Risk-Neutral Pricing

• Assume the local expectations theory.

• The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  
  – For all $t + 1 < T$,

$$
\frac{E_t[P(t + 1, T)]}{P(t, T)} = 1 + r(t). 
$$

  (101)

  – Relation (101) in fact follows from the risk-neutral valuation principle.\(^a\)

\(^a\)Theorem 16 on p. 457.
Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability $\pi$.

- Rewrite Eq. (101) as

$$\frac{E_t^\pi[P(t+1, T)]}{1 + r(t)} = P(t, T).$$

- It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.
Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

\[
P(t, T) = E_t^\pi \left[ \frac{P(t + 1, T)}{1 + r(t)} \right] = E_t^\pi \left[ \frac{E_{t+1}^\pi [P(t + 2, T)]}{(1 + r(t))(1 + r(t + 1))} \right] = \cdots
\]

\[
= E_t^\pi \left[ \frac{1}{(1 + r(t))(1 + r(t + 1)) \cdots (1 + r(T - 1))} \right]. \quad (102)
\]
Risk-Neutral Pricing (concluded)

• Equation (101) on p. 889 can also be expressed as

\[ E_t[P(t + 1, T)] = F(t, t + 1, T). \]

  – Verify that with, e.g., Eq. (96) on p. 884.

• Hence the forward price for the next period is an unbiased estimator of the expected bond price.
Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

\[ P(t, T) = E_t \left[ e^{-\int_t^T r(s) \, ds} \right], \quad t < T. \]  \hspace{1cm} (103)

- Note that \( e^{\int_t^T r(s) \, ds} \) is the bank account process, which denotes the rolled-over money market account.
Interest Rate Swaps

• Consider an interest rate swap made at time $t$ with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.

• The fixed rate is $c$ per annum.

• The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.

• For simplicity, assume $t_{i+1} - t_i$ is a fixed constant $\Delta t$ for all $i$, and the notional principal is one dollar.

• If $t < t_0$, we have a forward interest rate swap.

• The ordinary swap corresponds to $t = t_0$. 
Interest Rate Swaps (continued)

- The amount to be paid out at time $t_{i+1}$ is $(f_i - c) \Delta t$ for the floating-rate payer.
- Simple rates are adopted here.
- Hence $f_i$ satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$
Interest Rate Swaps (continued)

- The value of the swap at time $t$ is thus

$$\sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t_i}^{t} r(s) \, ds} \left( f_{i-1} - c \right) \Delta t \right]$$

$$= \sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t_i}^{t} r(s) \, ds} \left( \frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right]$$

$$= \sum_{i=1}^{n} [P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i)]$$

$$= P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).$$
Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.
Swap Rate

• The swap rate, which gives the swap zero value, equals

\[ S_n(t) \equiv \frac{P(t,t_0) - P(t,t_n)}{\sum_{i=1}^{n} P(t,t_i) \Delta t}. \]  \hspace{1cm} (104)

• The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.

• For an ordinary swap, \( P(t,t_0) = 1 \).
The Term Structure Equation

• Let us start with the zero-coupon bonds and the money market account.

• Let the zero-coupon bond price \( P(r, t, T) \) follow

\[
\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.
\]

• At time \( t \), short one unit of a bond maturing at time \( s_1 \) and buy \( \alpha \) units of a bond maturing at time \( s_2 \).
The Term Structure Equation (continued)

• The net wealth change follows

\[-dP(r, t, s_1) + \alpha dP(r, t, s_2)\]

\[= \left(-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2)\right) dt\]

\[+ \left(-P(r, t, s_1) \sigma_p(r, t, s_1) + \alpha P(r, t, s_2) \sigma_p(r, t, s_2)\right) dW.\]

• Pick

\[\alpha \equiv \frac{P(r, t, s_1) \sigma_p(r, t, s_1)}{P(r, t, s_2) \sigma_p(r, t, s_2)}.\]
The Term Structure Equation (continued)

• Then the net wealth has no volatility and must earn the riskless return:

\[ \frac{-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2)}{-P(r, t, s_1) + \alpha P(r, t, s_2)} = r. \]

• Simplify the above to obtain

\[ \frac{\sigma_p(r, t, s_1) \mu_p(r, t, s_2) - \sigma_p(r, t, s_2) \mu_p(r, t, s_1)}{\sigma_p(r, t, s_1) - \sigma_p(r, t, s_2)} = r. \]

• This becomes

\[ \frac{\mu_p(r, t, s_2) - r}{\sigma_p(r, t, s_2)} = \frac{\mu_p(r, t, s_1) - r}{\sigma_p(r, t, s_1)} \]

after rearrangement.
The Term Structure Equation (continued)

- Since the above equality holds for any \( s_1 \) and \( s_2 \),

\[
\frac{\mu_p(r, t, s) - r}{\sigma_p(r, t, s)} \equiv \lambda(r, t)
\] (105)

for some \( \lambda \) independent of the bond maturity \( s \).

- As \( \mu_p = r + \lambda \sigma_p \), all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset’s volatility.

- The term \( \lambda(r, t) \) is called the market price of risk.

- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.
The Term Structure Equation (continued)

• Assume a Markovian short rate model,

\[ dr = \mu(r, t) \, dt + \sigma(r, t) \, dW. \]

• Then the bond price process is also Markovian.

• By Eq. (14.15) on p. 202 in the text,

\[ p = \left( \frac{\partial P}{\partial T} + \mu(r, t) \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2} \frac{\partial^2 P}{\partial r^2} \right) / P, \]

\[ \mu_p = \left( -\frac{\partial P}{\partial T} + \mu(r, t) \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2} \frac{\partial^2 P}{\partial r^2} \right) / P, \]

\[ \sigma_p = \left( \sigma(r, t) \frac{\partial P}{\partial r} \right) / P, \]

subject to \( P(\cdot, T, T) = 1. \)
The Term Structure Equation (concluded)

• Substitute $\mu_p$ and $\sigma_p$ into Eq. (105) on p. 902 to obtain

$$-\frac{\partial P}{\partial T} + [\mu(r, t) - \chi(r, t) \sigma(r, t)] \frac{\partial P}{\partial r} + \frac{1}{2} \sigma(r, t)^2 \frac{\partial^2 P}{\partial r^2} = rP.$$  \hspace{1cm} (107)

• This is called the term structure equation.

• Once $P$ is available, the spot rate curve emerges via

$$r(t, T) = -\frac{\ln P(t, T)}{T - t}.$$  

• Equation (107) applies to all interest rate derivatives, the difference being the terminal and the boundary conditions.
The Binomial Model

• The analytical framework can be nicely illustrated with the binomial model.

• Suppose the bond price $P$ can move with probability $q$ to $Pu$ and probability $1 - q$ to $Pd$, where $u > d$:

$$
\begin{array}{c}
\begin{array}{c}
P \\
q
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
Pd \\
1 - q
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
P \\
q
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
Pu
\end{array}
\end{array}
$$
The Binomial Model (continued)

- Over the period, the bond’s expected rate of return is
  \[
  \hat{\mu} \equiv \frac{qP_u + (1 - q)P_d}{P} - 1 = qu + (1 - q)d - 1. 
  \tag{108}
  \]

- The variance of that return rate is
  \[
  \hat{\sigma}^2 \equiv q(1 - q)(u - d)^2. 
  \tag{109}
  \]
The Binomial Model (continued)

• In particular, the bond whose maturity is one period away will move from a price of $1/(1 + r)$ to its par value $1$.

• This is the money market account modeled by the short rate $r$.

• The market price of risk is defined as $\lambda \equiv (\hat{\mu} - r)/\hat{\sigma}$.

• As in the continuous-time case, it can be shown that $\lambda$ is independent of the maturity of the bond (see text).
The Binomial Model (concluded)

• Now change the probability from $q$ to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1 + r) - d}{u - d},$$

(110)

which is independent of bond maturity and $q$.

– Recall the BOPM.

• The bond’s expected rate of return becomes

$$\frac{pPu + (1 - p) Pd}{P} - 1 = pu + (1 - p) d - 1 = r.$$

• The local expectations theory hence holds under the new probability measure $p$. 
Numerical Examples

- Assume this spot rate curve:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate</td>
<td>4%</td>
<td>5%</td>
</tr>
</tbody>
</table>

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:
Numerical Examples (continued)

• No real-world probabilities are specified.

• The prices of one- and two-year zero-coupon bonds are, respectively,

\[
\frac{100}{1.04} = 96.154, \\
\frac{100}{(1.05)^2} = 90.703.
\]

• They follow the binomial processes on p. 911.
Numerical Examples (continued)

90.703 \quad 92.593 \quad 96.154
\quad 98.039 \quad 100

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.
Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.

- Suppose all securities have the same expected one-period rate of return, the riskless rate.

- Then

\[
(1 - p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4%,
\]

where \( p \) denotes the risk-neutral probability of a down move in rates.
Numerical Examples (concluded)

• Solving the equation leads to $p = 0.319$.

• Interest rate contingent claims can be priced under this probability.
Numerical Examples: Fixed-Income Options

• A one-year European call on the two-year zero with a $95 strike price has the payoffs,

\[
C \begin{cases} 0.000 \\ 3.039 \end{cases}
\]

• To solve for the option value \( C \), we replicate the call by a portfolio of \( x \) one-year and \( y \) two-year zeros.
Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

\[ x \times 100 + y \times 92.593 = 0.000, \]
\[ x \times 100 + y \times 98.039 = 3.039. \]

- They give \( x = -0.5167 \) and \( y = 0.5580 \).

- Consequently,

\[ C = x \times 96.154 + y \times 90.703 \approx 0.93 \]

to prevent arbitrage.
Numerical Examples: Fixed-Income Options
(continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.
Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

\[ C = \frac{(1 - p) \times 0 + p \times 3.039}{1.04} \approx 0.93, \]

the same as before.

- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.
Numerical Examples: Futures and Forward Prices

• A one-year futures contract on the one-year rate has a payoff of $100 - r$, where $r$ is the one-year rate at maturity:

\[
F = 92 \quad (= 100 - 8) \quad \quad \quad 98 \quad (= 100 - 2)
\]

• As the futures price $F$ is the expected future payoff (see text or p. 458), $F = (1 - p) \times 92 + p \times 98 = 93.914$.

• The forward price for a one-year forward contract on a one-year zero-coupon bond is $90.703/96.154 = 94.331\%$.

• The forward price exceeds the futures price.\(^a\)

\(^a\)Recall p. 404.
Numerical Examples: Mortgage-Backed Securities

- Consider a 5%-coupon, two-year mortgage-backed security without amortization, prepayments, and default risk.

- Its cash flow and price process are illustrated on p. 920.

- Its fair price is

\[
M = \frac{(1 - p) \times 102.222 + p \times 107.941}{1.04} = 100.045.
\]

- Identical results could have been obtained via arbitrage considerations.
The left diagram depicts the cash flow; the right diagram illustrates the price process.
Numerical Examples: MBSs (continued)

- Suppose that the security can be prepaid at par.
- It will be prepaid only when its price is higher than par.
- Prepayment will hence occur only in the “down” state when the security is worth 102.941 (excluding coupon).
- The price therefore follows the process,

\[ M \xleftarrow{102.222} \xrightarrow{105} \]

- The security is worth

\[
M = \frac{(1 - p) \times 102.222 + p \times 105}{1.04} = 99.142.
\]
Numerical Examples: MBSs (continued)

- The cash flow of the principal-only (PO) strip comes from the mortgage’s principal cash flow.

- The cash flow of the interest-only (IO) strip comes from the interest cash flow (p. 923(a)).

- Their prices hence follow the processes on p. 923(b).

- The fair prices are

$$\text{PO} = \frac{(1 - p) \times 92.593 + p \times 100}{1.04} = 91.304,$$

$$\text{IO} = \frac{(1 - p) \times 9.630 + p \times 5}{1.04} = 7.839.$$
The price $9.630$ is derived from $5 + (5/1.08)$. 
Numerical Examples: MBSs (continued)

- Suppose the mortgage is split into half floater and half inverse floater.
- Let the floater (FLT) receive the one-year rate.
- Then the inverse floater (INV) must have a coupon rate of
  \[(10\% - \text{one-year rate})\]
to make the overall coupon rate 5%.
- Their cash flows as \textit{percentages of par} and values are shown on p. 925.
\( \text{FLT:} \quad 108 \quad \text{INV:} \quad 102 \)

\( \quad 4 \quad \quad 6 \)

\( \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \)

\( \quad 108 \quad 102 \quad 104 \quad 106 \quad 104 \quad 100.444 \quad 106 \)

(a) (b)
Numerical Examples: MBSs (concluded)

- On p. 925, the floater’s price in the up node, 104, is derived from $4 + (108/1.08)$.
- The inverse floater’s price 100.444 is derived from $6 + (102/1.08)$.
- The current prices are
  \[
  \text{FLT} = \frac{1}{2} \times \frac{104}{1.04} = 50, \\
  \text{INV} = \frac{1}{2} \times \frac{(1 - p) \times 100.444 + p \times 106}{1.04} = 49.142.
  \]
Equilibrium Term Structure Models
8. What’s your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*
Introduction

• This chapter surveys equilibrium models.

• Since the spot rates satisfy

\[ r(t, T) = -\frac{\ln P(t, T)}{T - t}, \]

the discount function \( P(t, T) \) suffices to establish the spot rate curve.

• All models to follow are short rate models.

• Unless stated otherwise, the processes are risk-neutral.
The Vasicek Model$^a$

- The short rate follows

\[ dr = \beta(\mu - r) \, dt + \sigma \, dW. \]

- The short rate is pulled to the long-term mean level $\mu$ at rate $\beta$.

- Superimposed on this “pull” is a normally distributed stochastic term $\sigma \, dW$.

- Since the process is an Ornstein-Uhlenbeck process,

\[ E[r(T) | r(t) = r] = \mu + (r - \mu) \, e^{-\beta(T-t)} \]

from Eq. (55) on p. 517.

$^a$Vasicek (1977).
The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

\[ P(t, T) = A(t, T) e^{-B(t, T) r(t)}, \quad (111) \]

where

\[
A(t, T) = \begin{cases} 
\exp \left[ \frac{(B(t, T) - T + t) (\beta^2 \mu - \sigma^2 / 2) - \frac{\sigma^2 B(t, T)^2}{4 \beta}}{\beta^2} \right] & \text{if } \beta \neq 0, \\
\exp \left[ \frac{\sigma^2 (T - t)^3}{6} \right] & \text{if } \beta = 0.
\end{cases}
\]

and

\[
B(t, T) = \begin{cases} 
\frac{1 - e^{-\beta (T - t)}}{\beta} & \text{if } \beta \neq 0, \\
T - t & \text{if } \beta = 0.
\end{cases}
\]
The Vasicek Model (concluded)

- If $\beta = 0$, then $P$ goes to infinity as $T \to \infty$.
- Sensibly, $P$ goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, $P$ may exceed one for a finite $T$.
- The spot rate volatility structure is the curve
  $$
  \left( \frac{\partial r(t,T)}{\partial r} \right) \sigma = \sigma B(t,T)/(T - t).
  $$
- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, $\beta$, controls the shape of the curve.
- Indeed, higher $\beta$ leads to greater attenuation of volatility with maturity.
The Vasicek Model: Options on Zeros\textsuperscript{a}

- Consider a European call with strike price $X$ expiring at time $T$ on a zero-coupon bond with par value $1$ and maturing at time $s > T$.

- Its price is given by

$$P(t, s) N(x) - XP(t, T) N(x - \sigma_v).$$

\textsuperscript{a}Jamshidian (1989).
The Vasicek Model: Options on Zeros (concluded)

- Above

\[ x \equiv \frac{1}{\sigma_v} \ln \left( \frac{P(t, s)}{P(t, T) X} \right) + \frac{\sigma_v}{2}, \]
\[ \sigma_v \equiv v(t, T) B(T, s), \]
\[ v(t, T)^2 \equiv \begin{cases} \frac{\sigma^2[1-e^{-2\beta(T-t)}]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T - t), & \text{if } \beta = 0 \end{cases}. \]

- By the put-call parity, the price of a European put is

\[ XP(t, T) N(-x + \sigma_v) - P(t, s) N(-x). \]
Binomial Vasicek

• Consider a binomial model for the short rate in the time interval $[0, T]$ divided into $n$ identical pieces.

• Let $\Delta t \equiv T/n$ and

\[
p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r) \sqrt{\Delta t}}{2\sigma}.
\]

• The following binomial model converges to the Vasicek model,$^a$

\[
r(k + 1) = r(k) + \sigma \sqrt{\Delta t} \xi(k), \quad 0 \leq k < n.
\]

---

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\xi(k) = 1] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases}.$$ 

- Observe that the probability of an up move, $p$, is a decreasing function of the interest rate $r$.

- This is consistent with mean reversion.
Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its constant volatility, $\sigma$.
- For a general process $Y$ with nonconstant volatility, the resulting binomial tree may not combine, as we will see next.
The Cox-Ingersoll-Ross Model\textsuperscript{a}

- It is the following square-root short rate model:

\begin{equation}
    dr = \beta (\mu - r) \, dt + \sigma \sqrt{r} \, dW.
\end{equation}  \hfill (112)

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter $\beta$ determines the speed of adjustment.
- The short rate can reach zero only if $2\beta \mu < \sigma^2$.
- See text for the bond pricing formula.

\textsuperscript{a}Cox, Ingersoll, and Ross (1985).
Binomial CIR

• We want to approximate the short rate process in the time interval \([0, T]\).

• Divide it into \(n\) periods of duration \(\Delta t \equiv T/n\).

• Assume \(\mu, \beta \geq 0\).

• A direct discretization of the process is problematic because the resulting binomial tree will not combine.
Binomial CIR (continued)

- Instead, consider the transformed process

\[ x(r) \equiv 2\sqrt{r}/\sigma. \]

- It follows

\[ dx = m(x) \, dt + dW, \]

where

\[ m(x) \equiv 2\beta \mu/(\sigma^2 x) - (\beta x/2) - 1/(2x). \]

- Since this new process has a constant volatility, its associated binomial tree combines.
Binomial CIR (continued)

- Construct the combining tree for $r$ as follows.
- First, construct a tree for $x$.
- Then transform each node of the tree into one for $r$ via the inverse transformation $r = f(x) \equiv x^2\sigma^2/4$ (p. 943).
\[ f(x + 2\sqrt{\Delta t}) \]
\[ f(x + \sqrt{\Delta t}) \]
\[ f(x) \]
\[ f(x - \sqrt{\Delta t}) \]
\[ f(x - 2\sqrt{\Delta t}) \]
Binomial CIR (concluded)

- The probability of an up move at each node $r$ is

$$p(r) \equiv \frac{\beta(\mu - r) \Delta t + r - r^-}{r^+ - r^-}. \quad (113)$$

- $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.

- $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

- Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.
Numerical Examples

• Consider the process,

\[ 0.2 (0.04 - r) \, dt + 0.1 \sqrt{r} \, dW, \]

for the time interval \([0, 1]\) given the initial rate \(r(0) = 0.04\).

• We shall use \(\Delta t = 0.2\) (year) for the binomial approximation.

• See p. 946(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.
Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has \( x = 2 \sqrt{r(0)/\sigma} = 4 \), this particular node’s \( x \) value equals \( 4 + \sqrt{\Delta t} = 4.4472135955 \).
- Use the inverse transformation to obtain the short rate \( x^2 \times (0.1)^2/4 \approx 0.0494442719102 \).
Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.

- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.

- This phenomenon agrees with mean reversion.

- Convergence is quite good (see text).
A General Method for Constructing Binomial Models\textsuperscript{a}

- We are given a continuous-time process,
  \[ dy = \alpha(y, t) \, dt + \sigma(y, t) \, dW. \]

- Make sure the binomial model’s drift and diffusion converge to the above process by setting the probability of an up move to
  \[ \frac{\alpha(y, t) \, \Delta t + y - y_d}{y_u - y_d}. \]

- Here \( y_u \equiv y + \sigma(y, t) \sqrt{\Delta t} \) and \( y_d \equiv y - \sigma(y, t) \sqrt{\Delta t} \) represent the two rates that follow the current rate \( y \).

- The displacements are identical, at \( \sigma(y, t) \sqrt{\Delta t} \).

\textsuperscript{a}Nelson and Ramaswamy (1990).
A General Method (continued)

• But the binomial tree may not combine as

\[
\sigma(y, t)\sqrt{\Delta t} - \sigma(y_u, t + \Delta t)\sqrt{\Delta t} \\
\neq -\sigma(y, t)\sqrt{\Delta t} + \sigma(y_d, t + \Delta t)\sqrt{\Delta t}
\]

in general.

• When \(\sigma(y, t)\) is a constant independent of \(y\), equality holds and the tree combines.
A General Method (continued)

• To achieve this, define the transformation

\[ x(y, t) \equiv \int_{y}^{y} \sigma(z, t)^{-1} \, dz. \]

• Then \( x \) follows \( dx = m(y, t) \, dt + dW \) for some \( m(y, t) \) (see text).

• The key is that the diffusion term is now a constant, and the binomial tree for \( x \) combines.

• The transformation that turns a 1-dim stochastic process into one with a constant diffusion term is unique.\(^a\)

\(^a\)Chiu (R98723059) (2012).
A General Method (concluded)

• The probability of an up move remains

\[
\frac{\alpha(y(x, t), t) \Delta t + y(x, t) - y_d(x, t)}{y_u(x, t) - y_d(x, t)},
\]

where \( y(x, t) \) is the inverse transformation of \( x(y, t) \) from \( x \) back to \( y \).

• Note that \( y_u(x, t) \equiv y(x + \sqrt{\Delta t}, t + \Delta t) \) and \( y_d(x, t) \equiv y(x - \sqrt{\Delta t}, t + \Delta t) \).
Examples

• The transformation is

\[ \int_{r}^{\sqrt{z}} (\sigma \sqrt{z})^{-1} \, dz = 2\sqrt{r}/\sigma \]

for the CIR model.

• The transformation is

\[ \int_{S}^{S} (\sigma z)^{-1} \, dz = (1/\sigma) \ln S \]

for the Black-Scholes model.

• The familiar binomial option pricing model in fact discretizes \( \ln S \) not \( S \).
On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate levels only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.
On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.

- Derivatives whose values depend on the correlation structure will be mispriced.

- The calibrated models may not generate term structures as concave as the data suggest.

- The term structure empirically changes in slope and curvature as well as makes parallel moves.

- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.
On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two-factor ones.
Options on Coupon Bonds\textsuperscript{a}

- Assume a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time $T$ on a bond with par value $1$.
- Let $X$ denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all $i$.

\textsuperscript{a}Jamshidian (1989).
Options on Coupon Bonds (continued)

- The payoff for the option is
  \[
  \max \left( \sum_{i=1}^{n} c_i P(r(T), T, t_i) - X, 0 \right).
  \]

- At time \( T \), there is a unique value \( r^* \) for \( r(T) \) that renders the coupon bond’s price equal the strike price \( X \).

- This \( r^* \) can be obtained by solving
  \[
  X = \sum_{i=1}^{n} c_i P(r, T, t_i)
  \]
  numerically for \( r \).
Options on Coupon Bonds (continued)

- The solution is unique for one-factor models whose bond price is a monotonically decreasing function of $r$.

- Let

$$X_i \equiv P(r^*, T, t_i),$$

the value at time $T$ of a zero-coupon bond with par value $1$ and maturing at time $t_i$ if $r(T) = r^*$.

- Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$.
Options on Coupon Bonds (concluded)

• As $X = \sum_i c_i X_i$, the option’s payoff equals

$$\max \left( \sum_{i=1}^n c_i P(r(T), T, t_i) - \sum_i c_i X_i, 0 \right)$$

$$= \sum_{i=1}^n c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

• Thus the call is a package of $n$ options on the underlying zero-coupon bond.

• Why can’t we do the same thing for Asian options?\(^a\)

\(^a\)Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.
No-Arbitrage Term Structure Models
How much of the structure of our theories really tells us about things in nature, and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)
Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.
No-Arbitrage Models\textsuperscript{a}

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

\textsuperscript{a}Ho and Lee (1986). Thomas Lee is a “billionaire founder” of Thomas H. Lee Partners LP, according to \textit{Bloomberg} on May 26, 2012.
No-Arbitrage Models (concluded)

• No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.

• Bond price and forward rate models are usually non-Markovian (path dependent).

• In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).

• Markovian models are easier to handle computationally.
The Ho-Lee Model\textsuperscript{a}

- The short rates at any given time are evenly spaced.
- Let $p$ denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

\textsuperscript{a}Ho and Lee (1986).
The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t + 1), P(t, t + 2), \ldots$ at time $t$ identified with the root of the tree.
- Let the discount factors in the next period be
  
  \[ P_d(t + 1, t + 2), P_d(t + 1, t + 3), \ldots \quad \text{if short rate moves down} \]
  \[ P_u(t + 1, t + 2), P_u(t + 1, t + 3), \ldots \quad \text{if short rate moves up} \]

- By backward induction, it is not hard to see that for $n \geq 2$,
  
  \[ P_u(t + 1, t + n) = P_d(t + 1, t + n) e^{-(v_2 + \cdots + v_n)} \]

(see text).
The Ho-Lee Model (continued)

- It is also not hard to check that the $n$-period zero-coupon bond has yields

\[
y_d(n) \equiv -\frac{\ln P_d(t+1,t+n)}{n-1}
\]
\[
y_u(n) \equiv -\frac{\ln P_u(t+1,t+n)}{n-1} = y_d(n) + \frac{v_2 + \cdots + v_n}{n-1}
\]

- The volatility of the yield to maturity for this bond is therefore

\[
\kappa_n \equiv \sqrt{py_u(n)^2 + (1-p) y_d(n)^2 - [py_u(n) + (1-p) y_d(n)]^2}
\]
\[
= \sqrt{p(1-p)} \left( y_u(n) - y_d(n) \right)
\]
\[
= \sqrt{p(1-p)} \frac{v_2 + \cdots + v_n}{n-1}.
\]
The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)} \, v_2. \quad (115)$$

- The variance of the short rate therefore equals

$$p(1-p)(r_u - r_d)^2,$$

where $r_u$ and $r_d$ are the two successor rates.\(^a\)

\(^a\)Contrast this with the lognormal model.
The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of $\kappa_2, \kappa_3, \ldots$
  - It is independent of the $r_i$.
- It is easy to compute the $v_i$s from the volatility structure, and vice versa.
- The $r_i$s can be computed by forward induction.
- The volatility structure is supplied by the market.
The Ho-Lee Model: Bond Price Process

• In a risk-neutral economy, the initial discount factors satisfy

\[ P(t, t+n) = (p P_u(t+1, t+n) + (1-p) P_d(t+1, t+n)) P(t, t+1) \]

• Combine the above with Eq. (114) on p. 968 and assume \( p = 1/2 \) to obtain\(^a\)

\[ P_d(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2 \times \exp[v_2 + \cdots + v_n]}{1 + \exp[v_2 + \cdots + v_n]}, \]

\[ P_u(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1 + \exp[v_2 + \cdots + v_n]}. \]

\(^a\)In the limit, only the volatility matters.
The Ho-Lee Model: Bond Price Process (concluded)

• The bond price tree combines.

• Suppose all $v_i$ equal some constant $v$ and $\delta \equiv e^v > 0$.

• Then

\[
P_d(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2\delta^{n-1}}{1 + \delta^{n-1}},
\]

\[
P_u(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2}{1 + \delta^{n-1}}.
\]

• Short rate volatility $\sigma$ equals $v/2$ by Eq. (115) on p. 970.

• Price derivatives by taking expectations under the risk-neutral probability.
The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an \( n \)-period zero-coupon bond is

\[
r(t, t + n) \equiv \ln \left( \frac{P(t + 1, t + n)}{P(t, t + n)} \right).
\]

- Its value is either \( \ln \frac{P_d(t+1,t+n)}{P(t,t+n)} \) or \( \ln \frac{P_u(t+1,t+n)}{P(t,t+n)} \).

- Thus the variance of return is

\[
\text{Var}[r(t, t + n)] = p(1 - p)((n - 1)v)^2 = (n - 1)^2 \sigma^2.
\]
The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between \( r(t, t + n) \) and \( r(t, t + m) \) is \((n - 1)(m - 1)\sigma^2\) (see text).

- As a result, the correlation between any two one-period rates of return is unity.

- Strong correlation between rates is inherent in all one-factor Markovian models.
The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is
  \[ dr = \theta(t) \, dt + \sigma \, dW. \]

- This is Vasicek’s model with the mean-reverting drift replaced by a deterministic, time-dependent drift.

- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying, i.e.,
  \[ dr = \theta(t) \, dt + \sigma(t) \, dW. \]

- This corresponds to the discrete-time model in which \( v_i \) are not all identical.
The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model’s state variables (factors) not its parameters.
- Model parameters, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.
Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.

- Consequently, a model’s intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.