Matrix Computation

To set up a philosophy against physics is rash;
philosophers who have done so
have always ended in disaster.
— Bertrand Russell

Definitions and Basic Results

- Let $A \equiv [\, a_{ij} \,]_{1 \leq i \leq m, 1 \leq j \leq n}$, or simply $A \in \boldsymbol{R}^{m \times n}$, denote an $m \times n$ matrix.
- It can also be represented as $[\, a_1, a_2, \ldots, a_n \,]$ where $a_i \in \boldsymbol{R}^m$ are vectors.
  - Vectors are column vectors unless stated otherwise.
- A is a square matrix when $m = n$.
- The rank of a matrix is the largest number of linearly independent columns.

Definitions and Basic Results (continued)

- A square matrix A is said to be symmetric if $A^{\mathrm{T}} = A$.
- A real $n \times n$ matrix

$$A \equiv [\, a_{ij} \,]_{i,j}$$

  is diagonally dominant if $|\, a_{ii} \,| > \sum_{j \neq i} |\, a_{ij} \,|$ for $1 \leq i \leq n$.
  - Such matrices are nonsingular.
- The identity matrix is the square matrix

$$I \equiv \mathrm{diag}[\, 1, 1, \dots, 1 \,].$$

Definitions and Basic Results (concluded)

- A matrix has full column rank if its columns are linearly independent.
- A real symmetric matrix A is positive definite if

$$x^{\mathrm{T}} A x = \sum_{i,j} a_{ij} x_i x_j > 0$$

  for any nonzero vector x.
- A matrix A is positive definite if and only if there exists a matrix W such that $A = W^{\mathrm{T}} W$ and W has full column rank.

Cholesky Decomposition

- Positive definite matrices can be factored as

  $$A = LL^{\mathrm{T}},$$

  called the Cholesky decomposition.

  - Above, L is a lower triangular matrix.

Generation of Multivariate Distribution

- Let $\boldsymbol{x} \equiv [\, x_1, x_2, \ldots, x_n \,]^{\mathrm{T}}$ be a vector random variable with a positive definite covariance matrix C.
- As usual, assume $E[\, \boldsymbol{x} \,] = \mathbf{0}$.
- This covariance structure can be matched by $P\boldsymbol{y}$.
  - $C = PP^{\mathrm{T}}$ is the Cholesky decomposition of C.[a]
  - $\boldsymbol{y} \equiv [\, y_1, y_2, \ldots, y_n \,]^{\mathrm{T}}$ is a vector random variable with a covariance matrix equal to the identity matrix.

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[a] What if C is not positive definite? See Lai (`R93942114`) and Lyuu (2007).

Generation of Multivariate Normal Distribution

- Suppose we want to generate the multivariate normal distribution with a covariance matrix $C = PP^{\mathrm{T}}$.
  - First, generate independent standard normal distributions $y_1, y_2, \ldots, y_n$.
  - Then
$$P[\, y_1, y_2, \ldots, y_n \,]^{\mathrm{T}}$$
    has the desired distribution.
  - These steps can then be repeated.

Multivariate Derivatives Pricing

- Generating the multivariate normal distribution is essential for the Monte Carlo pricing of multivariate derivatives (pp. 633ff).
- For example, the rainbow option on k assets has payoff

$$\max(\max(S_1, S_2, \dots, S_k) - X, 0)$$

at maturity.

- The closed-form formula is a multi-dimensional integral.[a]

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[a] Johnson (1987); Chen (`D95723006`) and Lyuu (2009).

Multivariate Derivatives Pricing (concluded)

- Suppose $dS_j/S_j = r\,dt + \sigma_j\,dW_j$, $1 \leq j \leq k$, where C is the correlation matrix for $dW_1, dW_2, \ldots, dW_k$.
- Let $C = PP^{\mathrm{T}}$.
- Let ξ consist of k independent random variables from $N(0,1)$.
- Let $\xi' = P\xi$.
- Similar to Eq. (78) on p. 675,

$$S_{i+1} = S_i e^{(r-\sigma_j^2/2)\,\Delta t + \sigma_j \sqrt{\Delta t}\;\xi'_j}, \quad 1 \leq j \leq k.$$

Least-Squares Problems

- The least-squares (LS) problem is concerned with

  $$\min_{x \in R^n} \| Ax - b \|,$$

  where $A \in \boldsymbol{R}^{m \times n}$, $b \in \boldsymbol{R}^m$, $m \geq n$.

- The LS problem is called regression analysis in statistics and is equivalent to minimizing the mean-square error.

- Often written as

  $$Ax = b.$$

Polynomial Regression

- In polynomial regression, $x_0 + x_1 x + \cdots + x_n x^n$ is used to fit the data $\{ (a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m) \}$.
- This leads to the LS problem,

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^n \\ 1 & a_2 & a_2^2 & \cdots & a_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \cdots & a_m^n \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

- Consult the text for solutions.

American Option Pricing by Simulation

- The continuation value of an American option is the conditional expectation of the payoff from keeping the option alive now.
- The option holder must compare the immediate exercise value and the continuation value.
- In standard Monte Carlo simulation, each path is treated independently of other paths.
- But the decision to exercise the option cannot be reached by looking at one path alone.

The Least-Squares Monte Carlo Approach

- The continuation value can be estimated from the cross-sectional information in the simulation by using least squares.[a]
- The result is a function (of the state) for estimating the continuation values.
- Use the function to estimate the continuation value for each path to determine its cash flow.
- This is called the least-squares Monte Carlo (LSM) approach and is provably convergent.[b]

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[a] Longstaff and Schwartz (2001).

[b] Clément, Lamberton, and Protter (2002).

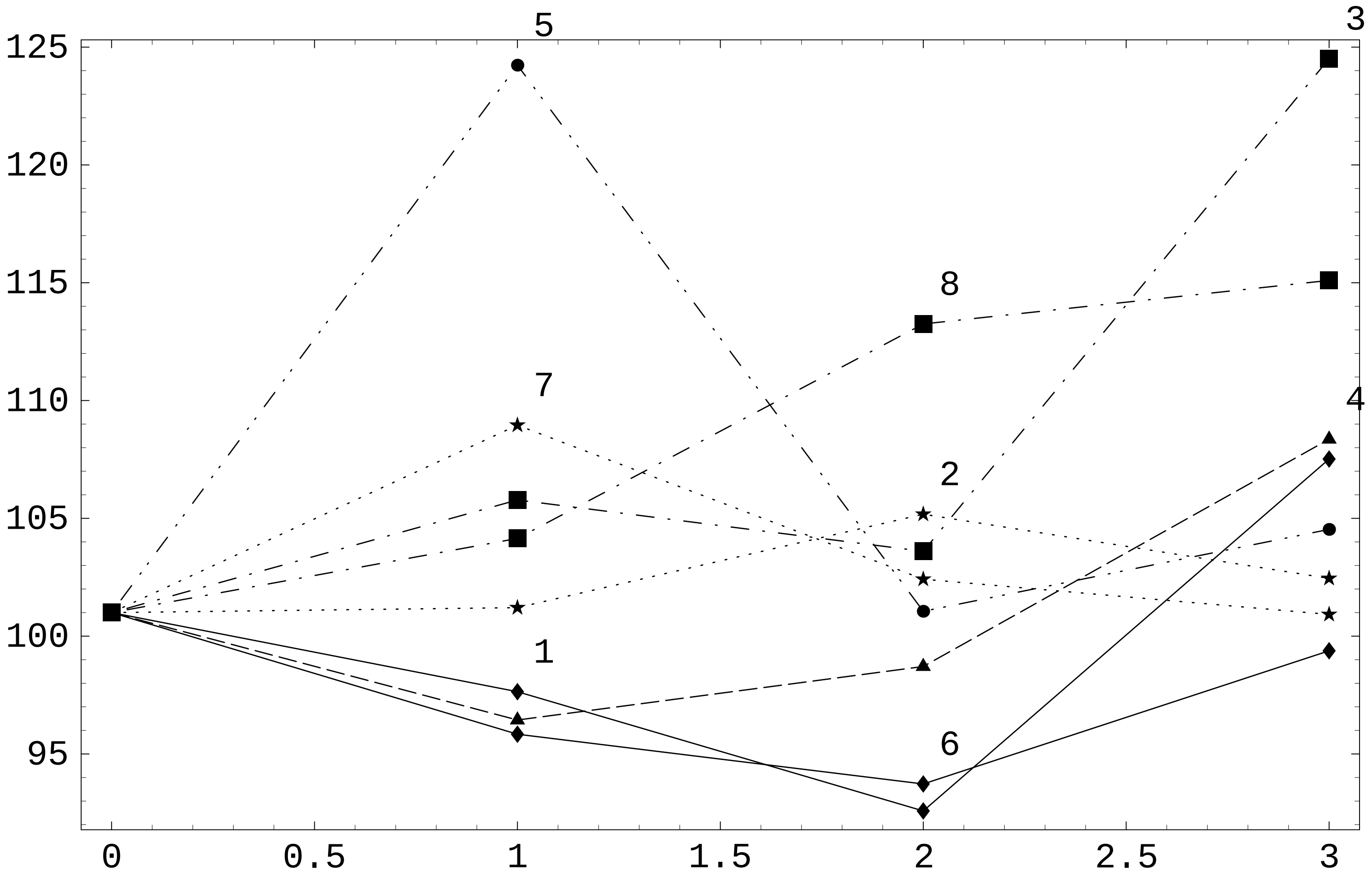
A Numerical Example

- Consider a 3-year American put on a non-dividend-paying stock.
- The put is exercisable at years 0, 1, 2, and 3.
- The strike price $X = 105$.
- The annualized riskless rate is $r = 5\%$.
- The spot stock price is 101.
  - The annual discount factor hence equals 0.951229.
- We use only 8 price paths to illustrate the algorithm.

A Numerical Example (continued)

Stock price paths

Path	Year 0	Year 1	Year 2	Year 3
1	**101**	**97.6424**	**92.5815**	107.5178
2	**101**	**101.2103**	105.1763	**102.4524**
3	**101**	105.7802	**103.6010**	124.5115
4	**101**	**96.4411**	**98.7120**	108.3600
5	**101**	124.2345	**101.0564**	**104.5315**
6	**101**	**95.8375**	**93.7270**	**99.3788**
7	**101**	108.9554	**102.4177**	**100.9225**
8	**101**	**104.1475**	113.2516	115.0994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4
2
1
6

A Numerical Example (continued)

- We use the basis functions $1, x, x^2$.
  - Other basis functions are possible.[a]
- The plot next page shows the final estimated optimal exercise strategy given by LSM.
- We now proceed to tackle our problem.
- The idea is to calculate the cash flow along each path, using information from all paths.

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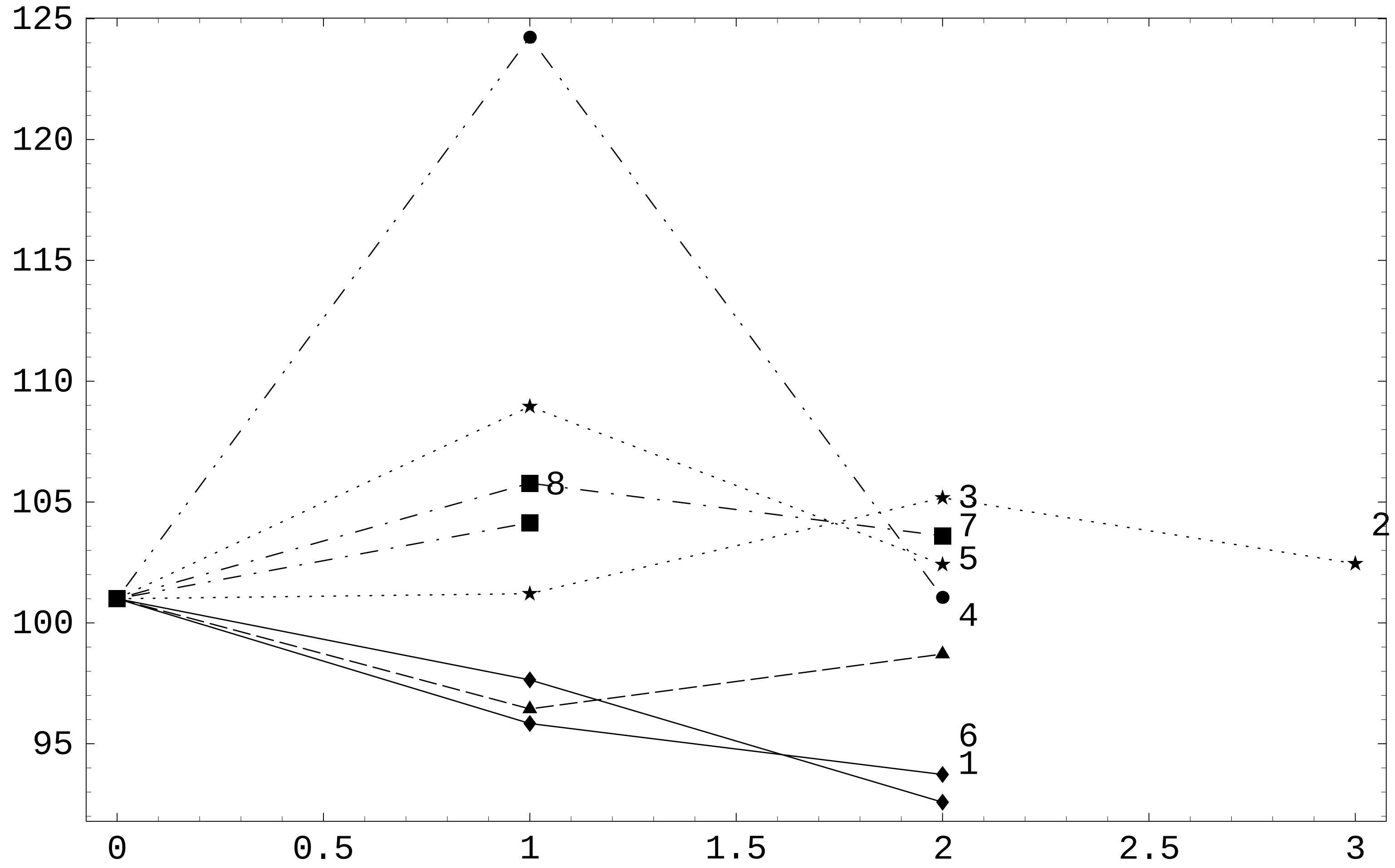
[a] Laguerre polynomials, Hermite polynomials, Legendre polynomials, Chebyshev polynomials, Gedenbauer polynomials, and Jacobi polynomials.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A Numerical Example (continued)

Cash flows at year 3

Path	Year 0	Year 1	Year 2	Year 3
1	—	—	—	0
2	—	—	—	2.5476
3	—	—	—	0
4	—	—	—	0
5	—	—	—	0.4685
6	—	—	—	5.6212
7	—	—	—	4.0775
8	—	—	—	0

A Numerical Example (continued)

- The cash flows at year 3 are the exercise value if the put is in the money.
- Only 4 paths are in the money: 2, 5, 6, 7.
- Some of the cash flows may not occur if the put is exercised earlier, which we will find out step by step.
- Incidentally, the European counterpart has a value of

$$0.951229^3 \times \frac{2.5476 + 0.4685 + 5.6212 + 4.0775}{8} = 1.3680.$$

A Numerical Example (continued)

- We move on to year 2.
- For each state that is in the money at year 2, we must decide whether to exercise it.
- There are 6 paths for which the put is in the money: 1, 3, 4, 5, 6, 7.
- Only in-the-money paths will be used in the regression because they are where early exercise is relevant.
  - If there were none, we would move on to year 1.

A Numerical Example (continued)

- Let x denote the stock prices at year 2 for those 6 paths.
- Let y denote the corresponding discounted future cash flows (at year 3) if the put is not exercised at year 2.

A Numerical Example (continued)

Regression at year 2

Path	x	y
1	92.5815	0×0.951229
2	—	—
3	103.6010	0×0.951229
4	98.7120	0×0.951229
5	101.0564	0.4685×0.951229
6	93.7270	5.6212×0.951229
7	102.4177	4.0775×0.951229
8	—	—

A Numerical Example (continued)

- We regress y on 1, x, and x^2.
- The result is

$$f(x) = 22.08 - 0.313114 \times x + 0.00106918 \times x^2.$$

- f estimates the continuation value conditional on the stock price at year 2.
- We next compare the immediate exercise value and the continuation value.

A Numerical Example (continued)

Optimal early exercise decision at year 2

Path	Exercise	Continuation
1	12.4185	$f(92.5815) = 2.2558$
2	—	—
3	1.3990	$f(103.6010) = 1.1168$
4	6.2880	$f(98.7120) = 1.5901$
5	3.9436	$f(101.0564) = 1.3568$
6	11.2730	$f(93.7270) = 2.1253$
7	2.5823	$f(102.4177) = 0.3326$
8	—	—

A Numerical Example (continued)

- Amazingly, the put should be exercised in all 6 paths: 1, 3, 4, 5, 6, 7.
- Now, any positive cash flow at year 3 should be set to zero for these paths as the put is exercised before year 3.
  - They are paths 5, 6, 7.
- Hence the cash flows on p. 734 become the next ones.

A Numerical Example (continued)

Cash flows at years 2 & 3

Path	Year 0	Year 1	Year 2	Year 3
1	—	—	12.4185	0
2	—	—	0	2.5476
3	—	—	1.3990	0
4	—	—	6.2880	0
5	—	—	3.9436	0
6	—	—	11.2730	0
7	—	—	2.5823	0
8	—	—	0	0

A Numerical Example (continued)

- We move on to year 1.
- For each state that is in the money at year 1, we must decide whether to exercise it.
- There are 5 paths for which the put is in the money: 1, 2, 4, 6, 8.
- Only in-the-money paths will be used in the regression because they are where early exercise is relevant.
  - If there were none, we would move on to year 0.

A Numerical Example (continued)

- Let x denote the stock prices at year 1 for those 5 paths.
- Let y denote the corresponding discounted future cash flows if the put is not exercised at year 1.
- From p. 742, we have the following table.

A Numerical Example (continued)

Regression at year 1

Path	x	y
1	97.6424	12.4185×0.951229
2	101.2103	2.5476×0.951229^2
3	—	—
4	96.4411	6.2880×0.951229
5	—	—
6	95.8375	11.2730×0.951229
7	—	—
8	104.1475	0

A Numerical Example (continued)

- We regress y on 1, x, and x^2.
- The result is

$$f(x) = -420.964 + 9.78113 \times x - 0.0551567 \times x^2.$$

- f estimates the continuation value conditional on the stock price at year 1.
- We next compare the immediate exercise value and the continuation value.

A Numerical Example (continued)

Optimal early exercise decision at year 1

Path	Exercise	Continuation
1	7.3576	$f(97.6424) = 8.2230$
2	3.7897	$f(101.2103) = 3.9882$
3	—	—
4	8.5589	$f(96.4411) = 9.3329$
5	—	—
6	9.1625	$f(95.8375) = 9.83042$
7	—	—
8	0.8525	$f(104.1475) = -0.551885$

A Numerical Example (continued)

- The put should be exercised for 1 path only: 8.
- Now, any positive future cash flow should be set to zero for this path.
  - But there is none.
- Hence the cash flows on p. 742 become the next ones.
- They also confirm the plot on p. 733.

A Numerical Example (continued)

Cash flows at years 1, 2, & 3

Path	Year 0	Year 1	Year 2	Year 3
1	—	0	12.4185	0
2	—	0	0	2.5476
3	—	0	1.3990	0
4	—	0	6.2880	0
5	—	0	3.9436	0
6	—	0	11.2730	0
7	—	0	2.5823	0
8	—	0.8525	0	0

A Numerical Example (continued)

- We move on to year 0.
- The continuation value is, from p 749,

$$
\begin{aligned}
& (12.4185 \times 0.951229^2 + 2.5476 \times 0.951229^3 \\
& +1.3990 \times 0.951229^2 + 6.2880 \times 0.951229^2 \\
& +3.9436 \times 0.951229^2 + 11.2730 \times 0.951229^2 \\
& +2.5823 \times 0.951229^2 + 0.8525 \times 0.951229)/8 \\
= \; & 4.66263.
\end{aligned}
$$

A Numerical Example (concluded)

- As this is larger than the immediate exercise value of $105 - 101 = 4$, the put should not be exercised at year 0.
- Hence the put's value is estimated to be 4.66263.
- Compare this to the European put's value of 1.3680 (p. 735).
- Why is the LSM estimate a lower bound?[a]

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[a] Contributed by Mr. Yang, Jui-Chung (`D97723002`) on April 29, 2009.

Time Series Analysis

The historian is a prophet in reverse.
— Friedrich von Schlegel (1772–1829)

GARCH Option Pricing[a]

- Options can be priced when the underlying asset's return follows a GARCH process.
- Let S_t denote the asset price at date t.
- Let h_t^2 be the conditional variance of the return over the period $[\,t, t+1\,]$ given the information at date t.
  - "One day" is merely a convenient term for any elapsed time Δt.

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[a] ARCH (autoregressive conditional heteroskedastic) is due to Engle (1982), co-winner of the 2003 Nobel Prize in Economic Sciences. GARCH (generalized ARCH) is due to Bollerslev (1986) and Taylor (1986). A Bloomberg quant said to me on Feb 29, 2008, that GARCH option pricing is seldom used in trading.

GARCH Option Pricing (continued)

- Adopt the following risk-neutral process for the price dynamics:[a]

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{h_t^2}{2} + h_t \epsilon_{t+1}, \tag{81}$$

where

$$\begin{aligned} h_{t+1}^2 &= \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 (\epsilon_{t+1} - c)^2, && (82) \\ \epsilon_{t+1} &\sim N(0,1) \text{ given information at date } t, \\ r &= \text{daily riskless return,} \\ c &\geq 0. \end{aligned}$$

---

[a] Duan (1995).

GARCH Option Pricing (continued)

- The five unknown parameters of the model are c, h_0, β_0, β_1, and β_2.
- It is postulated that $\beta_0, \beta_1, \beta_2 \geq 0$ to make the conditional variance positive.
- There are other inequalities to satisfy (see text).
- The above process is called the nonlinear asymmetric GARCH (or NGARCH) model.

GARCH Option Pricing (continued)

- It captures the volatility clustering in asset returns first noted by Mandelbrot (1963).[a]
  - When $c = 0$, a large ϵ_{t+1} results in a large h_{t+1}, which in turns tends to yield a large h_{t+2}, and so on.
- It also captures the negative correlation between the asset return and changes in its (conditional) volatility.[b]
  - For $c > 0$, a positive ϵ_{t+1} (good news) tends to decrease h_{t+1}, whereas a negative ϵ_{t+1} (bad news) tends to do the opposite.

---

[a] "... large changes tend to be followed by large changes—of either sign—and small changes tend to be followed by small changes ..."

[b] Noted by Black (1976): Volatility tends to rise in response to "bad news" and fall in response to "good news."

GARCH Option Pricing (concluded)

- With $y_t \equiv \ln S_t$ denoting the logarithmic price, the model becomes
$$y_{t+1} = y_t + r - \frac{h_t^2}{2} + h_t \epsilon_{t+1}. \tag{83}$$
- The pair (y_t, h_t^2) completely describes the current state.
- The conditional mean and variance of y_{t+1} are clearly
$$\begin{aligned} E[\, y_{t+1} \,|\, y_t, h_t^2 \,] &= y_t + r - \frac{h_t^2}{2}, & (84) \\ \text{Var}[\, y_{t+1} \,|\, y_t, h_t^2 \,] &= h_t^2. & (85) \end{aligned}$$

GARCH Model: Inferences

- Suppose the parameters c, h_0, β_0, β_1, and β_2 are given.
- Then we can recover $h_1, h_2, \dots, h_n$ and $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ from the prices
$$S_0, S_1, \dots, S_n$$
under the GARCH model (81) on p. 755.
- This property is useful in statistical inferences.

The Ritchken-Trevor (RT) Algorithm[a]

- The GARCH model is a continuous-state model.
- To approximate it, we turn to trees with *discrete* states.
- Path dependence in GARCH makes the tree for asset prices explode exponentially (why?).
- We need to mitigate this combinatorial explosion.

---

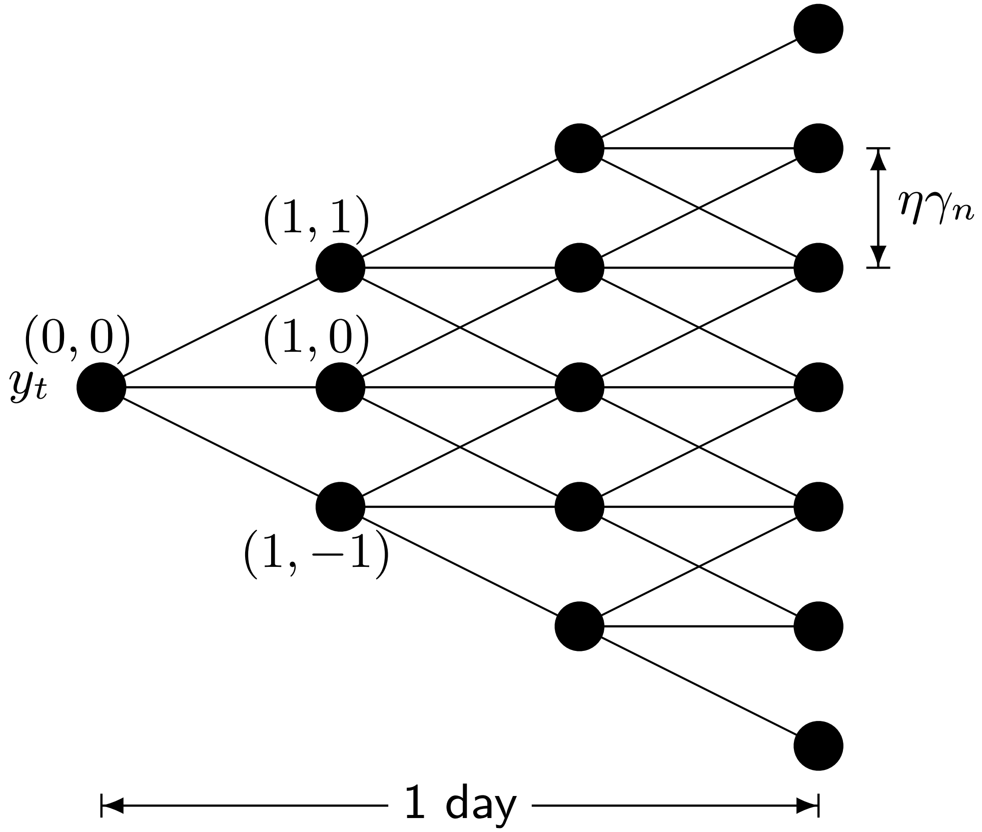
[a] Ritchken and Trevor (1999).

The Ritchken-Trevor Algorithm (continued)

- Partition a day into n periods.
- Three states follow each state (y_t, h_t^2) after a period.
- As the trinomial model combines, $2n + 1$ states at date $t + 1$ follow each state at date t (recall p. 588).
- These $2n + 1$ values must approximate the distribution of (y_{t+1}, h_{t+1}^2).
- So the conditional moments (84)–(85) at date $t + 1$ on p. 758 must be matched by the trinomial model to guarantee convergence to the continuous-state model.

The Ritchken-Trevor Algorithm (continued)

- It remains to pick the jump size and the three branching probabilities.
- The role of σ in the Black-Scholes option pricing model is played by h_t in the GARCH model.
- As a jump size proportional to $\sigma/\sqrt{n}$ is picked in the BOPM, a comparable magnitude will be chosen here.
- Define $\gamma \equiv h_0$, though other multiples of h_0 are possible, and
$$\gamma_n \equiv \frac{\gamma}{\sqrt{n}}.$$
- The jump size will be some integer multiple η of γ_n.
- We call η the jump parameter (p. 763).



The seven values on the right approximate the distribution of logarithmic price y_{t+1}.

The Ritchken-Trevor Algorithm (continued)

- The middle branch does not change the underlying asset's price.
- The probabilities for the up, middle, and down branches are

$$
\begin{aligned}
p_u &= \frac{h_t^2}{2\eta^2\gamma^2} + \frac{r-(h_t^2/2)}{2\eta\gamma\sqrt{n}}, && (86)\\
p_m &= 1-\frac{h_t^2}{\eta^2\gamma^2}, && (87)\\
p_d &= \frac{h_t^2}{2\eta^2\gamma^2} - \frac{r-(h_t^2/2)}{2\eta\gamma\sqrt{n}}. && (88)
\end{aligned}
$$

The Ritchken-Trevor Algorithm (continued)

- It can be shown that:
  - The trinomial model takes on $2n+1$ values at date $t+1$ for y_{t+1}.
  - These values have a matching mean for y_{t+1}.
  - These values have an asymptotically matching variance for y_{t+1}.
- The central limit theorem guarantees the desired convergence as n increases (if the probabilities are valid).

The Ritchken-Trevor Algorithm (continued)

- We can dispense with the intermediate nodes *between* dates to create a $(2n+1)$-nomial tree (p. 767).
- The resulting model is multinomial with $2n+1$ branches from any state (y_t, h_t^2).
- There are two reasons behind this manipulation.
  - Interdate nodes are created merely to approximate the continuous-state model after one day.
  - Keeping the interdate nodes results in a tree that can be as much as n times larger.[a]

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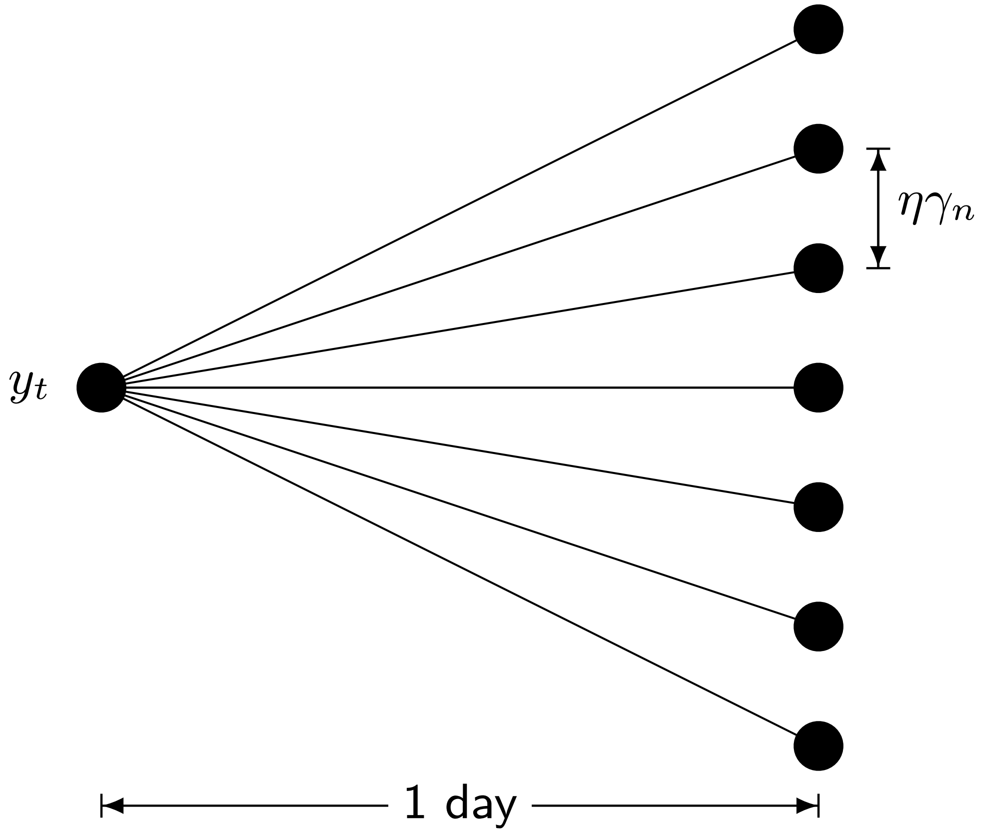
[a] Contrast that with the case on p. 334.



This heptanomial tree is the outcome of the trinomial tree on p. 763 after its intermediate nodes are removed.

The Ritchken-Trevor Algorithm (continued)

- A node with logarithmic price $y_t + \ell \eta \gamma_n$ at date $t+1$ follows the current node at date t with price y_t, where

$$-n \leq \ell \leq n.$$

- To reach that price in n periods, the number of up moves must exceed that of down moves by exactly ℓ.
- The probability that this happens is

$$P(\ell) \equiv \sum_{j_u, j_m, j_d} \frac{n!}{j_u! \, j_m! \, j_d!} \, p_u^{j_u} p_m^{j_m} p_d^{j_d},$$

with $j_u, j_m, j_d \geq 0$, $n = j_u + j_m + j_d$, and $\ell = j_u - j_d$.

The Ritchken-Trevor Algorithm (continued)

- A particularly simple way to calculate the $P(\ell)$s starts by noting that

$$(p_u x + p_m + p_d x^{-1})^n = \sum_{\ell=-n}^{n} P(\ell)\, x^{\ell}. \tag{89}$$

  - Convince yourself that this trick does the "accounting" correctly.

- So we expand $(p_u x + p_m + p_d x^{-1})^n$ and retrieve the probabilities by reading off the coefficients.

- It can be computed in $O(n^2)$ time.

The Ritchken-Trevor Algorithm (continued)

- The updating rule (82) on p. 755 must be modified to account for the adoption of the discrete-state model.
- The logarithmic price $y_t + \ell\eta\gamma_n$ at date $t+1$ following state (y_t, h_t^2) at date t has a variance equal to

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 (\epsilon'_{t+1} - c)^2, \tag{90}$$

  - Above,

$$\epsilon'_{t+1} = \frac{\ell\eta\gamma_n - (r - h_t^2/2)}{h_t}, \quad \ell = 0, \pm 1, \pm 2, \dots, \pm n,$$

    is a discrete random variable with $2n+1$ values.

The Ritchken-Trevor Algorithm (continued)

- Different conditional variances h_t^2 may require different η so that the probabilities calculated by Eqs. (86)–(88) on p. 764 lie between 0 and 1.
- This implies varying jump sizes.
- The necessary requirement $p_m \geq 0$ implies $\eta \geq h_t/\gamma$.
- Hence we try

$$\eta = \lceil h_t/\gamma \rceil, \lceil h_t/\gamma \rceil + 1, \lceil h_t/\gamma \rceil + 2, \ldots$$

until valid probabilities are obtained or until their nonexistence is confirmed.

The Ritchken-Trevor Algorithm (continued)

- The sufficient and necessary condition for valid probabilities to exist is[a]

$$\frac{|\, r - (h_t^2/2)\, |}{2\eta\gamma\sqrt{n}} \leq \frac{h_t^2}{2\eta^2\gamma^2} \leq \min\left(1 - \frac{|\, r - (h_t^2/2)\, |}{2\eta\gamma\sqrt{n}}, \frac{1}{2}\right).$$

- Obviously, the magnitude of η tends to grow with h_t.
- The plot on p. 773 uses $n = 1$ to illustrate our points for a 3-day model.
- For example, node $(1, 1)$ of date 1 and node $(2, 3)$ of date 2 pick $\eta = 2$.

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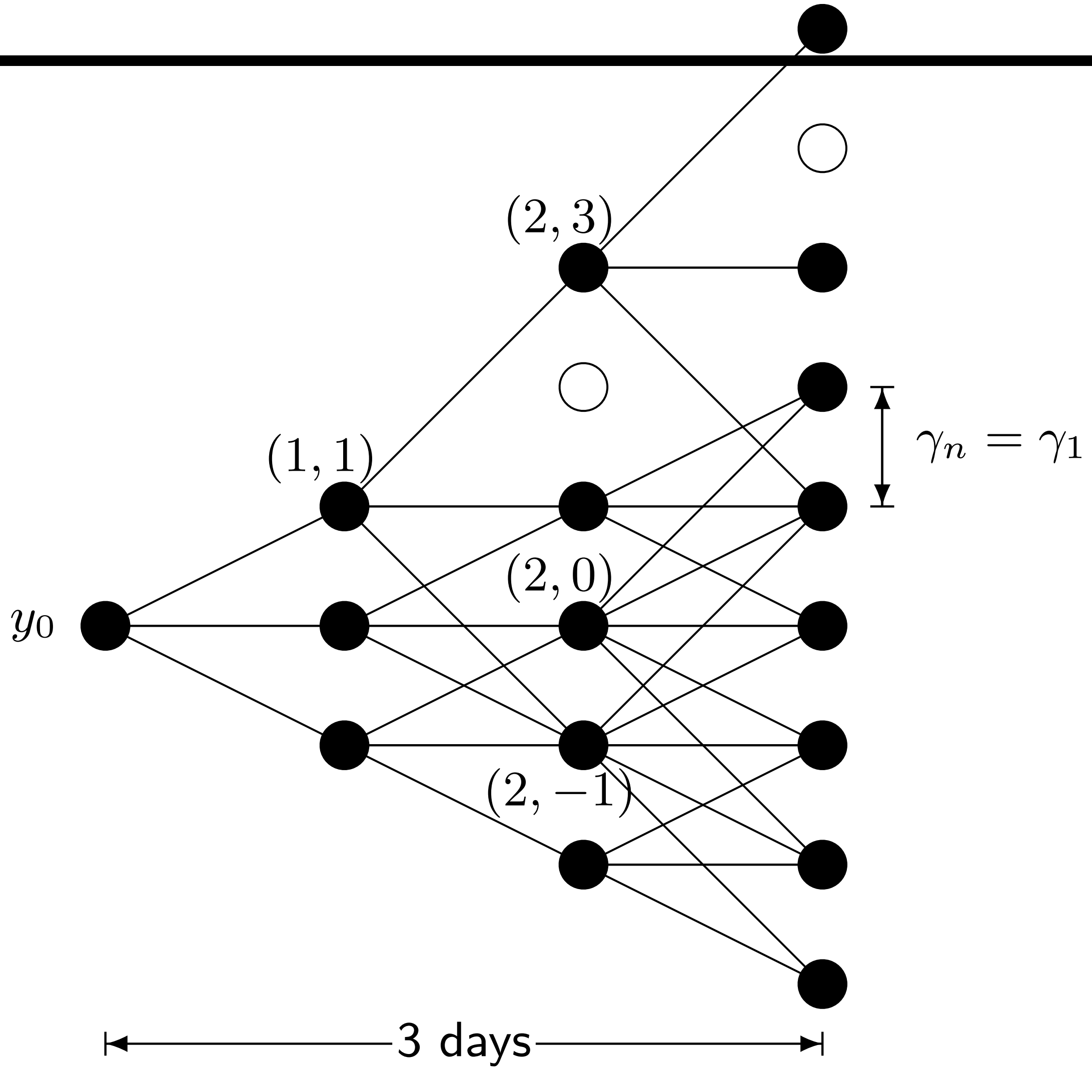
[a] Lyuu and Wu (`R90723065`) (2003).


$(2, 3)$
$(1, 1)$
$\gamma_n = \gamma_1$
$(2, 0)$
y_0
$(2, -1)$
3 days

The Ritchken-Trevor Algorithm (continued)

- The topology of the tree is not a standard combining multinomial tree.
- For example, a few nodes on p. 773 such as nodes $(2, 0)$ and $(2, -1)$ have multiple jump sizes.
- The reason is the path dependence of the model.
  - Two paths can reach node $(2, 0)$ from the root node, each with a different variance for the node.
  - One of the variances results in $\eta = 1$, whereas the other results in $\eta = 2$.

The Ritchken-Trevor Algorithm (concluded)

- The number of possible values of h_t^2 at a node can be exponential.
  - Because each path brings with it a different variance h_t^2.
- To address this problem, we record only the maximum and minimum h_t^2 at each node.[a]
- Therefore, each node on the tree contains only two states $(y_t, h_{\max}^2)$ and $(y_t, h_{\min}^2)$.
- Each of $(y_t, h_{\max}^2)$ and $(y_t, h_{\min}^2)$ carries its own η and set of $2n + 1$ branching probabilities.

---

[a] Cakici and Topyan (2000). But see p. 809 for a potential problem.

Negative Aspects of the Ritchken-Trevor Algorithm[a]

- A small n may yield inaccurate option prices.
- But the tree will grow exponentially if n is large enough.
  - Specifically, $n > (1 - \beta_1)/\beta_2$ when $r = c = 0$.
- A large n has another serious problem: The tree cannot grow beyond a certain date.
- Thus the choice of n may be quite limited in practice.
- The RT algorithm can be modified to be free of shortened maturity and exponential complexity.[b]

---

[a] Lyuu and Wu (`R90723065`) (2003, 2005).

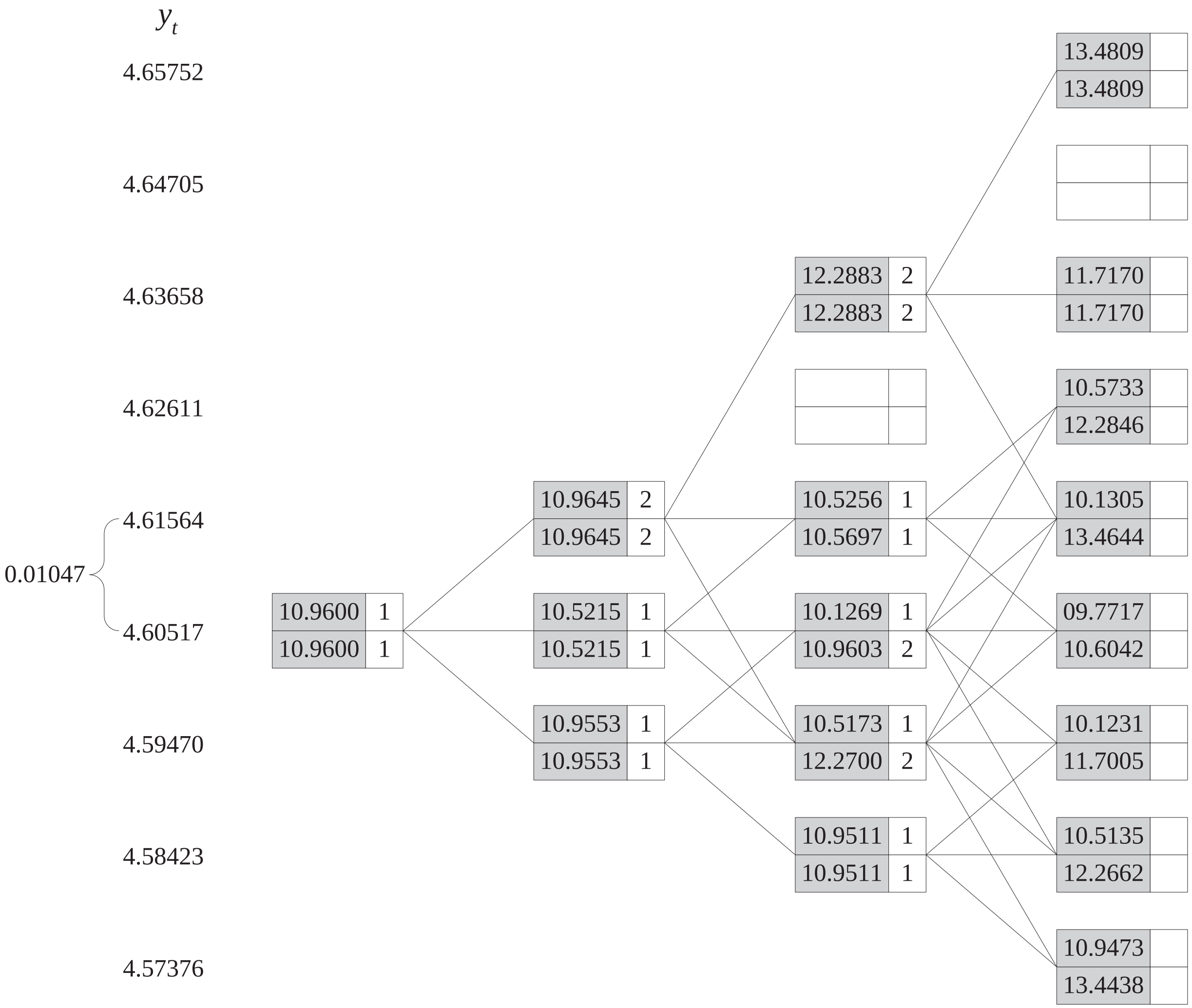
[b] It is only $O(n^2)$ if $n \leq (\sqrt{(1 - \beta_1)/\beta_2} - c)^2$!

Numerical Examples

- Assume $S_0 = 100$, $y_0 = \ln S_0 = 4.60517$, $r = 0$, $h_0^2 = 0.0001096$, $\gamma = h_0 = 0.010469$, $n = 1$, $\gamma_n = \gamma/\sqrt{n} = 0.010469$, $\beta_0 = 0.000006575$, $\beta_1 = 0.9$, $\beta_2 = 0.04$, and $c = 0$.
- A daily variance of 0.0001096 corresponds to an annual volatility of $\sqrt{365 \times 0.0001096} \approx 20\%$.
- Let $h^2(i, j)$ denote the variance at node (i, j).
- Initially, $h^2(0, 0) = h_0^2 = 0.0001096$.

Numerical Examples (continued)

- Let $h^2_{\max}(i, j)$ denote the maximum variance at node (i, j).
- Let $h^2_{\min}(i, j)$ denote the minimum variance at node (i, j).
- Initially, $h^2_{\max}(0, 0) = h^2_{\min}(0, 0) = h^2_0$.
- The resulting three-day tree is depicted on p. 779.



y_t

4.65752

4.64705

4.63658

4.62611

4.61564

0.01047

4.60517

4.59470

4.58423

4.57376

10.9600 1
10.9600 1

10.9645 2
10.9645 2

10.5215 1
10.5215 1

10.9553 1
10.9553 1

12.2883 2
12.2883 2

10.5256 1
10.5697 1

10.1269 1
10.9603 2

10.5173 1
12.2700 2

10.9511 1
10.9511 1

13.4809
13.4809

11.7170
11.7170

10.5733
12.2846

10.1305
13.4644

09.7717
10.6042

10.1231
11.7005

10.5135
12.2662

10.9473
13.4438

A top (bottom) number inside a gray box refers to the minimum (maximum, resp.) variance $h^2_{\min}$ ($h^2_{\max}$, resp.) for the node. Variances are multiplied by 100,000 for readability. A top (bottom) number inside a white box refers to η corresponding to $h^2_{\min}$ ($h^2_{\max}$, resp.).

Numerical Examples (continued)

- Let us see how the numbers are calculated.
- Start with the root node, node $(0, 0)$.
- Try $\eta = 1$ in Eqs. (86)–(88) on p. 764 first to obtain

$$\begin{aligned} p_u &= 0.4974, \\ p_m &= 0, \\ p_d &= 0.5026. \end{aligned}$$

- As they are valid probabilities, the three branches from the root node use single jumps.

Numerical Examples (continued)

- Move on to node $(1,1)$.
- It has one predecessor node—node $(0,0)$—and it takes an up move to reach the current node.
- So apply updating rule (90) on p. 770 with $\ell = 1$ and $h_t^2 = h^2(0,0)$.
- The result is $h^2(1,1) = 0.000109645$.

Numerical Examples (continued)

- Because $\lceil h(1,1)/\gamma \rceil = 2$, we try $\eta = 2$ in Eqs. (86)–(88) on p. 764 first to obtain

$$\begin{aligned} p_u &= 0.1237, \\ p_m &= 0.7499, \\ p_d &= 0.1264. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(1,1)$ use double jumps.

Numerical Examples (continued)

- Carry out similar calculations for node $(1, 0)$ with $\ell = 0$ in updating rule (90) on p. 770.
- Carry out similar calculations for node $(1, -1)$ with $\ell = -1$ in updating rule (90).
- Single jump $\eta = 1$ works for both nodes.
- The resulting variances are

$$\begin{aligned} h^2(1,0) &= 0.000105215, \\ h^2(1,-1) &= 0.000109553. \end{aligned}$$

Numerical Examples (continued)

- Node $(2,0)$ has 2 predecessor nodes, $(1,0)$ and $(1,-1)$.
- Both have to be considered in deriving the variances.
- Let us start with node $(1,0)$.
- Because it takes a middle move to reach the current node, we apply updating rule (90) on p. 770 with $\ell = 0$ and $h_t^2 = h^2(1,0)$.
- The result is $h_{t+1}^2 = 0.000101269$.

Numerical Examples (continued)

- Now move on to the other predecessor node $(1, -1)$.
- Because it takes an up move to reach the current node, apply updating rule (90) on p. 770 with $\ell = 1$ and $h_t^2 = h^2(1, -1)$.
- The result is $h_{t+1}^2 = 0.000109603$.
- We hence record

$$\begin{aligned} h_{\min}^2(2,0) &= 0.000101269, \\ h_{\max}^2(2,0) &= 0.000109603. \end{aligned}$$

Numerical Examples (continued)

- Consider state $h_{\max}^2(2,0)$ first.
- Because $\lceil h_{\max}(2,0)/\gamma \rceil = 2$, we first try $\eta = 2$ in Eqs. (86)–(88) on p. 764 to obtain

$$\begin{aligned} p_u &= 0.1237, \\ p_m &= 0.7500, \\ p_d &= 0.1263. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,0)$ with the maximum variance use double jumps.

Numerical Examples (continued)

- Now consider state $h^2_{\min}(2,0)$.
- Because $\lceil\, h_{\min}(2,0)/\gamma\,\rceil = 1$, we first try $\eta = 1$ in Eqs. (86)–(88) on p. 764 to obtain

$$\begin{aligned} p_u &= 0.4596, \\ p_m &= 0.0760, \\ p_d &= 0.4644. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,0)$ with the minimum variance use single jumps.

Numerical Examples (continued)

- Node $(2, -1)$ has 3 predecessor nodes.
- Start with node $(1, 1)$.
- Because it takes a down move to reach the current node, we apply updating rule (90) on p. 770 with $\ell = -1$ and $h_t^2 = h^2(1, 1)$.[a]
- The result is $h_{t+1}^2 = 0.0001227$.

---

[a] Note that it is not $\ell = -2$ because $-n \leq \ell \leq n$.

Numerical Examples (continued)

- Now move on to predecessor node $(1, 0)$.
- Because it also takes a down move to reach the current node, we apply updating rule (90) on p. 770 with $\ell = -1$ and $h_t^2 = h^2(1, 0)$.
- The result is $h_{t+1}^2 = 0.000105609$.

Numerical Examples (continued)

- Finally, consider predecessor node $(1, -1)$.
- Because it takes a middle move to reach the current node, we apply updating rule (90) on p. 770 with $\ell = 0$ and $h_t^2 = h^2(1, -1)$.
- The result is $h_{t+1}^2 = 0.000105173$.
- We hence record

$$\begin{aligned} h_{\min}^2(2, -1) &= 0.000105173, \\ h_{\max}^2(2, -1) &= 0.0001227. \end{aligned}$$

Numerical Examples (continued)

- Consider state $h^2_{\max}(2, -1)$.
- Because $\lceil h_{\max}(2, -1)/\gamma \rceil = 2$, we first try $\eta = 2$ in Eqs. (86)–(88) on p. 764 to obtain

$$\begin{aligned} p_u &= 0.1385, \\ p_m &= 0.7201, \\ p_d &= 0.1414. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2, -1)$ with the maximum variance use double jumps.

Numerical Examples (continued)

- Next, consider state $h^2_{\min}(2,-1)$.
- Because $\lceil h_{\min}(2,-1)/\gamma \rceil = 1$, we first try $\eta = 1$ in Eqs. (86)–(88) on p. 764 to obtain

$$\begin{aligned} p_u &= 0.4773, \\ p_m &= 0.0404, \\ p_d &= 0.4823. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,-1)$ with the minimum variance use single jumps.

Numerical Examples (concluded)

- Other nodes at dates 2 and 3 can be handled similarly.
- In general, if a node has k predecessor nodes, then $2k$ variances will be calculated using the updating rule.
  - This is because each predecessor node keeps two variance numbers.
- But only the maximum and minimum variances will be kept.

Negative Aspects of the RT Algorithm Revisited[a]

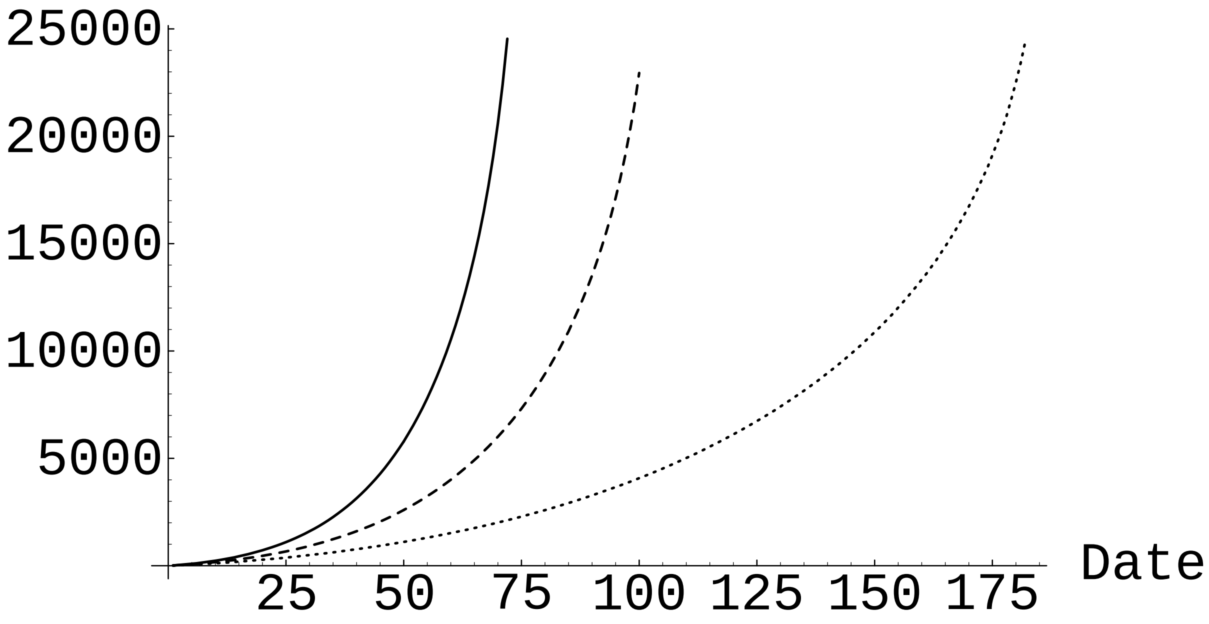
- Recall the problems mentioned on p. 776.
- In our case, combinatorial explosion occurs when

$$n > \frac{1 - \beta_1}{\beta_2} = \frac{1 - 0.9}{0.04} = 2.5.$$

- Suppose we are willing to accept the exponential running time and pick $n = 100$ to seek accuracy.
- But the problem of shortened maturity forces the tree to stop at date 9!

---

[a] Lyuu and Wu (`R90723065`) (2003).



Dotted line: $n = 3$; dashed line: $n = 4$; solid line: $n = 5$.

Backward Induction on the RT Tree

- After the RT tree is constructed, it can be used to price options by backward induction.
- Recall that each node keeps two variances $h^2_{\max}$ and $h^2_{\min}$.
- We now increase that number to K equally spaced variances between $h^2_{\max}$ and $h^2_{\min}$ at each node.
- Besides the minimum and maximum variances, the other $K-2$ variances in between are linearly interpolated.[a]

---

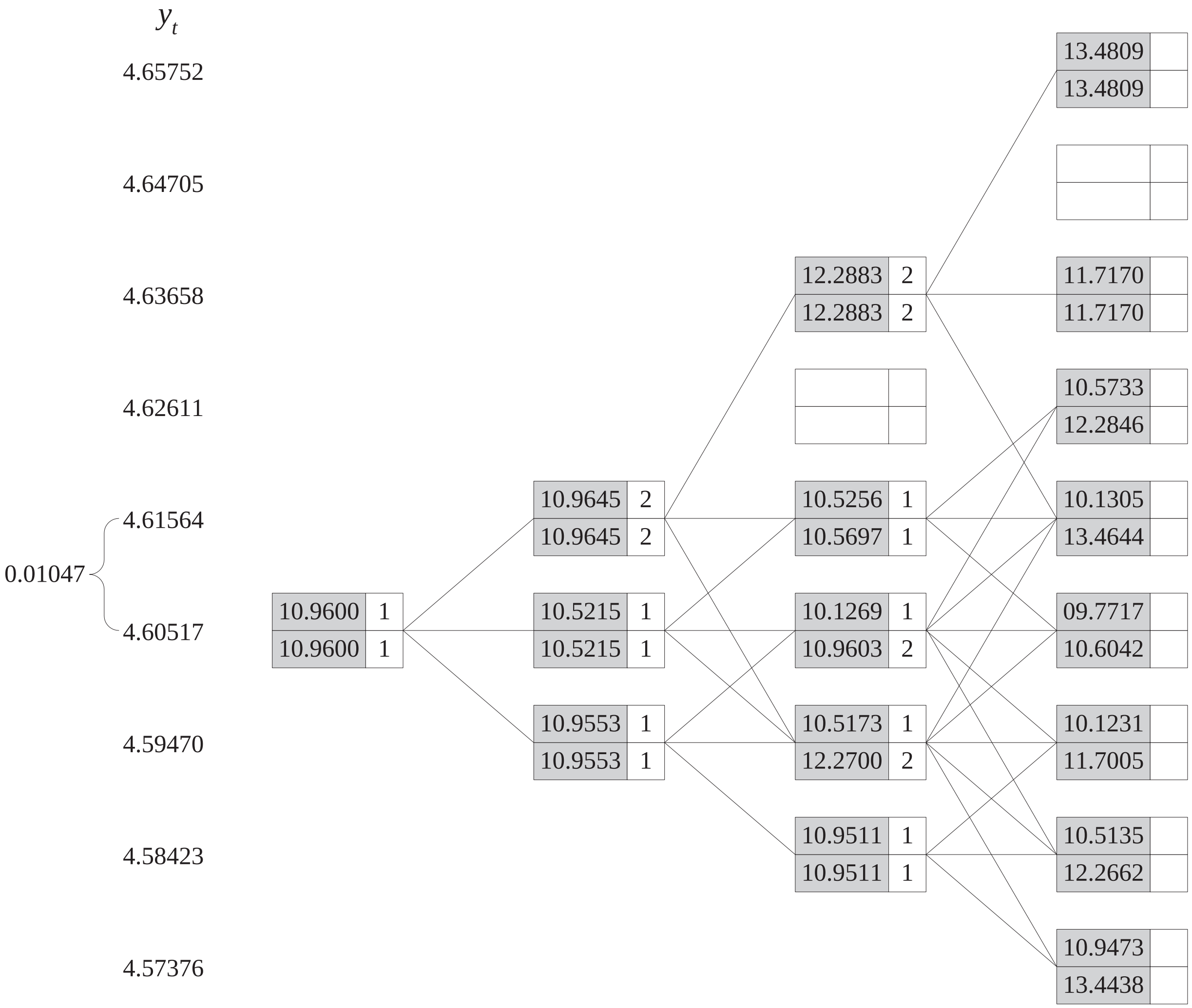
[a] In practice, log-linear interpolation works better (Lyuu and Wu (`R90723065`) (2005)). Log-cubic interpolation works even better (Liu (`R92922123`) (2005)).

Backward Induction on the RT Tree (continued)

- For example, if $K = 3$, then a variance of 10.5436×10^{-6} will be added between the maximum and minimum variances at node $(2, 0)$ on p. 779.[a]
- In general, the kth variance at node (i, j) is
$$h^2_{\min}(i,j) + k\,\frac{h^2_{\max}(i,j) - h^2_{\min}(i,j)}{K-1},$$
$k = 0, 1, \ldots, K-1$.
- Each interpolated variance's jump parameter and branching probabilities can be computed as before.

---

[a] Repeated on p. 799.



y_t

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12.2846

10.1305
13.4644

09.7717
10.6042

10.1231
11.7005

10.5135
12.2662

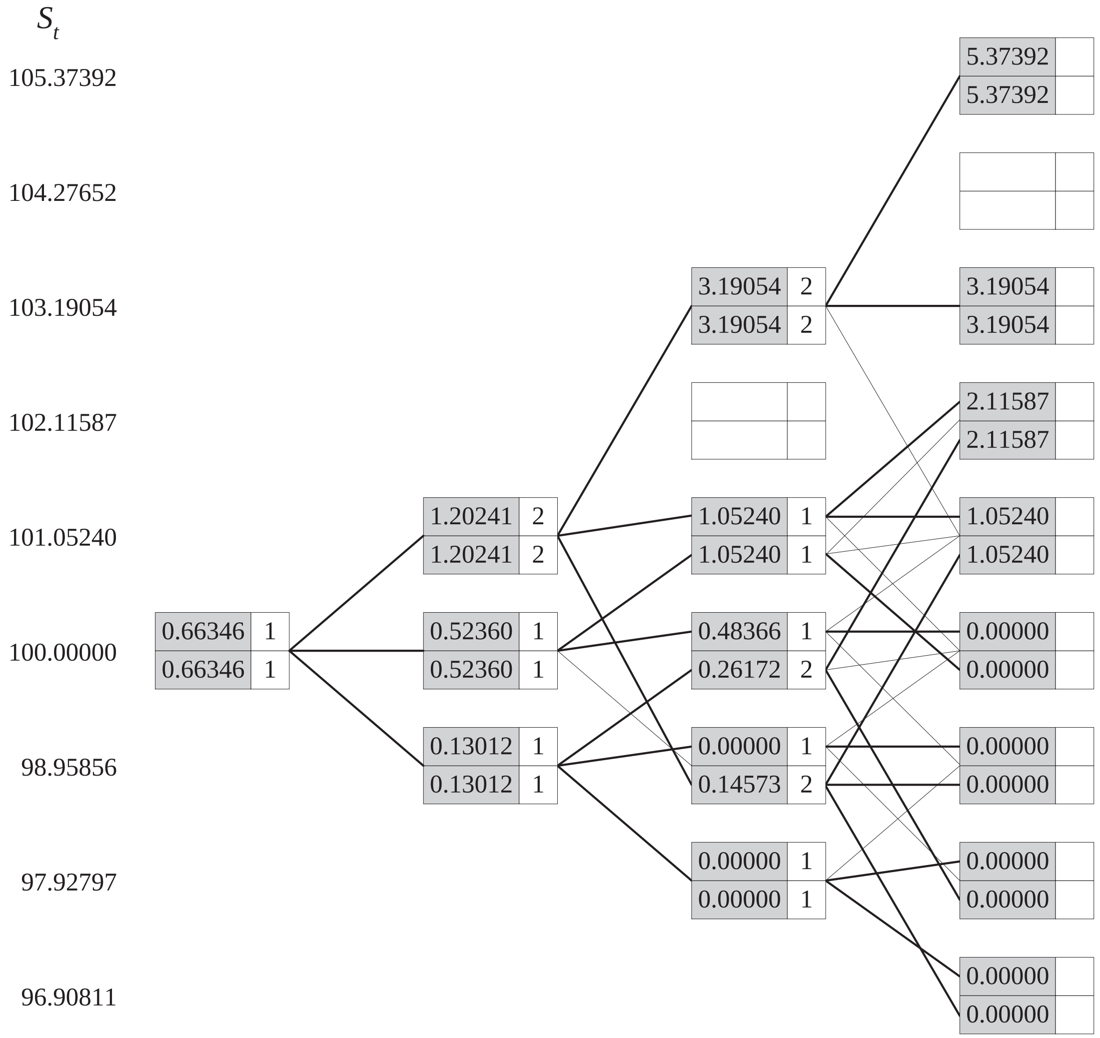
10.9473
13.4438

Backward Induction on the RT Tree (concluded)

- Suppose a variance falls between two of the K variances during backward induction.
- Linear interpolation of the option prices corresponding to the two bracketing variances will be used as the approximate option price.
- The above ideas are reminiscent of the ones on p. 355, where we dealt with arithmetic average-rate options.

Numerical Examples

- We next use the numerical example on p. 799 to price a European call option with a strike price of 100 and expiring at date 3.
- Recall that the riskless interest rate is zero.
- Assume $K = 2$; hence there are no interpolated variances.
- The pricing tree is shown on p. 802 with a call price of 0.66346.
  - The branching probabilities needed in backward induction can be found on p. 803.


S_t
105.37392
104.27652
103.19054
102.11587
101.05240
100.00000
98.95856
97.92797
96.90811
0.66346 1
0.66346 1
1.20241 2
1.20241 2
0.52360 1
0.52360 1
0.13012 1
0.13012 1
3.19054 2
3.19054 2
1.05240 1
1.05240 1
0.48366 1
0.26172 2
0.00000 1
0.14573 2
0.00000 1
0.00000 1
5.37392
5.37392
3.19054
3.19054
2.11587
2.11587
1.05240
1.05240
0.00000
0.00000
0.00000
0.00000
0.00000
0.00000
0.00000
0.00000

rb[i][0]
rb[i][1]

	rb[0][ ]	rb[1][ ]	rb[2][ ]	rb[3][ ]
rb[i][0]	0	−1	−2	−3
rb[i][1]	0	1	3	5

$h^2[i][j][0]$
$h^2[i][j][1]$

j	$h^2[0][\ ][\ ]$	$h^2[1][\ ][\ ]$	$h^2[2][\ ][\ ]$	$h^2[3][\ ][\ ]$
5				13.4809 13.4809
4				
3			12.2883 12.2883	11.7170 11.7170
2				10.5733 12.2846
1		10.9645 10.9645	10.5256 10.5697	10.1305 13.4644
0	10.9600 10.9600	10.5215 10.5215	10.1269 10.9603	09.7717 10.6042
−1		10.9553 10.9553	10.5173 12.2700	10.1231 11.7005
−2			10.9511 10.9511	10.5135 12.2662
−3				10.9473 13.4438

$\eta[i][j][0]$
$\eta[i][j][1]$

j	$\eta[0][\ ][\ ]$	$\eta[1][\ ][\ ]$	$\eta[2][\ ][\ ]$
3			2 2
2			
1		2 2	1 1
0	1 1	1 1	1 2
−1		1 1	1 2
−2			1 1

$p[i][j][0][1]$	$p[i][j][1][1]$
$p[i][j][0][0]$	$p[i][j][1][0]$
$p[i][j][0][-1]$	$p[i][j][1][-1]$

j	$p[0][\ ][\ ][\ ]$	$p[1][\ ][\ ][\ ]$	$p[2][\ ][\ ][\ ]$
3			0.1387 0.1387 0.7197 0.7197 0.1416 0.1416
2			
1		0.1237 0.1237 0.7499 0.7499 0.1264 0.1264	0.4777 0.4797 0.0396 0.0356 0.4827 0.4847
0	0.4974 0.4974 0.0000 0.0000 0.5026 0.5026	0.4775 0.4775 0.0400 0.0400 0.4825 0.4825	0.4596 0.1237 0.0760 0.7500 0.4644 0.1263
−1		0.4972 0.4972 0.0004 0.0004 0.5024 0.5024	0.4773 0.1385 0.0404 0.7201 0.4823 0.1414
−2			0.4970 0.4970 0.0008 0.0008 0.5022 0.5022

Numerical Examples (continued)

- Let us derive some of the numbers on p. 802.
- A gray line means the updated variance falls strictly between $h^2_{\max}$ and $h^2_{\min}$.
- The option price for a terminal node at date 3 equals $\max(S_3 - 100, 0)$, independent of the variance level.
- Now move on to nodes at date 2.
- The option price at node $(2,3)$ depends on those at nodes $(3,5)$, $(3,3)$, and $(3,1)$.
- It therefore equals

$$0.1387 \times 5.37392 + 0.7197 \times 3.19054 + 0.1416 \times 1.05240 = 3.19054.$$

Numerical Examples (continued)

- Option prices for other nodes at date 2 can be computed similarly.
- For node $(1, 1)$, the option price for both variances is

$$0.1237 \times 3.19054 + 0.7499 \times 1.05240 + 0.1264 \times 0.14573 = 1.20241.$$

- Node $(1, 0)$ is most interesting.
- We knew that a down move from it gives a variance of 0.000105609.
- This number falls between the minimum variance 0.000105173 and the maximum variance 0.0001227 at node $(2, -1)$ on p. 799.

Numerical Examples (continued)

- The option price corresponding to the minimum variance is 0.
- The option price corresponding to the maximum variance is 0.14573.
- The equation
$$x \times 0.000105173 + (1 - x) \times 0.0001227 = 0.000105609$$
is satisfied by $x = 0.9751$.
- So the option for the down state is approximated by
$$x \times 0 + (1 - x) \times 0.14573 = 0.00362.$$

Numerical Examples (continued)

- The up move leads to the state with option price 1.05240.
- The middle move leads to the state with option price 0.48366.
- The option price at node $(1, 0)$ is finally calculated as

$$0.4775 \times 1.05240 + 0.0400 \times 0.48366 + 0.4825 \times 0.00362 = 0.52360.$$

Numerical Examples (continued)

- A variance following an interpolated variance may exceed the maximum variance or be exceeded by the minimum variance.
- When this happens, the option price corresponding to the maximum or minimum variance will be used during backward induction.[a]

---

[a] Cakici and Topyan (2000).

Numerical Examples (concluded)

- But an interpolated variance may choose a branch that goes into a node that is *not* reached in forward induction.[a]
- In this case, the algorithm fails.
- It may be hard to calculate the implied β_1 and β_2 from option prices.[b]

---

[a] Lyuu and Wu (`R90723065`) (2005).

[b] Chang (`R93922034`) (2006).

Complexities of GARCH Models[a]

- The Ritchken-Trevor algorithm explodes exponentially if n is big enough (p. 776).
- The mean-tracking algorithm of Lyuu and Wu (2005) will make sure explosion does not happen if n is not too large.[b]
- The next page summarizes the situations for many GARCH option pricing models.
  - Our earlier treatment is for NGARCH only.

---

[a] Lyuu and Wu (`R90723065`) (2003, 2005).

[b] Similar to, but earlier than, the idea behind the binomial-trinomial tree on pp. 606ff.

Complexities of GARCH Models (concluded)[a]

Model	Explosion	Non-explosion
NGARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + \beta_2(\sqrt{n} + \lambda + c)^2 \leq 1$
LGARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + \beta_2(\sqrt{n} + \lambda)^2 \leq 1$
AGARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + \beta_2(\sqrt{n} + \lambda)^2 \leq 1$
GJR-GARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + (\beta_2 + \beta_3)(\sqrt{n} + \lambda)^2 \leq 1$
TS-GARCH	$\beta_1 + \beta_2 \sqrt{n} > 1$	$\beta_1 + \beta_2(\lambda + \sqrt{n}) \leq 1$
TGARCH	$\beta_1 + \beta_2 \sqrt{n} > 1$	$\beta_1 + (\beta_2 + \beta_3)(\lambda + \sqrt{n}) \leq 1$
Heston-Nandi	$\beta_1 + \beta_2(c - \frac{1}{2})^2 > 1$ & $c \leq \frac{1}{2}$	$\beta_1 + \beta_2 c^2 \leq 1$
VGARCH	$\beta_1 + (\beta_2/4) > 1$	$\beta_1 \leq 1$

---

[a] Chen (`R95723051`) (2008); Chen (`R95723051`), Lyuu, and Wen (`D94922003`) (2011).