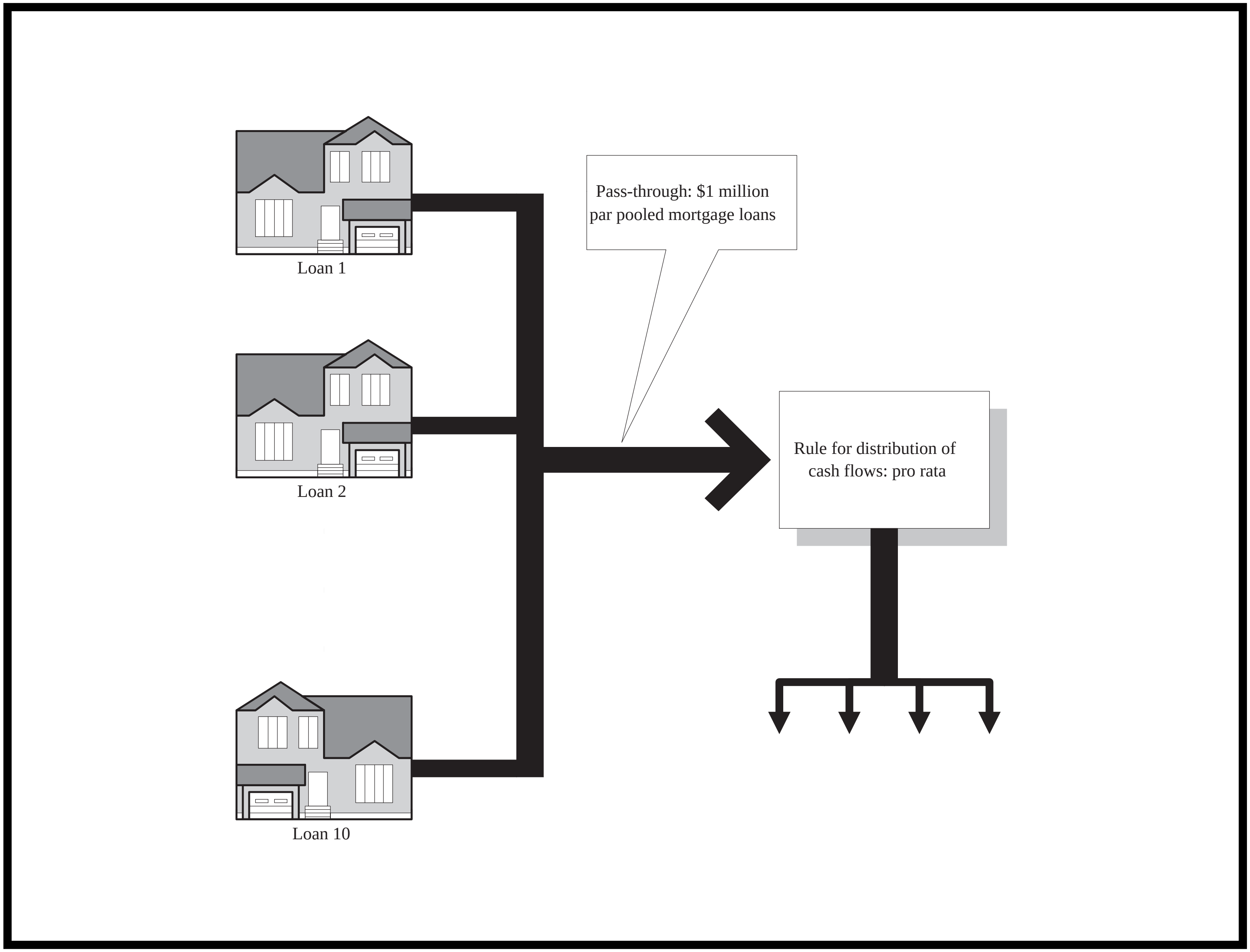
- The mortgage sector is one of the largest in the debt market (see text).
- Individual mortgages are unattractive for many investors.
- Often at hundreds of thousands of U.S. dollars or more, they demand too much investment.
- Most investors lack the resources and knowledge to assess the credit risk involved.

Problems Investing in Mortgages (concluded)

- Recall that a traditional mortgage is fixed rate, level payment, and fully amortized.
- So the percentage of principal and interest (P&I) varying from month to month, creating accounting headaches.
- Prepayment levels fluctuate with a host of factors, making the size and the timing of the cash flows unpredictable.

Mortgage Pass-Throughs

- The simplest kind of MBS.
- Payments from the underlying mortgages are passed from the mortgage holders through the servicing agency, after a fee is subtracted.
- They are distributed to the security holder on a pro rata basis.
  - The holder of a \$25,000 certificate from a \$1 million pool is entitled to $2\frac{1}{2}\%$ (or 1/40th) of the cash flow.
- Because of higher marketability, a pass-through is easier to sell than its individual loans.


Pass-through: $1 million
par pooled mortgage loans
Loan 1
Loan 2
Loan 10
Rule for distribution of
cash flows: pro rata

Collateralized Mortgage Obligations (CMOs)

- A pass-through exposes the investor to the total prepayment risk.
- Such risk is undesirable from an asset/liability perspective.
- To deal with prepayment uncertainty, CMOs were created.[a]
- Mortgage pass-throughs have a single maturity and are backed by individual mortgages.

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[a] In June 1983 by Freddie Mac with the help of First Boston.

Collateralized Mortgage Obligations (CMOs) (concluded)

- CMOs are *multiple*-maturity, *multi*class debt instruments collateralized by pass-throughs, stripped mortgage-backed securities, and whole loans.
- The total prepayment risk is now divided among classes of bonds called classes or tranches.[a]
- The principal, scheduled and prepaid, is allocated on a *prioritized* basis so as to redistribute the prepayment risk among the tranches in an unequal way.

[a] *Tranche* is a French word for "slice."

Sequential Tranche Paydown

- In the sequential tranche paydown structure, Class A receives principal paydown and prepayments before Class B, which in turn does it before Class C, and so on.
- Each tranche thus has a different effective maturity.
- Each tranche may even have a different coupon rate.
- CMOs were the first successful attempt to alter mortgage cash flows in a security form that attracts a wide range of investors