Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t+1<T$,
    $$\frac{E_t[\,P(t+1,T)\,]}{P(t,T)}=1+r(t). \tag{101}$$
  - Relation (101) in fact follows from the risk-neutral valuation principle.[a]

[a] Theorem 16 on p. 457.

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π.
- Rewrite Eq. (101) as

$$\frac{E_t^{\pi}[\,P(t+1,T)\,]}{1+r(t)}=P(t,T).$$

  - It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.

Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

$$\begin{aligned}
& P(t,T) \\
= \;& E_t^{\pi}\left[\,\frac{P(t+1,T)}{1+r(t)}\,\right] \\
= \;& E_t^{\pi}\left[\,\frac{E_{t+1}^{\pi}[\,P(t+2,T)\,]}{(1+r(t))(1+r(t+1))}\,\right] = \cdots \\
= \;& E_t^{\pi}\left[\,\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))}\,\right]. \qquad (102)
\end{aligned}$$

Risk-Neutral Pricing (concluded)

- Equation (101) on p. 889 can also be expressed as

$$E_t[\, P(t+1, T)\,] = F(t, t+1, T).$$

  - Verify that with, e.g., Eq. (96) on p. 884.
- Hence the forward price for the next period is an unbiased estimator of the expected bond price.

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

$$P(t,T) = E_t \left[e^{-\int_t^T r(s)\,ds} \right], \quad t < T. \tag{103}$$

- Note that $e^{\int_t^T r(s)\,ds}$ is the bank account process, which denotes the rolled-over money market account.

Interest Rate Swaps

- Consider an interest rate swap made at time t with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is c per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant Δt for all i, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

Interest Rate Swaps (continued)

- The amount to be paid out at time t_{i+1} is $(f_i - c)\,\Delta t$ for the floating-rate payer.
- Simple rates are adopted here.
- Hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

Interest Rate Swaps (continued)

- The value of the swap at time t is thus

$$
\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} (f_{i-1} - c)\,\Delta t \right] \\
= & \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds} \left(\frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\
= & \sum_{i=1}^{n} [\, P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \,] \\
= & \; P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}
$$

Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.

Swap Rate

- The swap rate, which gives the swap zero value, equals

$$S_n(t) \equiv \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i)\, \Delta t}. \tag{104}$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t, t_0) = 1$.

The Term Structure Equation

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price $P(r, t, T)$ follow

$$\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.$$

- At time t, short one unit of a bond maturing at time s_1 and buy α units of a bond maturing at time s_2.

The Term Structure Equation (continued)

- The net wealth change follows

$$\begin{aligned} & -dP(r,t,s_1) + \alpha\, dP(r,t,s_2) \\ = \quad & (-P(r,t,s_1)\,\mu_p(r,t,s_1) + \alpha P(r,t,s_2)\,\mu_p(r,t,s_2))\,dt \\ & +(-P(r,t,s_1)\,\sigma_p(r,t,s_1) + \alpha P(r,t,s_2)\,\sigma_p(r,t,s_2))\,dW. \end{aligned}$$

- Pick

$$\alpha \equiv \frac{P(r,t,s_1)\,\sigma_p(r,t,s_1)}{P(r,t,s_2)\,\sigma_p(r,t,s_2)}.$$

The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

$$\frac{-P(r,t,s_1)\,\mu_p(r,t,s_1)+\alpha P(r,t,s_2)\,\mu_p(r,t,s_2)}{-P(r,t,s_1)+\alpha P(r,t,s_2)}=r.$$

- Simplify the above to obtain

$$\frac{\sigma_p(r,t,s_1)\,\mu_p(r,t,s_2)-\sigma_p(r,t,s_2)\,\mu_p(r,t,s_1)}{\sigma_p(r,t,s_1)-\sigma_p(r,t,s_2)}=r.$$

- This becomes

$$\frac{\mu_p(r,t,s_2)-r}{\sigma_p(r,t,s_2)}=\frac{\mu_p(r,t,s_1)-r}{\sigma_p(r,t,s_1)}$$

after rearrangement.

The Term Structure Equation (continued)

- Since the above equality holds for any s_1 and s_2,

$$\frac{\mu_p(r,t,s)-r}{\sigma_p(r,t,s)} \equiv \lambda(r,t) \tag{105}$$

for some λ independent of the bond maturity s.

- As $\mu_p = r + \lambda\sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.
- The term $\lambda(r,t)$ is called the market price of risk.
- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.

The Term Structure Equation (continued)

- Assume a Markovian short rate model,

$$dr = \mu(r,t)\,dt + \sigma(r,t)\,dW.$$

- Then the bond price process is also Markovian.
- By Eq. (14.15) on p. 202 in the text,

$$\mu_p = \left(-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2}\right)/P, \tag{106}$$

$$\sigma_p = \left(\sigma(r,t)\,\frac{\partial P}{\partial r}\right)/P, \tag{106$'$}$$

subject to $P(\,\cdot\,,T,T) = 1$.

The Term Structure Equation (concluded)

- Substitute μ_p and σ_p into Eq. (105) on p. 902 to obtain

$$-\frac{\partial P}{\partial T} + [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = rP. \tag{107}$$

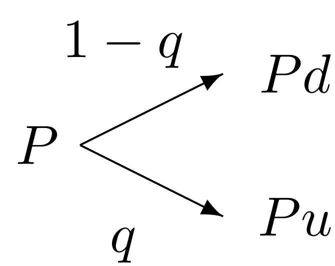
- This is called the term structure equation.
- Once P is available, the spot rate curve emerges via

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

- Equation (107) applies to all interest rate derivatives, the difference being the terminal and the boundary conditions.

The Binomial Model

- The analytical framework can be nicely illustrated with the binomial model.
- Suppose the bond price P can move with probability q to Pu and probability $1 - q$ to Pd, where $u > d$:



The Binomial Model (continued)

- Over the period, the bond's expected rate of return is

$$\widehat{\mu} \equiv \frac{qPu + (1-q)\,Pd}{P} - 1 = qu + (1-q)\,d - 1. \tag{108}$$

- The variance of that return rate is

$$\widehat{\sigma}^2 \equiv q(1-q)(u-d)^2. \tag{109}$$

The Binomial Model (continued)

- In particular, the bond whose maturity is one period away will move from a price of $1/(1+r)$ to its par value \$1.
- This is the money market account modeled by the short rate r.
- The market price of risk is defined as $\lambda \equiv (\widehat{\mu} - r)/\widehat{\sigma}$.
- As in the continuous-time case, it can be shown that λ is independent of the maturity of the bond (see text).

The Binomial Model (concluded)

- Now change the probability from q to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1+r)-d}{u-d}, \tag{110}$$

which is independent of bond maturity and q.

  – Recall the BOPM.

- The bond's expected rate of return becomes

$$\frac{pPu + (1-p)\,Pd}{P} - 1 = pu + (1-p)\,d - 1 = r.$$

- The local expectations theory hence holds under the new probability measure p.

Numerical Examples

- Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

$$
\begin{tikzcd}
 & 8\% \\
4\% \arrow[ur] \arrow[dr] & \\
 & 2\%
\end{tikzcd}
$$

Numerical Examples (continued)

- No real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$\begin{aligned} 100/1.04 &= 96.154, \\ 100/(1.05)^2 &= 90.703. \end{aligned}$$

- They follow the binomial processes on p. 911.

Numerical Examples (continued)

$$
\begin{tikzcd}[column sep=small]
 & 92.593\ (= 100/1.08) \\
90.703 \arrow[ur] \arrow[dr] & \\
 & 98.039\ (= 100/1.02)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & 100 \\
96.154 \arrow[ur] \arrow[dr] & \\
 & 100
\end{tikzcd}
$$

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

where p denotes the risk-neutral probability of a down move in rates.

Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,

$$
\begin{tikzcd}[row sep=small]
 & 0.000 \\
C \arrow[ur] \arrow[dr] & \\
 & 3.039
\end{tikzcd}
$$

- To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.

Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$\begin{aligned} x \times 100 + y \times 92.593 &= 0.000, \\ x \times 100 + y \times 98.039 &= 3.039. \end{aligned}$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

  the same as before.
- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity:

$$
\begin{tikzcd}
 & 92\ (= 100 - 8) \\
F \arrow[ur] \arrow[dr] & \\
 & 98\ (= 100 - 2)
\end{tikzcd}
$$

- As the futures price F is the expected future payoff (see text or p. 458), $F = (1 - p) \times 92 + p \times 98 = 93.914$.
- The forward price for a one-year forward contract on a one-year zero-coupon bond is $90.703/96.154 = 94.331\%$.
- The forward price exceeds the futures price.[a]

[a]Recall p. 404.

Numerical Examples: Mortgage-Backed Securities

- Consider a 5%-coupon, two-year mortgage-backed security without amortization, prepayments, and default risk.
- Its cash flow and price process are illustrated on p. 920.
- Its fair price is

$$M = \frac{(1-p) \times 102.222 + p \times 107.941}{1.04} = 100.045.$$

- Identical results could have been obtained via arbitrage considerations.

$$
\begin{array}{ccccc}
 & & & & 105 \\
 & & & \nearrow & \\
 & & 5 & & \\
 & \nearrow & & \searrow & \\
 & & & & 105 \\
0 & & & & \\
 & & & & 105 \\
 & \searrow & & \nearrow & \\
 & & 5 & & \\
 & & & \searrow & \\
 & & & & 105
\end{array}
\qquad\qquad
\begin{array}{ccl}
 & & 102.222 \; (= 5 + (105/1.08)) \\
 & \nearrow & \\
M & & \\
 & \searrow & \\
 & & 107.941 \; (= 5 + (105/1.02))
\end{array}
$$

The left diagram depicts the cash flow; the right diagram illustrates the price process.

Numerical Examples: MBSs (continued)

- Suppose that the security can be prepaid at par.
- It will be prepaid only when its price is higher than par.
- Prepayment will hence occur only in the "down" state when the security is worth 102.941 (excluding coupon).
- The price therefore follows the process,

$$\begin{tikzcd}[row sep=small] & 102.222 \\ M \arrow[ur] \arrow[dr] & \\ & 105 \end{tikzcd}$$

- The security is worth

$$M = \frac{(1-p) \times 102.222 + p \times 105}{1.04} = 99.142.$$

Numerical Examples: MBSs (continued)

- The cash flow of the principal-only (PO) strip comes from the mortgage's principal cash flow.
- The cash flow of the interest-only (IO) strip comes from the interest cash flow (p. 923(a)).
- Their prices hence follow the processes on p. 923(b).
- The fair prices are

$$
\begin{aligned}
\text{PO} &= \frac{(1-p)\times 92.593 + p\times 100}{1.04} = 91.304,\\
\text{IO} &= \frac{(1-p)\times 9.630 + p\times 5}{1.04} = 7.839.
\end{aligned}
$$

$$\begin{array}{ccccc} & & \text{PO:} & & 100\\ & & & \nearrow & \\ & & 0 & & \\ & \nearrow & & \searrow & \\ & & & & 100 \\ 0 & & & & \\ & & & & 0\\ & \searrow & & \nearrow & \\ & & 100 & & \\ & & & \searrow & \\ & & & & 0 \end{array} \qquad \begin{array}{ccccc} & & \text{IO:} & & 5\\ & & & \nearrow & \\ & & 5 & & \\ & \nearrow & & \searrow & \\ & & & & 5 \\ 0 & & & & \\ & & & & 0\\ & \searrow & & \nearrow & \\ & & 5 & & \\ & & & \searrow & \\ & & & & 0 \end{array}$$

(a)

$$\begin{array}{ccc} & & 92.593 \\ & \nearrow & \\ \text{PO} & & \\ & \searrow & \\ & & 100 \end{array} \qquad \begin{array}{ccc} & & 9.630 \\ & \nearrow & \\ \text{IO} & & \\ & \searrow & \\ & & 5 \end{array}$$

(b)

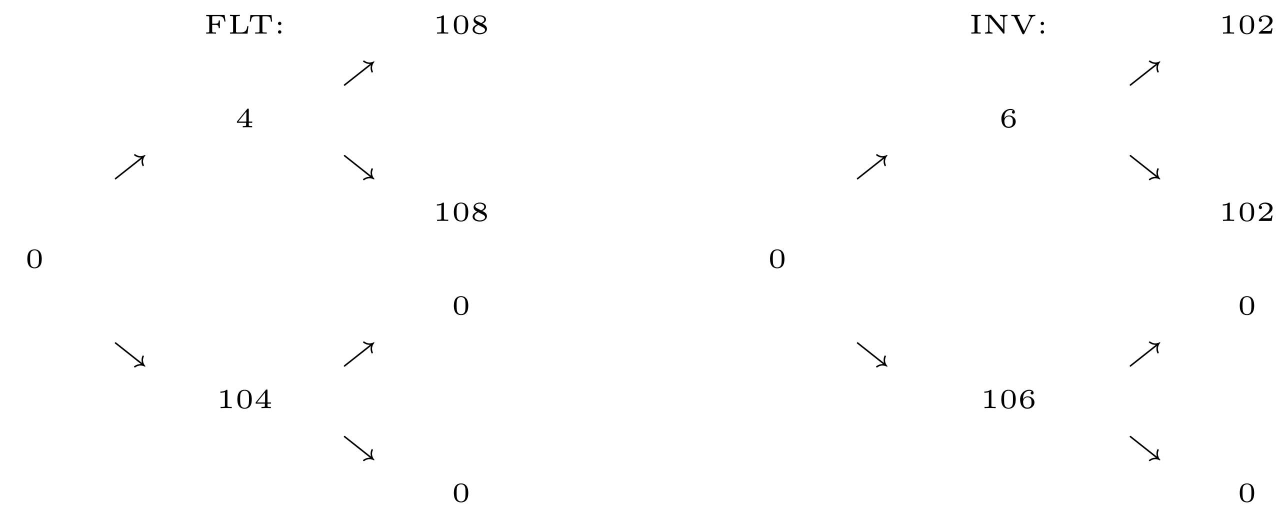
The price 9.630 is derived from $5 + (5/1.08)$.

Numerical Examples: MBSs (continued)

- Suppose the mortgage is split into half floater and half inverse floater.
- Let the floater (FLT) receive the one-year rate.
- Then the inverse floater (INV) must have a coupon rate of

$$(10\% - \textsf{one-year rate})$$

  to make the overall coupon rate 5%.
- Their cash flows as *percentages of par* and values are shown on p. 925.


FLT:
108
4
108
0
0
104
0
INV:
102
6
102
0
0
106
0

(a)


104
FLT
104
100.444
INV
106

(b)

Numerical Examples: MBSs (concluded)

- On p. 925, the floater's price in the up node, 104, is derived from $4 + (108/1.08)$.
- The inverse floater's price 100.444 is derived from $6 + (102/1.08)$.
- The current prices are

$$
\begin{aligned}
\text{FLT} &= \frac{1}{2} \times \frac{104}{1.04} = 50, \\
\text{INV} &= \frac{1}{2} \times \frac{(1-p) \times 100.444 + p \times 106}{1.04} = 49.142.
\end{aligned}
$$

Equilibrium Term Structure Models

8. What's your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*

Introduction

- This chapter surveys equilibrium models.
- Since the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t},$$

  the discount function $P(t,T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\, dt + \sigma\, dW.$$

- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\, dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\, r(T) \mid r(t) = r\,] = \mu + (r - \mu)\, e^{-\beta(T-t)}$$

from Eq. (55) on p. 517.

[a]Vasicek (1977).

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{111}$$

where

$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2 (T-t)^3}{6} \right] & \text{if } \beta = 0. \end{cases}$$

and

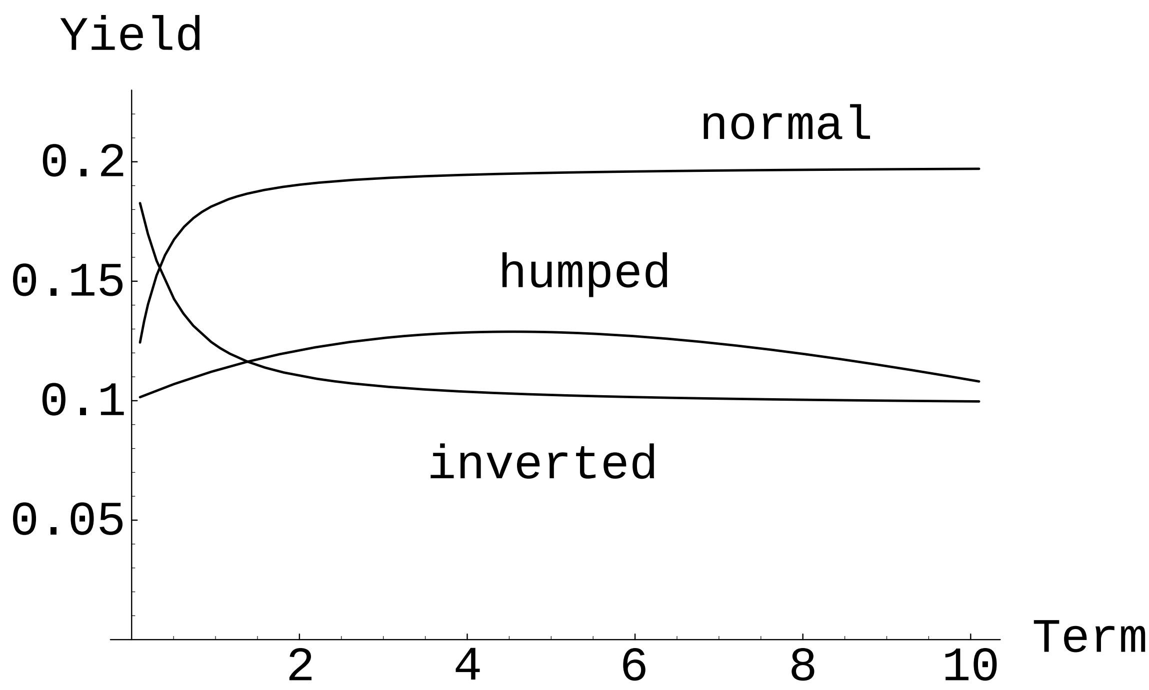
$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T-t & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (concluded)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, P may exceed one for a finite T.
- The spot rate volatility structure is the curve

$$(\partial r(t,T)/\partial r)\,\sigma = \sigma B(t,T)/(T-t).$$

- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, β, controls the shape of the curve.
- Indeed, higher β leads to greater attenuation of volatility with maturity.


Yield
normal
0.2
humped
0.15
0.1
inverted
0.05
2
4
6
8
10
Term

The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t, s)\, N(x) - X P(t, T)\, N(x - \sigma_v).$$

[a] Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$
\begin{aligned}
x &\equiv \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X}\right) + \frac{\sigma_v}{2}, \\
\sigma_v &\equiv v(t,T)\,B(T,s), \\
v(t,T)^2 &\equiv \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}.
\end{aligned}
$$

- By the put-call parity, the price of a European put is

$$
XP(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).
$$

Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[0, T]$ divided into n identical pieces.
- Let $\Delta t \equiv T/n$ and

$$p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[a]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\,\xi(k), \quad 0 \leq k < n.$$

---

[a] Nelson and Ramaswamy (1990).

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases} .$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.
- For a general process Y with nonconstant volatility, the resulting binomial tree may not combine, as we will see next.

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\, dt + \sigma\sqrt{r}\, dW. \qquad (112)$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.
- See text for the bond pricing formula.

---

[a] Cox, Ingersoll, and Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into n periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process

$$x(r) \equiv 2\sqrt{r}/\sigma.$$

- It follows

$$dx = m(x)\, dt + dW,$$

where

$$m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- Since this new process has a constant volatility, its associated binomial tree combines.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation $r = f(x) \equiv x^2 \sigma^2 / 4$ (p. 943).

$$
\begin{array}{ccccc}
 & & & & x + 2\sqrt{\Delta t} \\
 & & & \nearrow & \\
 & & x + \sqrt{\Delta t} & & \\
 & \nearrow & & \searrow & \\
x & & & & x \\
 & \searrow & & \nearrow & \\
 & & x - \sqrt{\Delta t} & & \\
 & & & \searrow & \\
 & & & & x - 2\sqrt{\Delta t}
\end{array}
\qquad
\begin{array}{ccccc}
 & & & & f(x + 2\sqrt{\Delta t}) \\
 & & & \nearrow & \\
 & & f(x + \sqrt{\Delta t}) & & \\
 & \nearrow & & \searrow & \\
f(x) & & & & f(x) \\
 & \searrow & & \nearrow & \\
 & & f(x - \sqrt{\Delta t}) & & \\
 & & & \searrow & \\
 & & & & f(x - 2\sqrt{\Delta t})
\end{array}
$$

Binomial CIR (concluded)

- The probability of an up move at each node r is

$$p(r) \equiv \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}. \tag{113}$$

  - $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Numerical Examples

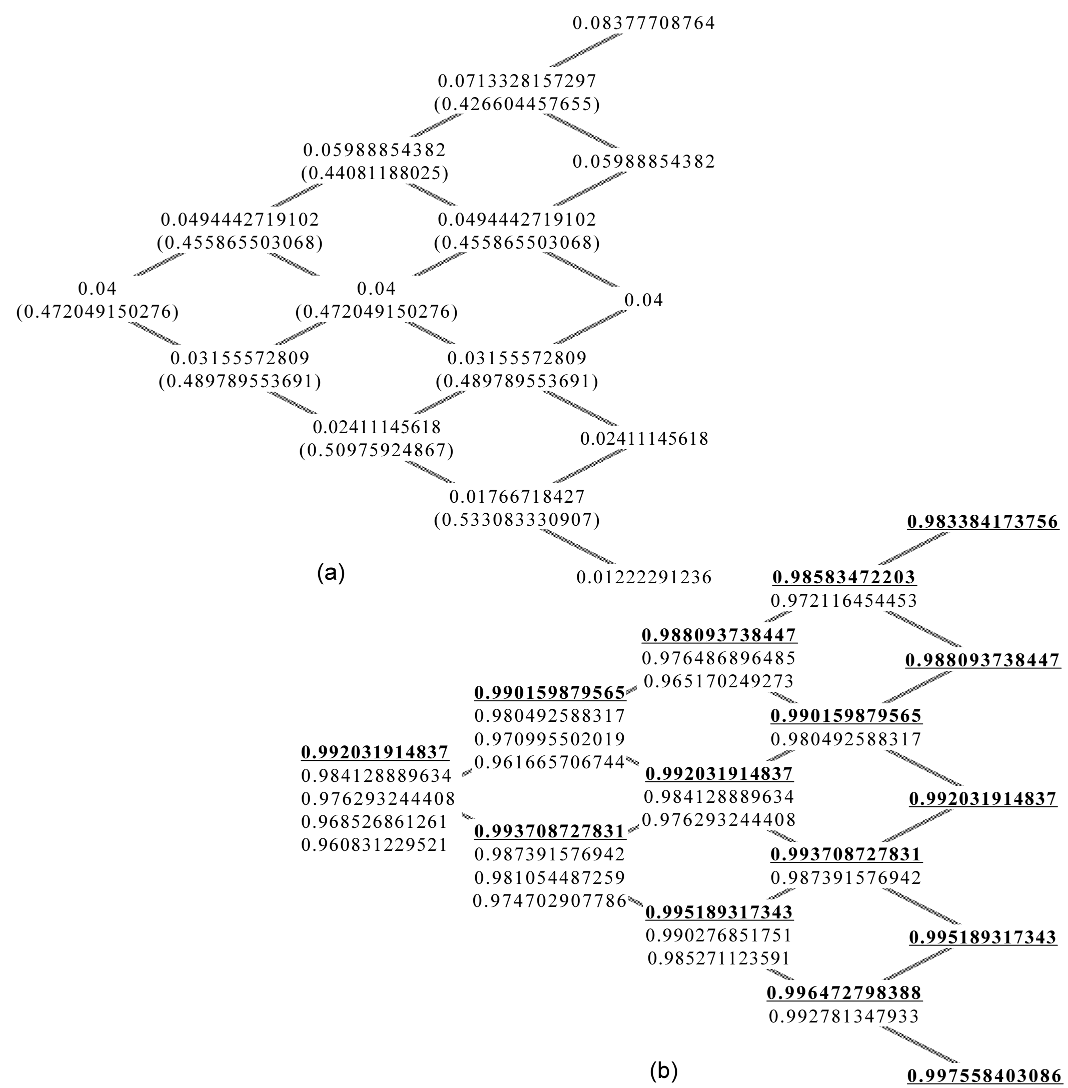
- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.

- See p. 946(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.


0.08377708764
0.0713328157297
(0.426604457655)
0.05988854382
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0.0494442719102
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0.01222291236
(a)
0.983384173756
0.98583472203
0.972116454453
0.988093738447
0.976486896485
0.965170249273
0.988093738447
0.990159879565
0.980492588317
0.970995502019
0.961665706744
0.990159879565
0.980492588317
0.992031914837
0.984128889634
0.976293244408
0.968526861261
0.960831229521
0.992031914837
0.984128889634
0.976293244408
0.992031914837
0.993708727831
0.987391576942
0.981054487259
0.974702907786
0.993708727831
0.987391576942
0.995189317343
0.990276851751
0.985271123591
0.995189317343
0.996472798388
0.992781347933
0.997558403086
(b)

Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$.

Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process,

$$dy = \alpha(y,t)\,dt + \sigma(y,t)\,dW.$$

- Make sure the binomial model's drift and diffusion converge to the above process by setting the probability of an up move to

$$\frac{\alpha(y,t)\,\Delta t + y - y_{\rm d}}{y_{\rm u} - y_{\rm d}}.$$

- Here $y_{\rm u} \equiv y + \sigma(y,t)\sqrt{\Delta t}$ and $y_{\rm d} \equiv y - \sigma(y,t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.
- The displacements are identical, at $\sigma(y,t)\sqrt{\Delta t}$.

[a]Nelson and Ramaswamy (1990).

A General Method (continued)

- But the binomial tree may not combine as

$$\begin{aligned} & \sigma(y,t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}}, t+\Delta t)\sqrt{\Delta t} \\ \neq\ & -\sigma(y,t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}}, t+\Delta t)\sqrt{\Delta t} \end{aligned}$$

  in general.

- When $\sigma(y,t)$ is a constant independent of y, equality holds and the tree combines.

A General Method (continued)

- To achieve this, define the transformation

$$x(y,t) \equiv \int^{y} \sigma(z,t)^{-1}\, dz.$$

- Then x follows $dx = m(y,t)\, dt + dW$ for some $m(y,t)$ (see text).
- The key is that the diffusion term is now a constant, and the binomial tree for x combines.
- The transformation that turns a 1-dim stochastic process into one with a constant diffusion term is unique.[a]

---

[a] Chiu (R98723059) (2012).

A General Method (concluded)

- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.

- Note that $y_{\mathrm{u}}(x,t) \equiv y(x+\sqrt{\Delta t}, t+\Delta t)$ and $y_{\mathrm{d}}(x,t) \equiv y(x-\sqrt{\Delta t}, t+\Delta t)$.

Examples

- The transformation is
$$\int^r (\sigma\sqrt{z})^{-1}\, dz = 2\sqrt{r}/\sigma$$
for the CIR model.
- The transformation is
$$\int^S (\sigma z)^{-1}\, dz = (1/\sigma)\ln S$$
for the Black-Scholes model.
- The familiar binomial option pricing model in fact discretizes $\ln S$ not S.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two-factor ones.

Options on Coupon Bonds[a]

- Assume a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time T on a bond with par value \$1.
- Let X denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all i.

---

[a] Jamshidian (1989).

Options on Coupon Bonds (continued)

- The payoff for the option is

$$\max\left(\sum_{i=1}^{n} c_i P(r(T), T, t_i) - X, 0\right).$$

- At time T, there is a unique value r^* for $r(T)$ that renders the coupon bond's price equal the strike price X.
- This r^* can be obtained by solving

$$X = \sum_{i=1}^{n} c_i P(r, T, t_i)$$

numerically for r.

Options on Coupon Bonds (continued)

- The solution is unique for one-factor models whose bond price is a monotonically decreasing function of r.
- Let
$$X_i \equiv P(r^*, T, t_i),$$
the value at time T of a zero-coupon bond with par value \$1 and maturing at time t_i if $r(T) = r^*$.
- Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$.

Options on Coupon Bonds (concluded)

- As $X = \sum_i c_i X_i$, the option's payoff equals

$$\begin{aligned} & \max\left(\sum_{i=1}^{n} c_i P(r(T), T, t_i) - \sum_i c_i X_i, 0\right) \\ = & \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0). \end{aligned}$$

- Thus the call is a package of n options on the underlying zero-coupon bond.
- Why can't we do the same thing for Asian options?[a]

[a] Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.

No-Arbitrage Term Structure Models

How much of the structure of our theories
really tells us about things in nature,
and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)

Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.

No-Arbitrage Models[a]

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

[a]Ho and Lee (1986). Thomas Lee is a "billionaire founder" of Thomas H. Lee Partners LP, according to *Bloomberg* on May 26, 2012.

No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.

The Ho-Lee Model[a]

- The short rates at any given time are evenly spaced.
- Let p denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

[a] Ho and Lee (1986).

$$
\begin{array}{cccccc}
 & & & & & \nearrow \\
 & & & & r_3 & \\
 & & & \nearrow & & \searrow \\
 & & r_2 & & & \\
 & \nearrow & & \searrow & & \nearrow \\
r_1 & & & & r_3 + v_3 & \\
 & \searrow & & \nearrow & & \searrow \\
 & & r_2 + v_2 & & & \\
 & & & \searrow & & \nearrow \\
 & & & & r_3 + 2v_3 & \\
 & & & & & \searrow
\end{array}
$$

The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t+1), P(t, t+2), \ldots$ at time t identified with the root of the tree.
- Let the discount factors in the next period be

$$\begin{array}{ll} P_{\mathrm{d}}(t+1, t+2), P_{\mathrm{d}}(t+1, t+3), \ldots & \text{if short rate moves down} \\ P_{\mathrm{u}}(t+1, t+2), P_{\mathrm{u}}(t+1, t+3), \ldots & \text{if short rate moves up} \end{array}$$

- By backward induction, it is not hard to see that for $n \geq 2$,

$$P_{\mathrm{u}}(t+1, t+n) = P_{\mathrm{d}}(t+1, t+n)\, e^{-(v_2 + \cdots + v_n)} \tag{114}$$

(see text).

The Ho-Lee Model (continued)

- It is also not hard to check that the n-period zero-coupon bond has yields

$$\begin{aligned}
y_{\mathrm{d}}(n) &\equiv -\frac{\ln P_{\mathrm{d}}(t+1,t+n)}{n-1} \\
y_{\mathrm{u}}(n) &\equiv -\frac{\ln P_{\mathrm{u}}(t+1,t+n)}{n-1} = y_{\mathrm{d}}(n) + \frac{v_2 + \cdots + v_n}{n-1}
\end{aligned}$$

- The volatility of the yield to maturity for this bond is therefore

$$\begin{aligned}
\kappa_n &\equiv \sqrt{p y_{\mathrm{u}}(n)^2 + (1-p)\, y_{\mathrm{d}}(n)^2 - [\, p y_{\mathrm{u}}(n) + (1-p)\, y_{\mathrm{d}}(n)\,]^2} \\
&= \sqrt{p(1-p)}\, (y_{\mathrm{u}}(n) - y_{\mathrm{d}}(n)) \\
&= \sqrt{p(1-p)}\, \frac{v_2 + \cdots + v_n}{n-1}.
\end{aligned}$$

The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)}\, v_2. \tag{115}$$

- The variance of the short rate therefore equals $p(1-p)(r_{\mathrm{u}} - r_{\mathrm{d}})^2$, where r_{u} and r_{d} are the two successor rates.[a]

[a] Contrast this with the lognormal model.

The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of $\kappa_2, \kappa_3, \ldots$.
  - It is independent of the r_i.
- It is easy to compute the v_is from the volatility structure, and vice versa.
- The r_is can be computed by forward induction.
- The volatility structure is supplied by the market.

The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy

$$P(t, t+n) = (pP_{\mathrm{u}}(t+1, t+n) + (1-p)\, P_{\mathrm{d}}(t+1, t+n))\, P(t, t+1)$$

- Combine the above with Eq. (114) on p. 968 and assume $p = 1/2$ to obtain[a]

$$P_{\mathrm{d}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2 \times \exp[\, v_2 + \cdots + v_n \,]}{1 + \exp[\, v_2 + \cdots + v_n \,]}, \tag{116}$$

$$P_{\mathrm{u}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1 + \exp[\, v_2 + \cdots + v_n \,]}. \tag{116$'$}$$

[a] In the limit, only the volatility matters.

The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.
- Suppose all v_i equal some constant v and $\delta \equiv e^v > 0$.
- Then

$$
\begin{aligned}
P_{\mathrm{d}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}}, \\
P_{\mathrm{u}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1+\delta^{n-1}}.
\end{aligned}
$$

- Short rate volatility σ equals $v/2$ by Eq. (115) on p. 970.
- Price derivatives by taking expectations under the risk-neutral probability.

The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an n-period zero-coupon bond is

$$r(t, t+n) \equiv \ln\left(\frac{P(t+1, t+n)}{P(t, t+n)}\right).$$

- Its value is either $\ln \frac{P_{\mathrm{d}}(t+1,t+n)}{P(t,t+n)}$ or $\ln \frac{P_{\mathrm{u}}(t+1,t+n)}{P(t,t+n)}$.
- Thus the variance of return is

$$\mathrm{Var}[\, r(t, t+n)\,] = p(1-p)((n-1)\, v)^2 = (n-1)^2 \sigma^2.$$

The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t, t+n)$ and $r(t, t+m)$ is $(n-1)(m-1)\,\sigma^2$ (see text).
- As a result, the correlation between any two one-period rates of return is unity.
- Strong correlation between rates is inherent in all one-factor Markovian models.

The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is

$$dr = \theta(t)\, dt + \sigma\, dW.$$

- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying, i.e., $dr = \theta(t)\, dt + \sigma(t)\, dW$.
- This corresponds to the discrete-time model in which v_i are not all identical.

The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.

Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model's state variables (factors) not its parameters.
- Model *parameters*, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.

Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model's intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.