The Black-Derman-Toy Model\textsuperscript{a}

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 820ff (repeated on next page).
- The volatility structure is given by the market.
- From it, the short rate volatilities (thus $v_i$) are determined together with $r_i$.

\textsuperscript{a}Black, Derman, and Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).
The Black-Derman-Toy Model (concluded)

• Our earlier binomial interest rate tree, in contrast, assumes $v_i$ are given a priori.
  – A related model of Salomon Brothers takes $v_i$ to be a given constant.\textsuperscript{a}

• Lognormal models preclude negative short rates.

\textsuperscript{a}Tuckman (2002).
The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the $i$-period zero-coupon bond be denoted by $\kappa_i$.
- $P_u$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes an up move.
- $P_d$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes a down move.
The BDT Model: Volatility Structure (concluded)

• Corresponding to these two prices are the following yields to maturity,

\[ y_u \equiv P_u^{-1/(i-1)} - 1, \]
\[ y_d \equiv P_d^{-1/(i-1)} - 1. \]

• The yield volatility is defined as

\[ \kappa_i \equiv \frac{\ln(y_u/y_d)}{2}. \]
The BDT Model: Calibration

• The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.

• For economy of expression, all numbers are period based.

• Suppose inductively that we have calculated

\[(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1}).\]

  – They define the binomial tree up to period \(i - 1\).

• We now proceed to calculate \(r_i\) and \(v_i\) to extend the tree to period \(i\).
The BDT Model: Calibration (continued)

- Assume the price of the $i$-period zero can move to $P_u$ or $P_d$ one period from now.

- Let $y$ denote the current $i$-period spot rate, which is known.

- In a risk-neutral economy,

$$
\frac{P_u + P_d}{2(1 + r_1)} = \frac{1}{(1 + y)^i}.
$$

(117)

- Obviously, $P_u$ and $P_d$ are functions of the unknown $r_i$ and $v_i$. 
• Viewed from now, the future \((i - 1)\)-period spot rate at time 1 is uncertain.

• Recall that \(y_u\) and \(y_d\) represent the spot rates at the up node and the down node, respectively (p. 984).

• With \(\kappa^2\) denoting their variance, we have

\[
\kappa_i = \frac{1}{2} \ln \left( \frac{P_u^{-1/(i-1)} - 1}{P_d^{-1/(i-1)} - 1} \right). \tag{118}
\]
The BDT Model: Calibration (continued)

• We will employ forward induction to derive a quadratic-time calibration algorithm.a

• Recall that forward induction inductively figures out, by moving forward in time, how much $1 at a node contributes to the price (review p. 846(a)).

• This number is called the state price and is the price of the claim that pays $1 at that node and zero elsewhere.

---

aChen (R84526007) and Lyuu (1997); Lyuu (1999).
The BDT Model: Calibration (continued)

• Let the unknown baseline rate for period \( i \) be \( r_i = r \).

• Let the unknown multiplicative ratio be \( v_i = v \).

• Let the state prices at time \( i - 1 \) be \( P_1, P_2, \ldots, P_i \), corresponding to rates \( r, rv, \ldots, rv^{i-1} \), respectively.

• One dollar at time \( i \) has a present value of

\[
f(r, v) \equiv \frac{P_1}{1 + r} + \frac{P_2}{1 + rv} + \frac{P_3}{1 + rv^2} + \cdots + \frac{P_i}{1 + rv^{i-1}}.
\]
The BDT Model: Calibration (continued)

• The yield volatility is

\[
g(r,v) \equiv \frac{1}{2} \ln \left( \frac{\left( \frac{P_{u,1}}{1+r} + \frac{P_{u,2}}{1+r^2v} + \cdots + \frac{P_{u,i-1}}{1+r^i v^{i-1}} \right)^{-1/(i-1)} - 1}{\left( \frac{P_{d,1}}{1+r} + \frac{P_{d,2}}{1+r^2v} + \cdots + \frac{P_{d,i-1}}{1+r^i v^{i-2}} \right)^{-1/(i-1)} - 1} \right).
\]

• Above, \( P_{u,1}, P_{u,2}, \ldots \) denote the state prices at time \( i-1 \) of the subtree rooted at the up node (like \( r_2 v_2 \) on p. 981).

• And \( P_{d,1}, P_{d,2}, \ldots \) denote the state prices at time \( i-1 \) of the subtree rooted at the down node (like \( r_2 \) on p. 981).
The BDT Model: Calibration (concluded)

- Note that every node maintains 3 state prices.
- Now solve

\[
\begin{align*}
  f(r, v) &= \frac{1}{(1+y)^i}, \\
  g(r, v) &= \kappa_i,
\end{align*}
\]

for \( r = r_i \) and \( v = v_i \).

- This \( O(n^2) \)-time algorithm appears in the text.
The BDT Model: Continuous-Time Limit

- The continuous-time limit of the BDT model is
  \[ d\ln r = \left( \theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t) dW. \]

- The short rate volatility clearly should be a declining function of time for the model to display mean reversion.
  - That makes \( \sigma'(t) < 0. \)

- In particular, constant volatility will not attain mean reversion.
The Black-Karasinski Model\textsuperscript{a}

- The BK model stipulates that the short rate follows
  \[d\ln r = \kappa(t)(\theta(t) - \ln r)\,dt + \sigma(t)\,dW.\]

- This explicitly mean-reverting model depends on time through \(\kappa(\cdot), \theta(\cdot),\) and \(\sigma(\cdot).\)

- The BK model hence has one more degree of freedom than the BDT model.

- The speed of mean reversion \(\kappa(t)\) and the short rate volatility \(\sigma(t)\) are independent.

\textsuperscript{a}Black and Karasinski (1991).
The Black-Karasinski Model: Discrete Time

- The discrete-time version of the BK model has the same representation as the BDT model.
- To maintain a combining binomial tree, however, requires some manipulations.
- The next plot illustrates the ideas in which

\[
\begin{align*}
t_2 & \equiv t_1 + \Delta t_1, \\
t_3 & \equiv t_2 + \Delta t_2.
\end{align*}
\]
\[ \ln r(t_1) \implies \ln r_d(t_2) \implies \ln r_{du}(t_3) = \ln r_{ud}(t_3) \implies \ln r_u(t_2) \]
The Black-Karasinski Model: Discrete Time (continued)

- Note that

\[ \ln r_d(t_2) = \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1)) \Delta t_1 - \sigma(t_1)\sqrt{\Delta t_1}, \]
\[ \ln r_u(t_2) = \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1)) \Delta t_1 + \sigma(t_1)\sqrt{\Delta t_1}. \]

- To ensure that an up move followed by a down move coincides with a down move followed by an up move, impose

\[ \ln r_d(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_d(t_2)) \Delta t_2 + \sigma(t_2)\sqrt{\Delta t_2}, \]
\[ = \ln r_u(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_u(t_2)) \Delta t_2 - \sigma(t_2)\sqrt{\Delta t_2}. \]
The Black-Karasinski Model: Discrete Time (concluded)

• They imply

\[ \kappa(t_2) = \frac{1 - (\sigma(t_2)/\sigma(t_1))\sqrt{\Delta t_2/\Delta t_1}}{\Delta t_2}. \] (119)

• So from \( \Delta t_1 \), we can calculate the \( \Delta t_2 \) that satisfies the combining condition and then iterate.

\[- t_0 \rightarrow \Delta t_0 \rightarrow t_1 \rightarrow \Delta t_1 \rightarrow t_2 \rightarrow \Delta t_2 \rightarrow \cdots \rightarrow T \]

(roughly).\(^a\)

\(^a\)As \( \kappa(t), \theta(t), \sigma(t) \) are independent of \( r \), the \( \Delta t_i \)'s will not depend on \( r \).
Problems with Lognormal Models in General

• Lognormal models such as BDT and BK share the problem that \( E^\pi[M(t)] = \infty \) for any finite \( t \) if they the continuously compounded rate.

• Hence periodic compounding should be used.

• Another issue is computational.

• Lognormal models usually do not give analytical solutions to even basic fixed-income securities.

• As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.
Problems with Lognormal Models in General (concluded)

- This problem can be somewhat mitigated by adopting different time steps: Use a fine time step up to the maturity of the short-dated derivative and a coarse time step beyond the maturity.\(^a\)

- A down side of this procedure is that it has to be tailor-made for each derivative.

- Finally, empirically, interest rates do not follow the lognormal distribution.

\(^a\)Hull and White (1993).
The Extended Vasicek Model\textsuperscript{a}

- Hull and White proposed models that extend the Vasicek model and the CIR model.
- They are called the extended Vasicek model and the extended CIR model.
- The extended Vasicek model adds time dependence to the original Vasicek model,

\[ dr = (\theta(t) - a(t) r) \, dt + \sigma(t) \, dW. \]

- Like the Ho-Lee model, this is a normal model, and the inclusion of \( \theta(t) \) allows for an exact fit to the current spot rate curve.

\textsuperscript{a}Hull and White (1990).
The Extended Vasicek Model (concluded)

- Function $\sigma(t)$ defines the short rate volatility, and $a(t)$ determines the shape of the volatility structure.

- Under this model, many European-style securities can be evaluated analytically, and efficient numerical procedures can be developed for American-style securities.
The Hull-White Model

• The Hull-White model is the following special case,

\[ dr = (\theta(t) - ar) \, dt + \sigma \, dW. \]

• When the current term structure is matched,\(^a\)

\[ \theta(t) = \frac{\partial f(0, t)}{\partial t} + af(0, t) + \frac{\sigma^2}{2a} \left(1 - e^{-2at}\right). \]

\(^{a}\)Hull and White (1993).
The Extended CIR Model

• In the extended CIR model the short rate follows

$$dr = (\theta(t) - a(t) r) dt + \sigma(t) \sqrt{r} dW.$$ 

• The functions $\theta(t)$, $a(t)$, and $\sigma(t)$ are implied from market observables.

• With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.
The Hull-White Model: Calibration\textsuperscript{a}

- We describe a trinomial forward induction scheme to calibrate the Hull-White model given \( a \) and \( \sigma \).

- As with the Ho-Lee model, the set of achievable short rates is evenly spaced.

- Let \( r_0 \) be the annualized, continuously compounded short rate at time zero.

- Every short rate on the tree takes on a value

\[
r_0 + j\Delta r
\]

for some integer \( j \).

\textsuperscript{a}Hull and White (1993).
The Hull-White Model: Calibration (continued)

- Time increments on the tree are also equally spaced at $\Delta t$ apart.
- Hence nodes are located at times $i\Delta t$ for $i = 0, 1, 2, \ldots$.
- We shall refer to the node on the tree with $t_i \equiv i\Delta t$, $r_j \equiv r_0 + j\Delta r$, as the $(i, j)$ node.
- The short rate at node $(i, j)$, which equals $r_j$, is effective for the time period $[t_i, t_{i+1})$. 
The Hull-White Model: Calibration (continued)

- Use

\[ \mu_{i,j} \equiv \theta(t_i) - ar_j \]  \hspace{1cm} (120)

...to denote the drift rate, or the expected change, of the short rate as seen from node \((i, j)\).

- The three distinct possibilities for node \((i, j)\) with three branches incident from it are displayed on p. 1007.

- The interest rate movement described by the middle branch may be an increase of \(\Delta r\), no change, or a decrease of \(\Delta r\).

\[ ^{\text{aA predecessor to Lyuu and Wu’s (R90723065) (2003, 2005) mean-tracking idea.}} \]
The Hull-White Model: Calibration (continued)

\[(i, j) \rightarrow (i + 1, j + 1) \rightarrow (i + 1, j) (i, j) \rightarrow (i + 1, j)(i, j) \rightarrow (i + 1, j) \rightarrow (i + 1, j + 1) \rightarrow (i + 1, j + 2) \]

\[(i + 1, j) \rightarrow (i + 1, j - 1) \rightarrow (i + 1, j - 2) \]
The Hull-White Model: Calibration (continued)

• The upper and the lower branches bracket the middle branch.

• Define

\[ p_1(i, j) \equiv \text{the probability of following the upper branch from node } (i, j) \]
\[ p_2(i, j) \equiv \text{the probability of following the middle branch from node } (i, j) \]
\[ p_3(i, j) \equiv \text{the probability of following the lower branch from node } (i, j) \]

• The root of the tree is set to the current short rate \( r_0 \).

• Inductively, the drift \( \mu_{i,j} \) at node \( (i, j) \) is a function of \( \theta(t_i) \).
The Hull-White Model: Calibration (continued)

- Once $\theta(t_i)$ is available, $\mu_{i,j}$ can be derived via Eq. (120) on p. 1006.
- This in turn determines the branching scheme at every node $(i,j)$ for each $j$, as we will see shortly.
- The value of $\theta(t_i)$ must thus be made consistent with the spot rate $r(0,t_{i+2})$. 
The Hull-White Model: Calibration (continued)

- The branches emanating from node \((i, j)\) with their accompanying probabilities\(^a\) must be chosen to be consistent with \(\mu_{i,j}\) and \(\sigma\).

- This is accomplished by letting the middle node be as close as possible to the current value of the short rate plus the drift.

- Let \(k\) be the number among \(\{j - 1, j, j + 1\}\) that makes the short rate reached by the middle branch, \(r_k\), closest to \(r_j + \mu_{i,j}\Delta t\).

\(^a\)\(p_1(i, j), p_2(i, j), \text{ and } p_3(i, j)\).
The Hull-White Model: Calibration (continued)

- Then the three nodes following node \((i, j)\) are nodes \((i + 1, k + 1)\), \((i + 1, k)\), and \((i + 1, k - 1)\).

- The resulting tree may have the geometry depicted on p. 1012.

- The resulting tree combines because of the constant jump sizes to reach \(k\).
The Hull-White Model: Calibration (continued)

- The probabilities for moving along these branches are functions of $\mu_{i,j}$, $\sigma$, $j$, and $k$:

$$p_1(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} + \frac{\eta}{2\Delta r}$$  \hspace{2cm} (121)

$$p_2(i,j) = 1 - \frac{\sigma^2 \Delta t + \eta^2}{(\Delta r)^2}$$  \hspace{2cm} (121')

$$p_3(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} - \frac{\eta}{2\Delta r}$$  \hspace{2cm} (121'')

where

$$\eta \equiv \mu_{i,j} \Delta t + (j - k) \Delta r.$$
The Hull-White Model: Calibration (continued)

• As trinomial tree algorithms are but explicit methods in disguise, certain relations must hold for $\Delta r$ and $\Delta t$ to guarantee stability.

• It can be shown that their values must satisfy

$$\frac{\sigma \sqrt{3\Delta t}}{2} \leq \Delta r \leq 2\sigma \sqrt{\Delta t}$$

for the probabilities to lie between zero and one.

  - For example, $\Delta r$ can be set to $\sigma \sqrt{3\Delta t}$.

---

\textsuperscript{a}Hull and White (1988).
The Hull-White Model: Calibration (continued)

• Now it only remains to determine $\theta(t_i)$.

• At this point at time $t_i$, $r(0,t_1)$, $r(0,t_2)$, \ldots, $r(0,t_{i+1})$ have already been matched.

• Let $Q(i,j)$ denote the value of the state contingent claim that pays one dollar at node $(i,j)$ and zero otherwise.

• By construction, the state prices $Q(i,j)$ for all $j$ are known by now.

• We begin with state price $Q(0,0) = 1$. 
The Hull-White Model: Calibration (continued)

- Let \( \hat{r}(i) \) refer to the short rate value at time \( t_i \).
- The value at time zero of a zero-coupon bond maturing at time \( t_{i+2} \) is then

\[
e^{-r(0,t_{i+2})(i+2) \Delta t} = \sum_j Q(i, j) e^{-r_j \Delta t} E^\pi \left[ e^{-\hat{r}(i+1) \Delta t} \bigg| \hat{r}(i) = r_j \right]. \tag{122}
\]

- The right-hand side represents the value of $1 obtained by holding a zero-coupon bond until time \( t_{i+1} \) and then reinvesting the proceeds at that time at the prevailing short rate \( \hat{r}(i+1) \), which is stochastic.
The Hull-White Model: Calibration (continued)

• The expectation (122) can be approximated by

\[ E^\pi \left[ e^{-\hat{r}(i+1)\Delta t} \mid \hat{r}(i) = r_j \right] \]

\[ \approx e^{-r_j\Delta t} \left( 1 - \mu_{i,j}(\Delta t)^2 + \frac{\sigma^2(\Delta t)^3}{2} \right). \quad (123) \]

• Substitute Eq. (123) into Eq. (122) and replace \( \mu_{i,j} \) with \( \theta(t_i) - ar_j \) to obtain

\[
\theta(t_i) \approx \frac{\sum_j Q(i, j) e^{-2r_j\Delta t} \left( 1 + ar_j(\Delta t)^2 + \sigma^2(\Delta t)^3 / 2 \right) - e^{-r(0,t_i+2)(i+2)\Delta t}}{(\Delta t)^2 \sum_j Q(i, j) e^{-2r_j\Delta t}}.
\]
The Hull-White Model: Calibration (continued)

- For the Hull-White model, the expectation in Eq. (123) on p. 1017 is actually known analytically by Eq. (19) on p. 152:

\[
E^{\pi} \left[ e^{-\hat{\tau}(i+1)\Delta t} \bigg| \hat{\tau}(i) = r_j \right] = e^{-r_j \Delta t + (-\theta(t_i) + ar_j + \sigma^2 \Delta t/2)(\Delta t)^2}.
\]

- Therefore, alternatively,

\[
\theta(t_i) = \frac{r(0, t_{i+2})(i + 2)}{\Delta t} + \frac{\sigma^2 \Delta t}{2} + \frac{\ln \sum_j Q(i, j) e^{-2r_j \Delta t + ar_j (\Delta t)^2}}{(\Delta t)^2}.
\]
The Hull-White Model: Calibration (concluded)

- With $\theta(t_i)$ in hand, we can compute $\mu_{i,j}$, the probabilities, and finally the state prices at time $t_{i+1}$:

$$Q(i+1, j) = \sum_{(i, j^*) \text{ is connected to } (i + 1, j) \text{ with probability } p_{j^*}} p_{j^*} e^{-r_{j^*} \Delta t} Q(i, j^*).$$

- There are at most 5 choices for $j^*$ (why?).
- The total running time is $O(n^2)$.
- The space requirement is $O(n)$ (why?).
Comments on the Hull-White Model

- One can try different values of $a$ and $\sigma$ for each option or have an $a$ value common to all options but use a different $\sigma$ value for each option.
- Either approach can match all the option prices exactly.
- If the demand is for a single set of parameters that replicate all option prices, the Hull-White model can be calibrated to all the observed option prices by choosing $a$ and $\sigma$ that minimize the mean-squared pricing error.\(^a\)

\(^a\)Hull and White (1995).
The Hull-White Model: Calibration with Irregular Trinomial Trees

- The previous calibration algorithm is quite general.
- For example, it can be modified to apply to cases where the diffusion term has the form $\sigma r^b$.
- But it has at least two shortcomings.
- First, the resulting trinomial tree is irregular (p. 1012).
  - So it is harder to program.
- The second shortcoming is again a consequence of the tree’s irregular shape.
The Hull-White Model: Calibration with Irregular Trinomial Trees (concluded)

- Recall that the algorithm figured out $\theta(t_i)$ that matches the spot rate $r(0, t_{i+2})$ in order to determine the branching schemes for the nodes at time $t_i$.

- But without those branches, the tree was not specified, and backward induction on the tree was not possible.

- To avoid this dilemma, the algorithm turned to the continuous-time model to evaluate Eq. (122) on p. 1016 that helps derive $\theta(t_i)$ later.

- The resulting $\theta(t_i)$ hence might not yield a tree that matches the spot rates exactly.
The Hull-White Model: Calibration with Regular Trinomial Trees\(^a\)

- The next, simpler algorithm exploits the fact that the Hull-White model has a constant diffusion term \(\sigma\).
- The resulting trinomial tree will be regular.
- All the \(\theta(t_i)\) terms can be chosen by backward induction to match the spot rates exactly.
- The tree is constructed in two phases.

\(^a\)Hull and White (1994).
The Hull-White Model: Calibration with Regular Trinomial Trees (continued)

- In the first phase, a tree is built for the $\theta(t) = 0$ case, which is an Ornstein-Uhlenbeck process:

$$dr = -ar\,dt + \sigma\,dW, \quad r(0) = 0.$$ 

- The tree is dagger-shaped (p. 1026).
- The number of nodes above the $r_0$-line, $j_{\text{max}}$, and that below the line, $j_{\text{min}}$, will be picked so that the probabilities (121) on p. 1013 are positive for all nodes.
- The tree’s branches and probabilities are in place.
The Hull-White Model: Calibration with Regular Trinomial Trees (concluded)

- Phase two fits the term structure.
  - Backward induction is applied to calculate the $\beta_i$ to add to the short rates on the tree at time $t_i$ so that the spot rate $r(0, t_{i+1})$ is matched.
The short rate at node \((0, 0)\) equals \(r_0 = 0\); here \(j_{\max} = 3\) and \(j_{\min} = 2\).
The Hull-White Model: Calibration

- Set $\Delta r = \sigma \sqrt{3 \Delta t}$ and assume that $a > 0$.
- Node $(i, j)$ is a top node if $j = j_{\text{max}}$ and a bottom node if $j = -j_{\text{min}}$.
- Because the root of the tree has a short rate of $r_0 = 0$, phase one adopts $r_j = j \Delta r$.
- Hence the probabilities in Eqs. (121) on p. 1013 use

$$\eta \equiv -a j \Delta r \Delta t + (j - k) \Delta r.$$
The Hull-White Model: Calibration (continued)

- The probabilities become

\[
p_1(i, j) = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2a j \Delta t (j - k) + (j - k)^2 - a j \Delta t + (j - k)}{2}, \quad (124)
\]

\[
p_2(i, j) = \frac{2}{3} - \left[ a^2 j^2 (\Delta t)^2 - 2a j \Delta t (j - k) + (j - k)^2 \right], \quad (125)
\]

\[
p_3(i, j) = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2a j \Delta t (j - k) + (j - k)^2 + a j \Delta t - (j - k)}{2}. \quad (126)
\]
The Hull-White Model: Calibration (continued)

• The dagger shape dictates this:
  - Let $k = j - 1$ if node $(i, j)$ is a top node.
  - Let $k = j + 1$ if node $(i, j)$ is a bottom node.
  - Let $k = j$ for the rest of the nodes.

• Note that the probabilities are identical for nodes $(i, j)$ with the same $j$.

• Furthermore, $p_1(i, j) = p_3(i, -j)$. 
The Hull-White Model: Calibration (continued)

- The inequalities

\[
\frac{3 - \sqrt{6}}{3} < ja\Delta t < \sqrt{\frac{2}{3}}
\]

(127)

ensure that all the branching probabilities are positive in the upper half of the tree, that is, \( j > 0 \) (verify this).

- Similarly, the inequalities

\[
-\sqrt{\frac{2}{3}} < ja\Delta t < -\frac{3 - \sqrt{6}}{3}
\]

ensure that the probabilities are positive in the lower half of the tree, that is, \( j < 0 \).
The Hull-White Model: Calibration (continued)

- To further make the tree symmetric across the $r_0$-line, we let $j_{\text{min}} = j_{\text{max}}$.
- As $\frac{3 - \sqrt{6}}{3} \approx 0.184$, a good choice is

$$j_{\text{max}} = \lceil 0.184/(a\Delta t) \rceil.$$  

- Phase two computes the $\beta_i$s to fit the spot rates.
- We begin with state price $Q(0, 0) = 1$.
- Inductively, suppose that spot rates $r(0, t_1)$, $r(0, t_2)$, $\ldots$, $r(0, t_i)$ have already been matched at time $t_i$.  

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The Hull-White Model: Calibration (continued)

• By construction, the state prices $Q(i, j)$ for all $j$ are known by now.

• The value of a zero-coupon bond maturing at time $t_{i+1}$ equals

$$e^{-r(0, t_{i+1})(i+1)\Delta t} = \sum_j Q(i, j) e^{-(\beta_i + r_j)\Delta t}$$

by risk-neutral valuation.

• Hence

$$\beta_i = \frac{r(0, t_{i+1})(i + 1)\Delta t + \ln \sum_j Q(i, j) e^{-r_j\Delta t}}{\Delta t},$$

and the short rate at node $(i, j)$ equals $\beta_i + r_j$. 
The Hull-White Model: Calibration (concluded)

- The state prices at time $t_{i+1}$, 
  
  $$Q(i + 1, j), \quad -\min(i + 1, j_{\text{max}}) \leq j \leq \min(i + 1, j_{\text{max}}),$$
  
  can now be calculated as before.

- The total running time is $O(nj_{\text{max}})$.

- The space requirement is $O(n)$. 
A Numerical Example

- Assume $a = 0.1$, $\sigma = 0.01$, and $\Delta t = 1$ (year).
- Immediately, $\Delta r = 0.0173205$ and $j_{\text{max}} = 2$.
- The plot on p. 1035 illustrates the 3-period trinomial tree after phase one.
- For example, the branching probabilities for node E are calculated by Eqs. (124)–(126) on p. 1028 with $j = 2$ and $k = 1$. 
<table>
<thead>
<tr>
<th>Node</th>
<th>A, C, G</th>
<th>B, F</th>
<th>E</th>
<th>D, H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$ (%)</td>
<td>0.00000</td>
<td>1.73205</td>
<td>3.46410</td>
<td>-1.73205</td>
<td>-3.46410</td>
</tr>
<tr>
<td>$p_1$</td>
<td>0.16667</td>
<td>0.12167</td>
<td>0.88667</td>
<td>0.22167</td>
<td>0.08667</td>
</tr>
<tr>
<td>$p_2$</td>
<td>0.66667</td>
<td>0.65667</td>
<td>0.02667</td>
<td>0.65667</td>
<td>0.02667</td>
</tr>
<tr>
<td>$p_3$</td>
<td>0.16667</td>
<td>0.22167</td>
<td>0.08667</td>
<td>0.12167</td>
<td>0.88667</td>
</tr>
</tbody>
</table>
A Numerical Example (continued)

• Suppose that phase two is to fit the spot rate curve

$$0.08 - 0.05 \times e^{-0.18t}. $$

• The annualized continuously compounded spot rates are

$$r(0, 1) = 3.82365\%, r(0, 2) = 4.51162\%, r(0, 3) = 5.08626\%. $$

• Start with state price $Q(0, 0) = 1$ at node A.
A Numerical Example (continued)

• Now,

\[ \beta_0 = r(0, 1) + \ln Q(0, 0) e^{-r_0} = r(0, 1) = 3.82365\%. \]

• Hence the short rate at node A equals

\[ \beta_0 + r_0 = 3.82365\%. \]

• The state prices at year one are calculated as

\[
\begin{align*}
Q(1, 1) &= p_1(0, 0) e^{-(\beta_0 + r_0)} = 0.160414, \\
Q(1, 0) &= p_2(0, 0) e^{-(\beta_0 + r_0)} = 0.641657, \\
Q(1, -1) &= p_3(0, 0) e^{-(\beta_0 + r_0)} = 0.160414.
\end{align*}
\]
A Numerical Example (continued)

- The 2-year rate spot rate \( r(0, 2) \) is matched by picking

\[
\beta_1 = r(0, 2) \times 2 + \ln \left[ Q(1, 1) e^{-\Delta r} + Q(1, 0) + Q(1, -1) e^{\Delta r} \right] = 5.20459\%.
\]

- Hence the short rates at nodes B, C, and D equal

\[
\beta_1 + r_j,
\]

where \( j = 1, 0, -1 \), respectively.

- They are found to be 6.93664\%, 5.20459\%, and 3.47254\%.
A Numerical Example (continued)

- The state prices at year two are calculated as

\[
\begin{align*}
Q(2, 2) &= p_1(1, 1) e^{-(\beta_1+r_1)} Q(1, 1) = 0.018209, \\
Q(2, 1) &= p_2(1, 1) e^{-(\beta_1+r_1)} Q(1, 1) + p_1(1, 0) e^{-(\beta_1+r_0)} Q(1, 0) \\
&= 0.199799, \\
Q(2, 0) &= p_3(1, 1) e^{-(\beta_1+r_1)} Q(1, 1) + p_2(1, 0) e^{-(\beta_1+r_0)} Q(1, 0) \\
&\quad + p_1(1, -1) e^{-(\beta_1+r-1)} Q(1, -1) = 0.473597, \\
Q(2, -1) &= p_3(1, 0) e^{-(\beta_1+r_0)} Q(1, 0) + p_2(1, -1) e^{-(\beta_1+r-1)} Q(1, -1) \\
&= 0.203263, \\
Q(2, -2) &= p_3(1, -1) e^{-(\beta_1+r-1)} Q(1, -1) = 0.018851.
\end{align*}
\]
A Numerical Example (concluded)

- The 3-year rate spot rate $r(0, 3)$ is matched by picking

$$\beta_2 = r(0, 3) \times 3 + \ln \left[ Q(2, 2) e^{-2 \times \Delta r} + Q(2, 1) e^{-\Delta r} + Q(2, 0) + Q(2, -1) e^{\Delta r} + Q(2, -2) e^{2 \times \Delta r} \right] = 6.25359\%.$$  

- Hence the short rates at nodes E, F, G, H, and I equal $\beta_2 + r_j$, where $j = 2, 1, 0, -1, -2$, respectively.

- They are found to be 9.71769%, 7.98564%, 6.25359%, 4.52154%, and 2.78949%.

- The figure on p. 1041 plots $\beta_i$ for $i = 0, 1, \ldots, 29$. 
The (Whole) Yield Curve Approach

- We have seen several Markovian short rate models.
- The Markovian approach is computationally efficient.
- But it is difficult to model the behavior of yields and bond prices of different maturities.
- The alternative yield curve approach regards the whole term structure as the state of a process and directly specifies how it evolves.
The Heath-Jarrow-Morton Model

• This influential model is a forward rate model.
• It is also a popular model.
• The HJM model specifies the initial forward rate curve and the forward rate volatility structure, which describes the volatility of each forward rate for a given maturity date.
• Like the Black-Scholes option pricing model, neither risk preference assumptions nor the drifts of forward rates are needed.

\(^a\)Heath, Jarrow, and Morton (HJM) (1992).
Introduction to Mortgage-Backed Securities
Anyone stupid enough to promise to be responsible for a stranger’s debts deserves to have his own property held to guarantee payment.

— Proverbs 27:13
Mortgages

• A mortgage is a loan secured by the collateral of real estate property.

• The lender — the mortgagee — can foreclose the loan by seizing the property if the borrower — the mortgagor — defaults, that is, fails to make the contractual payments.
Mortgage-Backed Securities

- A mortgage-backed security (MBS) is a bond backed by an undivided interest in a pool of mortgages.
- MBSs traditionally enjoy high returns, wide ranges of products, high credit quality, and liquidity.
- The mortgage market has witnessed tremendous innovations in product design.
Mortgage-Backed Securities (concluded)

- The complexity of the products and the prepayment option require advanced models and software techniques.
  - In fact, the mortgage market probably could not have operated efficiently without them.\(^a\)
- They also consume lots of computing power.
- Our focus will be on residential mortgages.
- But the underlying principles are applicable to other types of assets.

\(^a\)Merton (1994).
Types of MBSs

- An MBS is issued with pools of mortgage loans as the collateral.
- The cash flows of the mortgages making up the pool naturally reflect upon those of the MBS.
- There are three basic types of MBSs:
  1. Mortgage pass-through security (MPTS).
  2. Collateralized mortgage obligation (CMO).
Problems Investing in Mortgages

• The mortgage sector is one of the largest in the debt market (see text).

• Individual mortgages are unattractive for many investors.

• Often at hundreds of thousands of U.S. dollars or more, they demand too much investment.

• Most investors lack the resources and knowledge to assess the credit risk involved.
Problems Investing in Mortgages (concluded)

• Recall that a traditional mortgage is fixed rate, level payment, and fully amortized.

• So the percentage of principal and interest (P&I) varying from month to month, creating accounting headaches.

• Prepayment levels fluctuate with a host of factors, making the size and the timing of the cash flows unpredictable.
Mortgage Pass-Throughs

- The simplest kind of MBS.
- Payments from the underlying mortgages are passed from the mortgage holders through the servicing agency, after a fee is subtracted.
- They are distributed to the security holder on a pro rata basis.
  - The holder of a $25,000 certificate from a $1 million pool is entitled to 2\(\frac{1}{2}\)% (or 1/40th) of the cash flow.
- Because of higher marketability, a pass-through is easier to sell than its individual loans.
Rule for distribution of cash flows: pro rata

Pass-through: $1 million par pooled mortgage loans

Loan 1
Loan 2
Loan 10

Rule for distribution of cash flows: pro rata
Collateralized Mortgage Obligations (CMOs)

- A pass-through exposes the investor to the total prepayment risk.
- Such risk is undesirable from an asset/liability perspective.
- To deal with prepayment uncertainty, CMOs were created.\(^a\)
- Mortgage pass-throughs have a single maturity and are backed by individual mortgages.

\(^a\)In June 1983 by Freddie Mac with the help of First Boston.
Collateralized Mortgage Obligations (CMOs) (concluded)

- CMOs are *multiple*-maturity, *multiclass* debt instruments collateralized by pass-throughs, stripped mortgage-backed securities, and whole loans.

- The total prepayment risk is now divided among classes of bonds called classes or tranches.a

- The principal, scheduled and prepaid, is allocated on a *prioritized* basis so as to redistribute the prepayment risk among the tranches in an unequal way.

---

*a Tranche is a French word for “slice.”*
Sequential Tranche Paydown

- In the sequential tranche paydown structure, Class A receives principal paydown and prepayments before Class B, which in turn does it before Class C, and so on.
- Each tranche thus has a different effective maturity.
- Each tranche may even have a different coupon rate.
- CMOs were the first successful attempt to alter mortgage cash flows in a security form that attracts a wide range of investors.