Trees

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

And though the holes were rather small,
they had to count them all.
— The Beatles, *A Day in the Life* (1967)

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 244.
- We will now apply it to price barrier options.

The Reflection Principle[a]

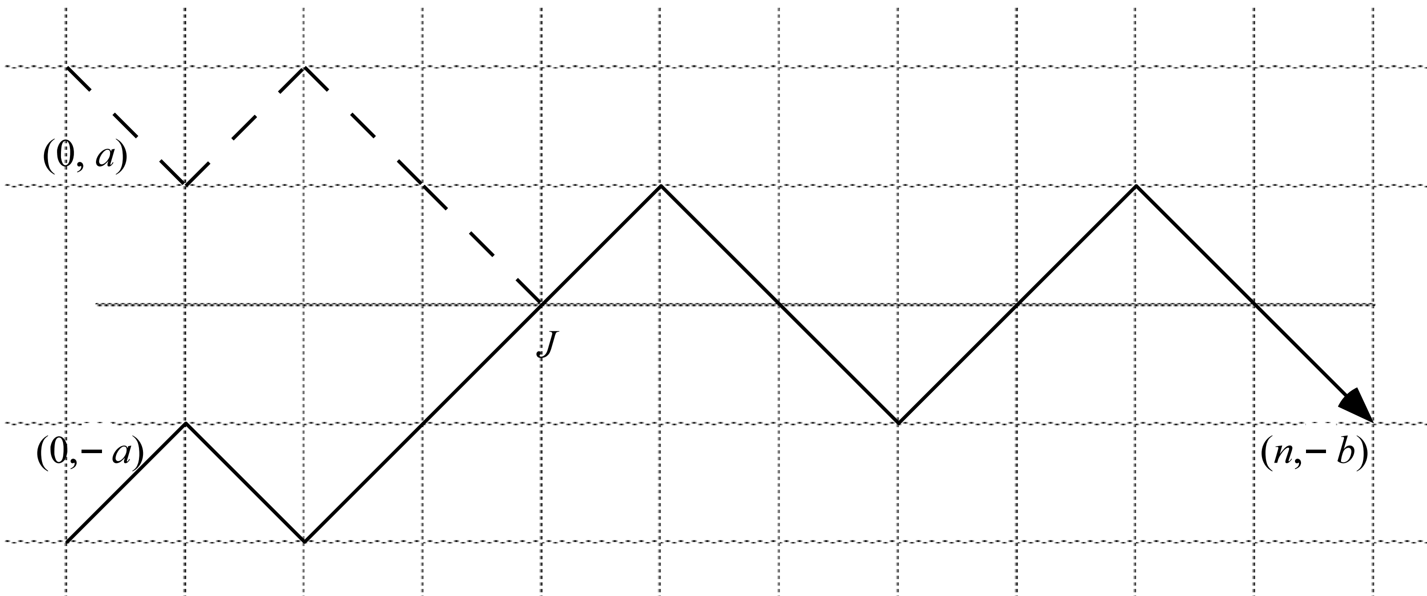
- Imagine a particle at position $(0, -\boldsymbol{a})$ on the integral lattice that is to reach $(n, -\boldsymbol{b})$.
- Without loss of generality, assume $\boldsymbol{a} > 0$ and $\boldsymbol{b} \geq 0$.
- This particle's movement:

$(i, j) \to (i + 1, j + 1)$ up move $S \to Su$

$(i, j) \to (i + 1, j - 1)$ down move $S \to Sd$

- How many paths touch the x axis?

[a] André (1887).


(0, a)
J
(0,− a)
(n,− b)

The Reflection Principle (continued)

- For a path from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touches the x axis, let J denote the first point this happens.
- Reflect the portion of the path from $(0, -\boldsymbol{a})$ to J.
- A path from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$ is constructed.
- It also hits the x axis at J for the first time.
- The one-to-one mapping shows the number of paths from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touch the x axis equals the number of paths from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$.

The Reflection Principle (concluded)

- A path of this kind has $(n + \boldsymbol{b} + \boldsymbol{a})/2$ down moves and $(n - \boldsymbol{b} - \boldsymbol{a})/2$ up moves.
- Hence there are

$$\binom{n}{\frac{n+\boldsymbol{a}+\boldsymbol{b}}{2}} \tag{59}$$

such paths for even $n + \boldsymbol{a} + \boldsymbol{b}$.

  – Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define

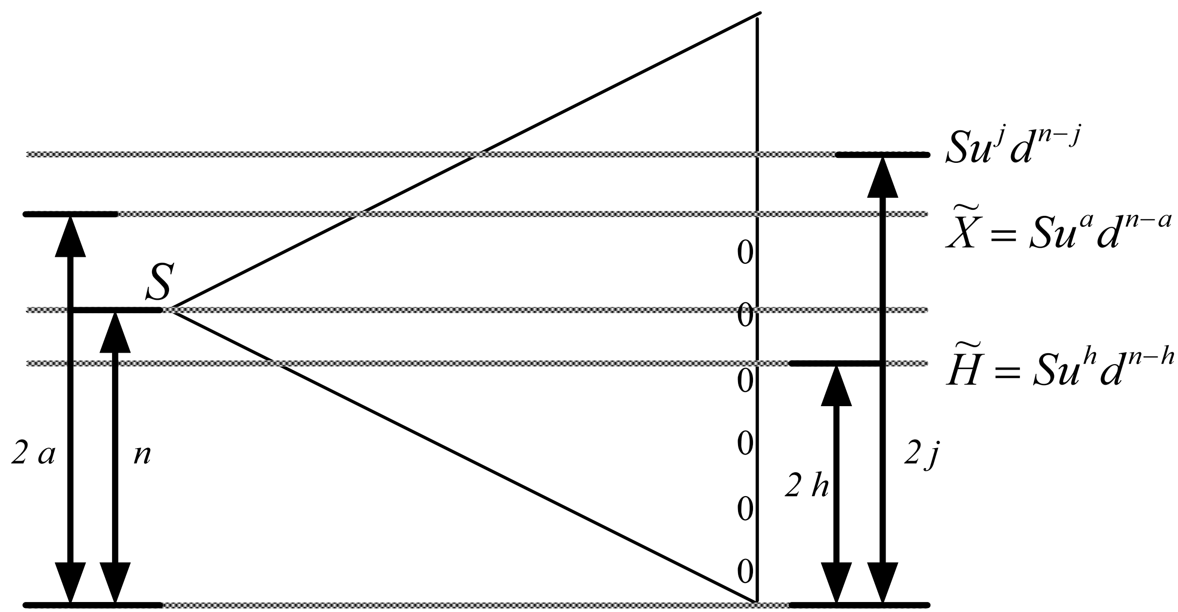
$$a \equiv \left\lceil \frac{\ln\left(X/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil,$$

$$h \equiv \left\lfloor \frac{\ln\left(H/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor.$$

  - h is such that $\tilde{H} \equiv Su^h d^{n-h}$ is the *terminal* price that is closest to, but does not exceed H.
  - a is such that $\tilde{X} \equiv Su^a d^{n-a}$ is the terminal price that is closest to, but is not exceeded by X.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with n moves hence ends up in the money if and only if the number of up moves is at least a.
- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price Sd^n on the binomial tree.
  –
$$Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-2k}. \quad (60)$$


$Su^j d^{n-j}$
$\tilde{X} = Su^a d^{n-a}$
S
$\tilde{H} = Su^h d^{n-h}$
0
0
0
0
0
0
2 a
n
2 h
2 j

Pricing Barrier Options (continued)

- The number of paths from S to the terminal price $Su^j d^{n-j}$ is $\binom{n}{j}$, each with probability $p^j(1-p)^{n-j}$.
- With reference to p. 574, the reflection principle can be applied with $\boldsymbol{a} = n - 2h$ and $\boldsymbol{b} = 2j - 2h$ in Eq. (59) on p. 571 by treating the $\tilde{H}$ line as the x axis.
- Therefore,

$$\binom{n}{\frac{n+(n-2h)+(2j-2h)}{2}} = \binom{n}{n-2h+j}$$

paths hit $\tilde{H}$ in the process for $h \leq n/2$.

Pricing Barrier Options (concluded)

- The terminal price $Su^j d^{n-j}$ is reached by a path that hits the effective barrier with probability

$$\binom{n}{n-2h+j} p^j (1-p)^{n-j}.$$

- The option value equals

$$\frac{\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - X \right)}{R^n}. \qquad (61)$$

  - $R \equiv e^{r\tau/n}$ is the riskless return per period.

- It implies a linear-time algorithm.[a]

[a]Lyuu (1998).

Convergence of BOPM

- Equation (61) results in the sawtooth-like convergence shown on p. 333.
- The reasons are not hard to see.
- The true barrier most likely does not equal the effective barrier.
- The same holds for the strike price and the effective strike price.
- The issue of the strike price is less critical.
- But the issue of the barrier is not negligible.

Convergence of BOPM (continued)

- Convergence is actually good if we limit n to certain values—191, for example.
- These values make the true barrier coincide with or just above one of the stock price levels, that is,

$$H \approx Sd^j = Se^{-j\sigma\sqrt{\tau/n}}$$

for some integer j.

- The preferred n's are thus

$$n = \left\lfloor \frac{\tau}{\left(\ln(S/H)/(j\sigma)\right)^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

Convergence of BOPM (continued)

- There is only one minor technicality left.
- We picked the effective barrier to be one of the $n+1$ possible terminal stock prices.
- However, the effective barrier above, Sd^j, corresponds to a terminal stock price only when $n-j$ is even.[a]
- To close this gap, we decrement n by one, if necessary, to make $n-j$ an even number.

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[a]This is because $j = n - 2k$ for some k by Eq. (60) on p. 573. Of course we could have adopted the form Sd^j $(-n \leq j \leq n)$ for the effective barrier. It makes a good exercise.

Convergence of BOPM (concluded)

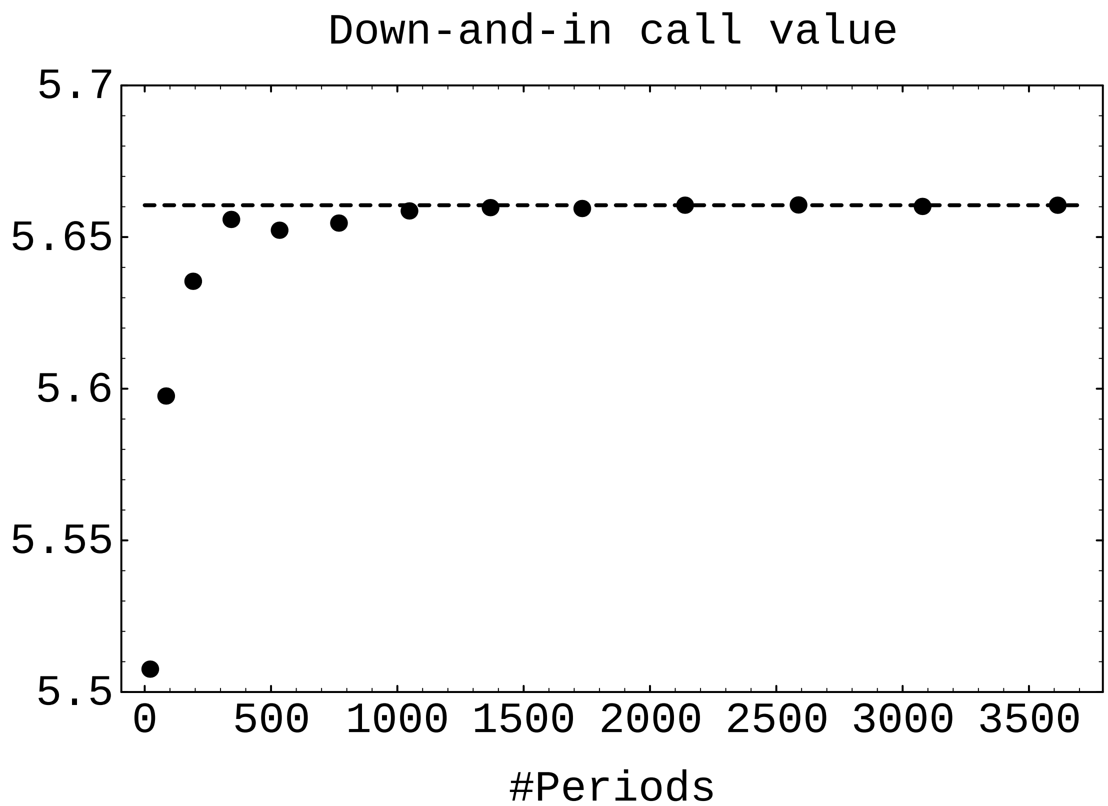
- The preferred n's are now

$$n = \begin{cases} \ell & \text{if } \ell - j \text{ is even} \\ \ell - 1 & \text{otherwise} \end{cases},$$

$j = 1, 2, 3, \ldots$, where

$$\ell \equiv \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.$$

- Evaluate pricing formula (61) on p. 576 only with the n's above.


Down-and-in call value
5.7
5.65
5.6
5.55
5.5
0
500
1000
1500
2000
2500
3000
3500
#Periods

Practical Implications

- Now that barrier options can be efficiently priced, we can afford to pick very large n's (p. 583).
- This has profound consequences (to describe later).

n	Combinatorial method	
	Value	Time (milliseconds)
21	5.507548	0.30
84	5.597597	0.90
191	5.635415	2.00
342	5.655812	3.60
533	5.652253	5.60
768	5.654609	8.00
1047	5.658622	11.10
1368	5.659711	15.00
1731	5.659416	19.40
2138	5.660511	24.70
2587	5.660592	30.20
3078	5.660099	36.70
3613	5.660498	43.70
4190	5.660388	44.10
4809	5.659955	51.60
5472	5.660122	68.70
6177	5.659981	76.70
6926	5.660263	86.90
7717	5.660272	97.20

Practical Implications (concluded)

- Pricing is prohibitively time consuming when $S \approx H$ because $n \sim 1/\ln^2(S/H)$.
  - This is called the barrier-too-close problem.
- This observation is indeed true of standard quadratic-time binomial tree algorithms.
- But it no longer applies to linear-time algorithms (see p. 585).
- This binomial model is $O(1/\sqrt{n})$ convergent in general but $O(1/n)$ convergent when the barrier is matched.[a]

[a] Lin (R95221010) (2008) and Lin (R95221010) and Palmer (2010).

	Barrier at 95.0			Barrier at 99.5			Barrier at 99.9	
n	Value	Time	n	Value	Time	n	Value	Time
	⋮		795	7.47761	8	19979	8.11304	253
2743	2.56095	31.1	3184	7.47626	38	79920	8.11297	1013
3040	2.56065	35.5	7163	7.47682	88	179819	8.11300	2200
3351	2.56098	40.1	12736	7.47661	166	319680	8.11299	4100
3678	2.56055	43.8	19899	7.47676	253	499499	8.11299	6300
4021	2.56152	48.1	28656	7.47667	368	719280	8.11299	8500
True	2.5615			7.4767			8.1130	

(All times in milliseconds.)

Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion $dS/S = r\,dt + \sigma\,dW$.[a]
- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Impose the matching of mean and that of variance:

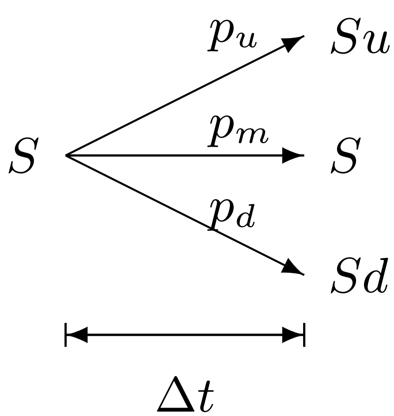
$$
\begin{aligned}
1 &= p_u + p_m + p_d, \\
SM &= (p_u u + p_m + (p_d/u))\,S, \\
S^2 V &= p_u(Su - SM)^2 + p_m(S - SM)^2 + p_d(Sd - SM)^2.
\end{aligned}
$$

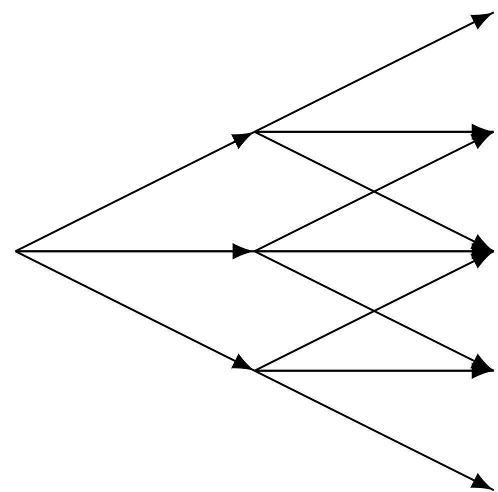
[a] Boyle (1988).

- Above,

$$
\begin{aligned}
M &\equiv e^{r\Delta t}, \\
V &\equiv M^2(e^{\sigma^2 \Delta t} - 1),
\end{aligned}
$$

by Eqs. (19) on p. 152.


p_u
Su
p_m
S
S
p_d
Sd
Δt



Trinomial Tree (concluded)

- Use linear algebra to verify that

$$p_u = \frac{u\left(V + M^2 - M\right) - (M - 1)}{(u - 1)\,(u^2 - 1)},$$

$$p_d = \frac{u^2\left(V + M^2 - M\right) - u^3(M - 1)}{(u - 1)\,(u^2 - 1)}.$$

  – In practice, must make sure the probabilities lie between 0 and 1.

- Countless variations.

A Trinomial Tree

- Use $u = e^{\lambda\sigma\sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$\begin{aligned} p_u &\to \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}, \\ p_d &\to \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}. \end{aligned}$$

- A nice choice for λ is $\sqrt{\pi/2}$.[a]

[a] Omberg (1988).

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting λ so that the barrier is hit exactly.[a]
- It takes

$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}$$

consecutive down moves to go from S to H if h is an integer, which is easy to achieve by adjusting λ.
  - This is because $Se^{-h\lambda\sigma\sqrt{\Delta t}} = H$.

[a] Ritchken (1995).

Barrier Options Revisited (continued)

- Typically, we find the smallest $\lambda \geq 1$ such that h is an integer.[a]
- That is, we find the largest integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ and then let

$$\lambda = \frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}}.$$

  - Such a λ may not exist for very small n's.
  - This is not hard to check.

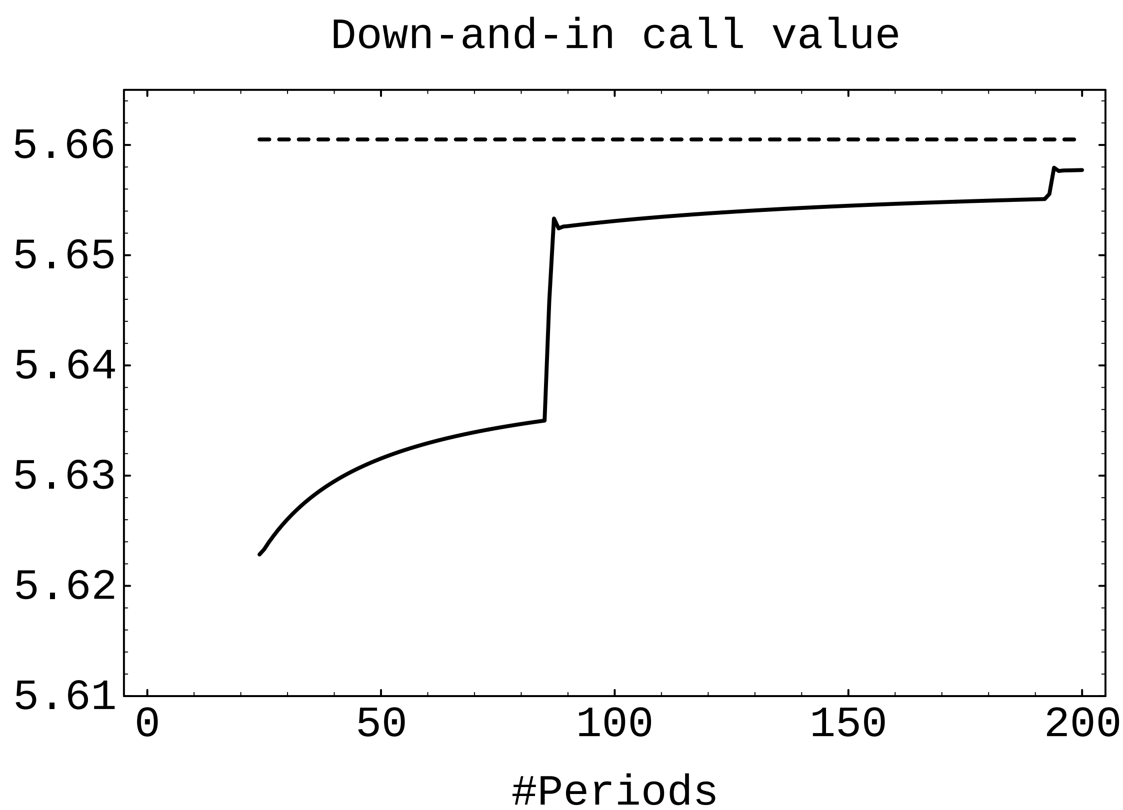
[a]Why must $\lambda \geq 1$?

Barrier Options Revisited (concluded)

- This done, one of the layers of the trinomial tree coincides with the barrier.
- The following probabilities may be used,

$$\begin{aligned} p_u &= \frac{1}{2\lambda^2} + \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}, \\ p_m &= 1 - \frac{1}{\lambda^2}, \\ p_d &= \frac{1}{2\lambda^2} - \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}. \end{aligned}$$

  – $\mu' \equiv r - \sigma^2/2$.


Down-and-in call value
5.66
5.65
5.64
5.63
5.62
5.61
0
50
100
150
200
#Periods

Algorithms Comparison[a]

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the n value at which they converge.
  - The one with the smallest n wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times, not n.

[a]Lyuu (1998).

Algorithms Comparison (continued)

- Pages 333 and 594 seem to show the trinomial model converges at a smaller n than BOPM.
- It is in this sense when people say trinomial models converge faster than binomial ones.
- But does it make the trinomial model better then?

Algorithms Comparison (concluded)

- The linear-time binomial tree algorithm actually performs better than the trinomial one.
- See the next page, expanded from p. 583.
- The barrier-too-close problem is too hard for a quadratic-time trinomial tree algorithm.[a]
- In fact, the trinomial model also has a linear-time algorithm![b]

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[a] Lyuu (1998).
[b] Chen (R94922003) (2007).

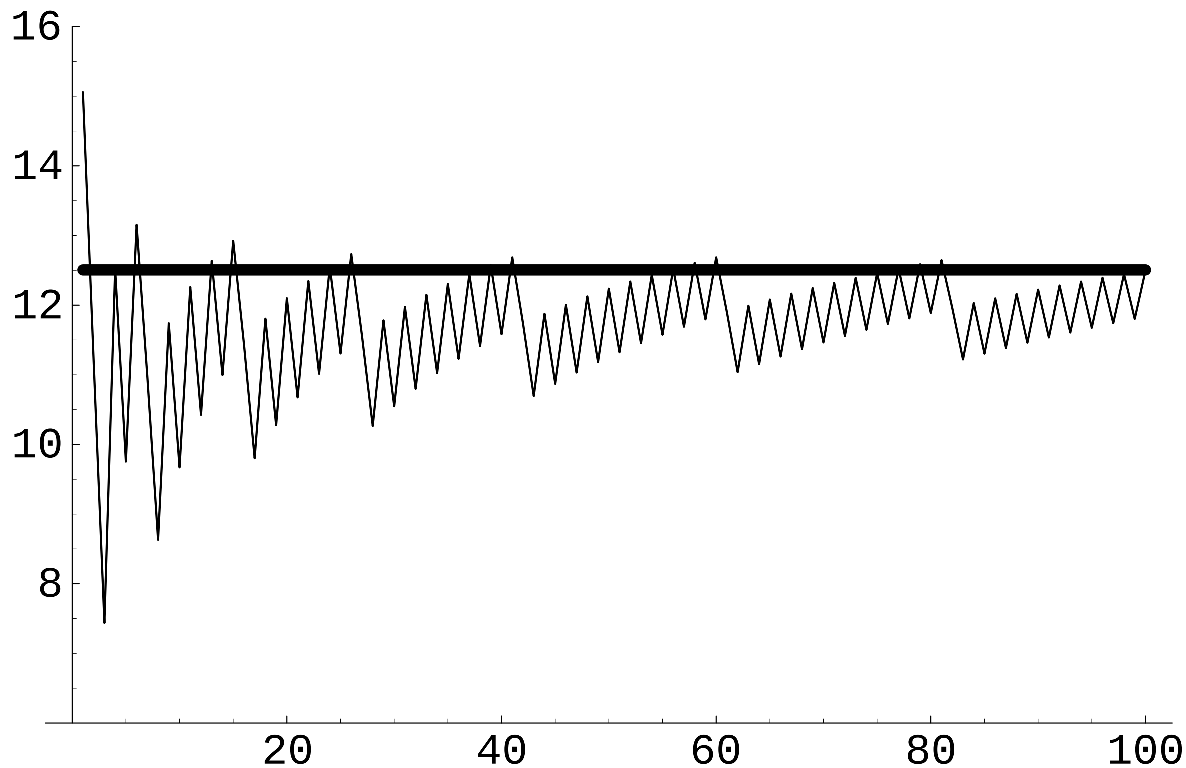
n	Combinatorial method		Trinomial tree algorithm	
	Value	Time	Value	Time
21	5.507548	0.30		
84	5.597597	0.90	5.634936	35.0
191	5.635415	2.00	5.655082	185.0
342	5.655812	3.60	5.658590	590.0
533	5.652253	5.60	5.659692	1440.0
768	5.654609	8.00	5.660137	3080.0
1047	5.658622	11.10	5.660338	5700.0
1368	5.659711	15.00	5.660432	9500.0
1731	5.659416	19.40	5.660474	15400.0
2138	5.660511	24.70	5.660491	23400.0
2587	5.660592	30.20	5.660493	34800.0
3078	5.660099	36.70	5.660488	48800.0
3613	5.660498	43.70	5.660478	67500.0
4190	5.660388	44.10	5.660466	92000.0
4809	5.659955	51.60	5.660454	130000.0
5472	5.660122	68.70		
6177	5.659981	76.70		

(All times in milliseconds.)

Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
- Assume $L < S < H$.
- The binomial model produces oscillating option values (see plot on next page).[a]
- The combinatorial method gives a linear-time algorithm (see text).

[a] Chao (R86526053) (1999); Dai (R86526008, D8852600) and Lyuu (2005).


16
14
12
10
8
20
40
60
80
100

Double-Barrier Knock-Out Options

- We knew how to pick the λ so that one of the layers of the trinomial tree coincides with one barrier, say H.
- This choice, however, does not guarantee that the other barrier, L, is also hit.
- One way to handle this problem is to lower the layer of the tree just above L to coincide with L.[a]
  - More general ways to make the trinomial model hit both barriers are available.[b]

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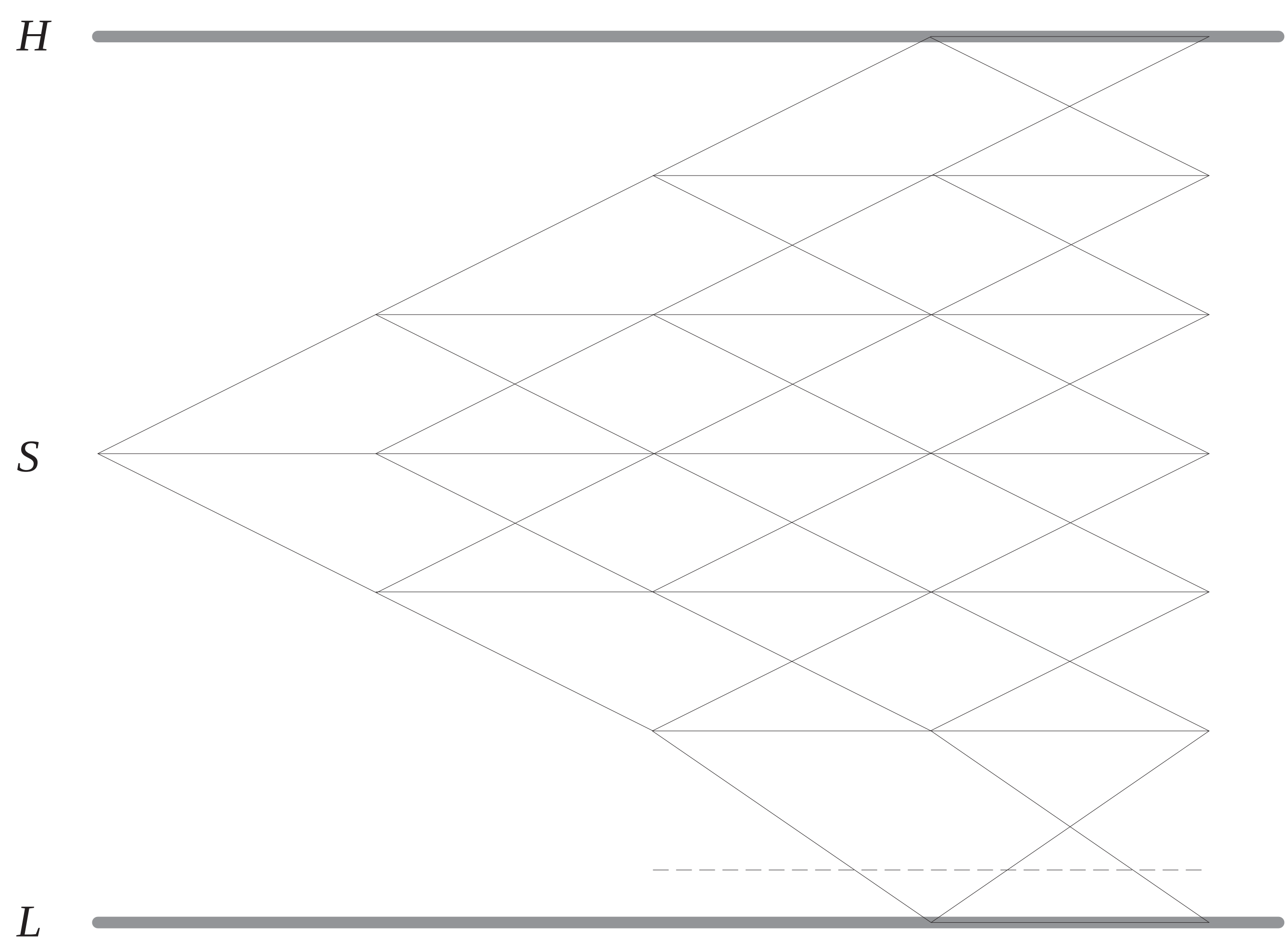
[a]Ritchken (1995).

[b]Hsu (R7526001) and Lyuu (2006). Dai (R86526008, D8852600) and Lyuu (2006) combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options (see pp. 606ff).



H
S
L

Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above L must be adjusted.
- Let ℓ be the positive integer such that

$$Sd^{\ell+1} < L < Sd^{\ell}.$$

- Hence the layer of the tree just above L has price Sd^{ℓ}.[a]

[a] You probably can do the same thing for binomial models. But the benefits are most likely nil (why?). Thanks to a lively discussion on April 25, 2012.

Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying

$$L = Sd^{\ell-1} e^{-\gamma\lambda\sigma\sqrt{\Delta t}}.$$

  - The prices between the barriers are

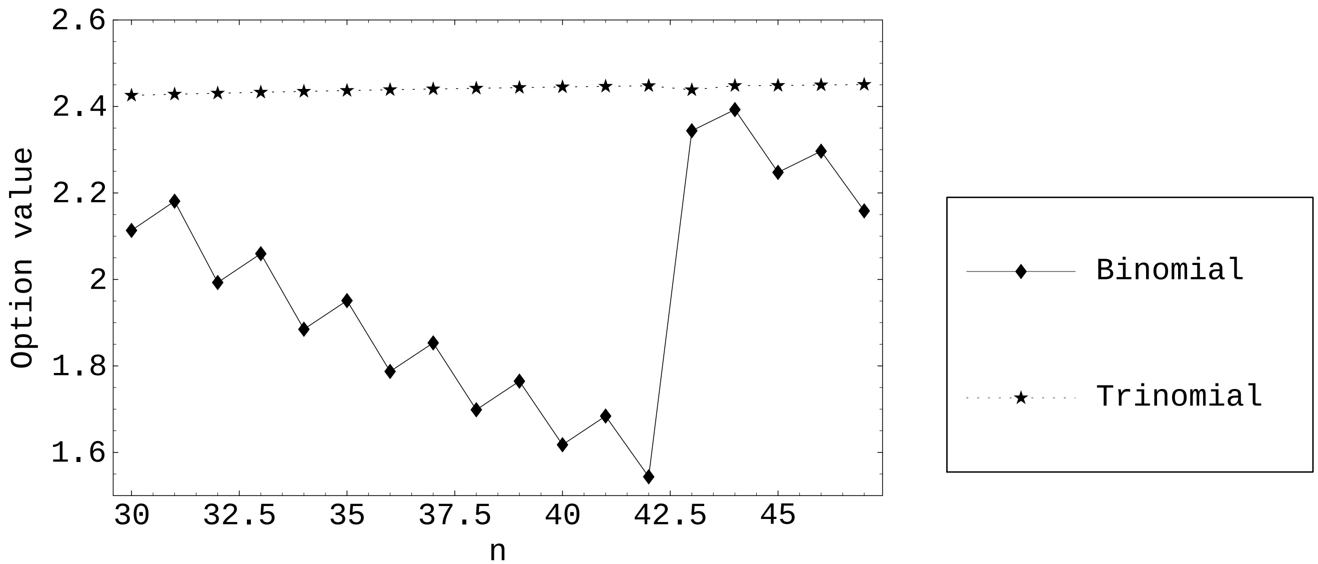
$$L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H.$$

- The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are

$$p_u' = \frac{b + a\gamma}{1+\gamma}, \quad p_d' = \frac{b-a}{\gamma+\gamma^2}, \quad \text{and} \quad p_m' = 1 - p_u' - p_d',$$

where $a \equiv \mu'\sqrt{\Delta t}/(\lambda\sigma)$ and $b \equiv 1/\lambda^2$.

Convergence: Binomial vs. Trinomial



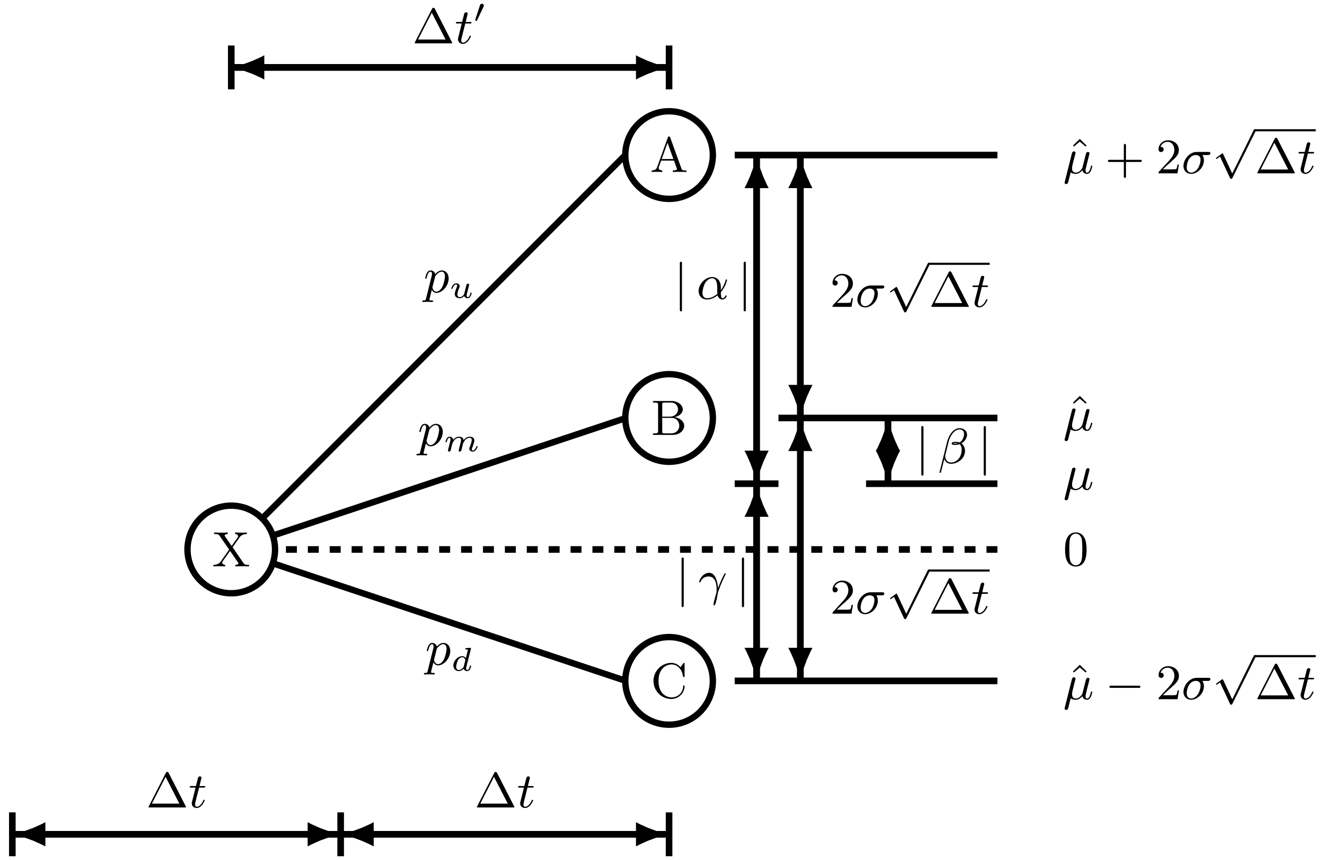
The Binomial-Trinomial Tree

- Append a trinomial structure to a binomial tree can lead to improved convergence and efficiency.[a]
- The resulting tree is called the binomial-trinomial tree.[b]
- Suppose a binomial tree will be built with Δt as the duration of one period.
- Node X at time t needs to pick three nodes on the binomial tree at time $t + \Delta t'$ as its successor nodes.
  - $\Delta t \leq \Delta t' < 2\Delta t$.

[a] Dai (R86526008, D8852600) and Lyuu (2006, 2008, 2010).

[b] The idea first emerged in a hotel in Muroran, Hokkaido, Japan, in May of 2005.

The Binomial-Trinomial Tree (continued)



The Binomial-Trinomial Tree (continued)

- These three nodes should guarantee:
  1. The mean and variance of the stock price are matched.
  2. The branching probabilities are between 0 and 1.
- Let S be the stock price at node X.
- Use $s(z)$ to denote the stock price at node z.

The Binomial-Trinomial Tree (continued)

- Recall (p. 255, e.g.) that the expected value of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ at time $t+\Delta t'$ equals
$$\mu \equiv \left(r - \sigma^2/2\right) \Delta t'. \tag{62}$$
- Its variance equals
$$\text{Var} \equiv \sigma^2 \Delta t'. \tag{63}$$
- Let node B be the node whose logarithmic return $\hat{\mu} \equiv \ln(s(\text{B})/S)$ is closest to μ among all the nodes on the binomial tree at time $t+\Delta t'$.

The Binomial-Trinomial Tree (continued)

- The middle branch from node X will end at node B.
- The two nodes A and C, which bracket node B, are the destinations of the other two branches from node X.
- Recall that adjacent nodes on the binomial tree are spaced at $2\sigma\sqrt{\Delta t}$ apart.
- See the figure on p. 607 for illustration.

The Binomial-Trinomial Tree (continued)

- The three branching probabilities from node X are obtained through matching the mean and variance of the logarithmic return $\ln(S_{t+\Delta t'}/S)$.
- Let $\hat{\mu} \equiv \ln(s(B)/S)$ be the logarithmic return of the middle node B.
- Also, let α, β, and γ be the differences between μ and the logarithmic returns $\ln(s(\mathrm{Z})/S)$ of nodes $\mathrm{Z} = \mathrm{A}, \mathrm{B}, \mathrm{C}$, in that order.

The Binomial-Trinomial Tree (continued)

- In other words,

$$\alpha \equiv \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t}\,, \tag{64}$$

$$\beta \equiv \hat{\mu} - \mu, \tag{65}$$

$$\gamma \equiv \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t}\,. \tag{66}$$

- The three branching probabilities p_u, p_m, p_d then satisfy

$$p_u\alpha + p_m\beta + p_d\gamma = 0, \tag{67}$$

$$p_u\alpha^2 + p_m\beta^2 + p_d\gamma^2 = \text{Var}, \tag{68}$$

$$p_u + p_m + p_d = 1. \tag{69}$$

The Binomial-Trinomial Tree (concluded)

- Equation (67) matches the mean (62) of the logarithmic return $\ln(S_{t+\Delta t'}/S)$ on p. 609.
- Equation (68) matches its variance (63) on p. 609.
- The three probabilities can be proved to lie between 0 and 1.

Pricing Double-Barrier Options

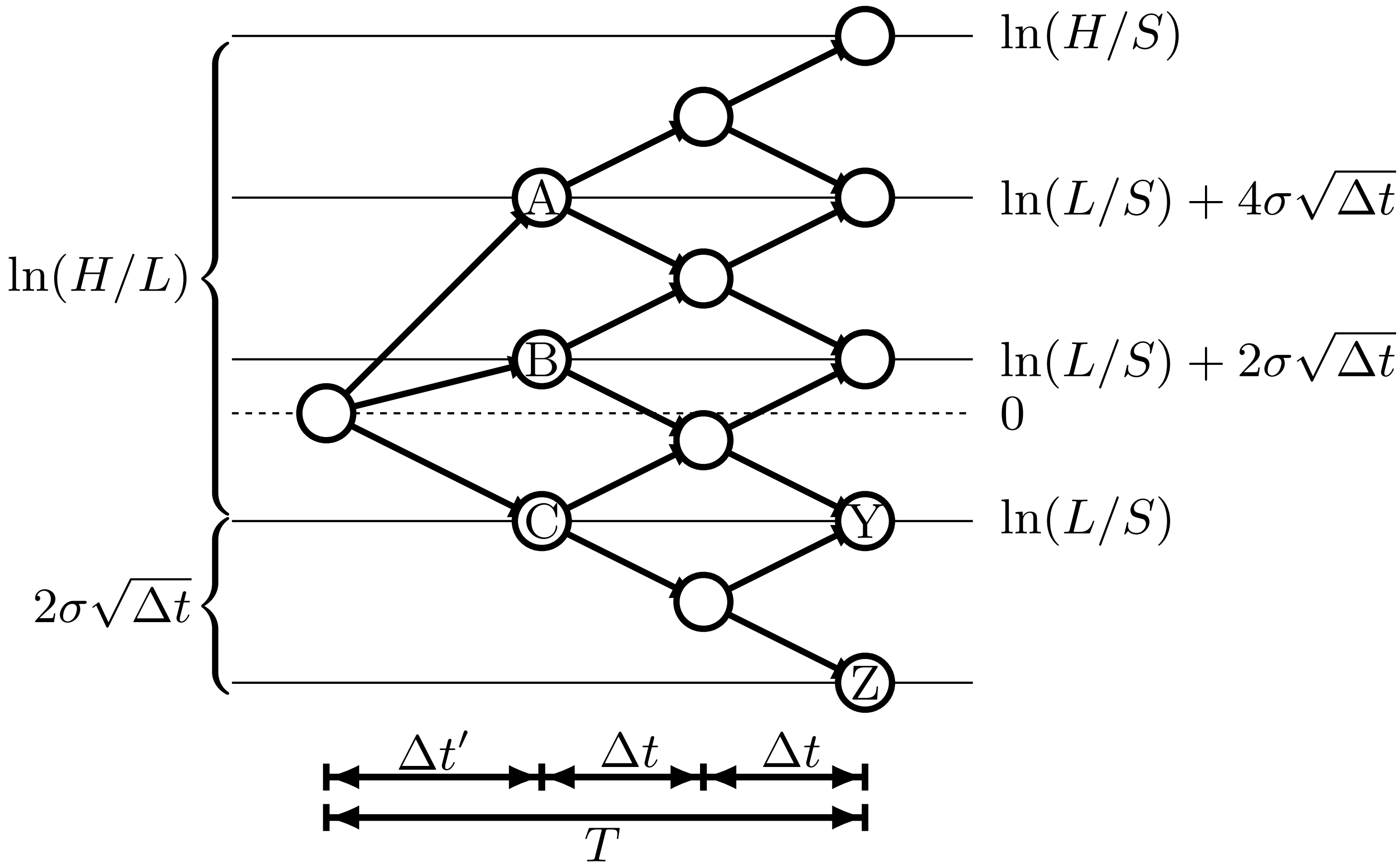
- Consider a double-barrier option with two barriers L and H, where $L < S < H$.
- We need to make each barrier coincide with a layer of the binomial tree for better convergence.
- This means choosing a Δt such that

$$\kappa \equiv \frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}$$

is a positive integer.

  – The distance between two adjacent nodes such as nodes Y and Z in the figure on p. 615 is $2\sigma\sqrt{\Delta t}$.

Pricing Double-Barrier Options (continued)



Pricing Double-Barrier Options (continued)

- Suppose that the goal is a tree with $\sim m$ periods.
- Suppose we pick $\Delta\tau \equiv T/m$ for the length of each period.
- There is no guarantee that $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}}$ is an integer.
- So we pick a Δt that is close to, but does not exceed, $\Delta\tau$ and makes $\frac{\ln(H/L)}{2\sigma\sqrt{\Delta t}}$ an integer.
- Specifically, we select

$$\Delta t = \left(\frac{\ln(H/L)}{2\kappa\sigma} \right)^2,$$

where $\kappa = \left\lceil \frac{\ln(H/L)}{2\sigma\sqrt{\Delta\tau}} \right\rceil$.

Pricing Double-Barrier Options (continued)

- We now proceed to build the binomial-trinomial tree.
- Start with the binomial part.
- Lay out the nodes from the low barrier L upward and downward.
- Automatically, a layer coincides with the high barrier H.
- It is unlikely that Δt divides T, however.
- As a consequence, the position at time 0 and with logarithmic return $\ln(S/S) = 0$ is not occupied by a binomial node to serve as the root node.

Pricing Double-Barrier Options (continued)

- The binomial-trinomial structure can address this problem as follows.
- Between time 0 and time T, the binomial tree spans $T/\Delta t$ periods.
- Keep only the last $\lfloor T/\Delta t \rfloor - 1$ periods and let the first period have a duration equal to

$$\Delta t' = T - \left(\left\lfloor \frac{T}{\Delta t} \right\rfloor - 1 \right) \Delta t.$$

- Then these $\lfloor T/\Delta t \rfloor$ periods span T years.
- It is easy to verify that $\Delta t \leq \Delta t' < 2\Delta t$.

Pricing Double-Barrier Options (continued)

- Start with the root node at time 0 and at a price with logarithmic return $\ln(S/S) = 0$.
- Find the three nodes on the binomial tree at time $\Delta t'$ as described earlier.
- Calculate the three branching probabilities to them.
- Grow the binomial tree from these three nodes until time T to obtain a binomial-trinomial tree with $\lfloor T/\Delta t \rfloor$ periods.
- See the figure on p. 615 for illustration.

Pricing Double-Barrier Options (continued)

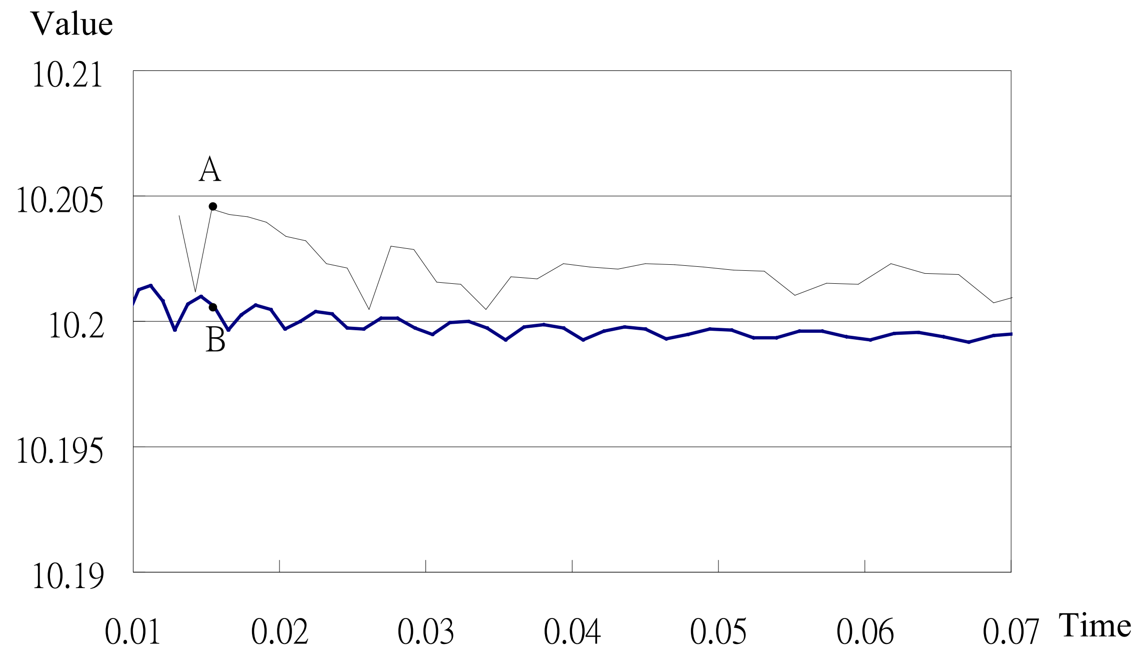
- Now the binomial-trinomial tree can be used to price double-barrier options by backward induction.
- That takes quadratic time.
- But we know a linear-time algorithm exists for double-barrier options on the binomial tree (see text).
- Apply that algorithm to price the double-barrier option's prices at the three nodes at time $\Delta t'$.
  - That is, nodes A, B, and C on p. 615.
- Then calculate their expected discounted value for the root node.
- The overall running time is only linear.

Pricing Double-Barrier Options (continued)

- Binomial trees have troubles with pricing barrier options (see p. 333 and p. 605).
- Even pit against the much better trinomial tree, the binomial-trinomial tree converges faster and smoother (see p. 622).
- In fact, the binomial-trinomial tree has an error of $O(1/n)$ for single-barrier options.[a]

[a] Lyuu and Palmer (2010).

Pricing Double-Barrier Options (concluded)



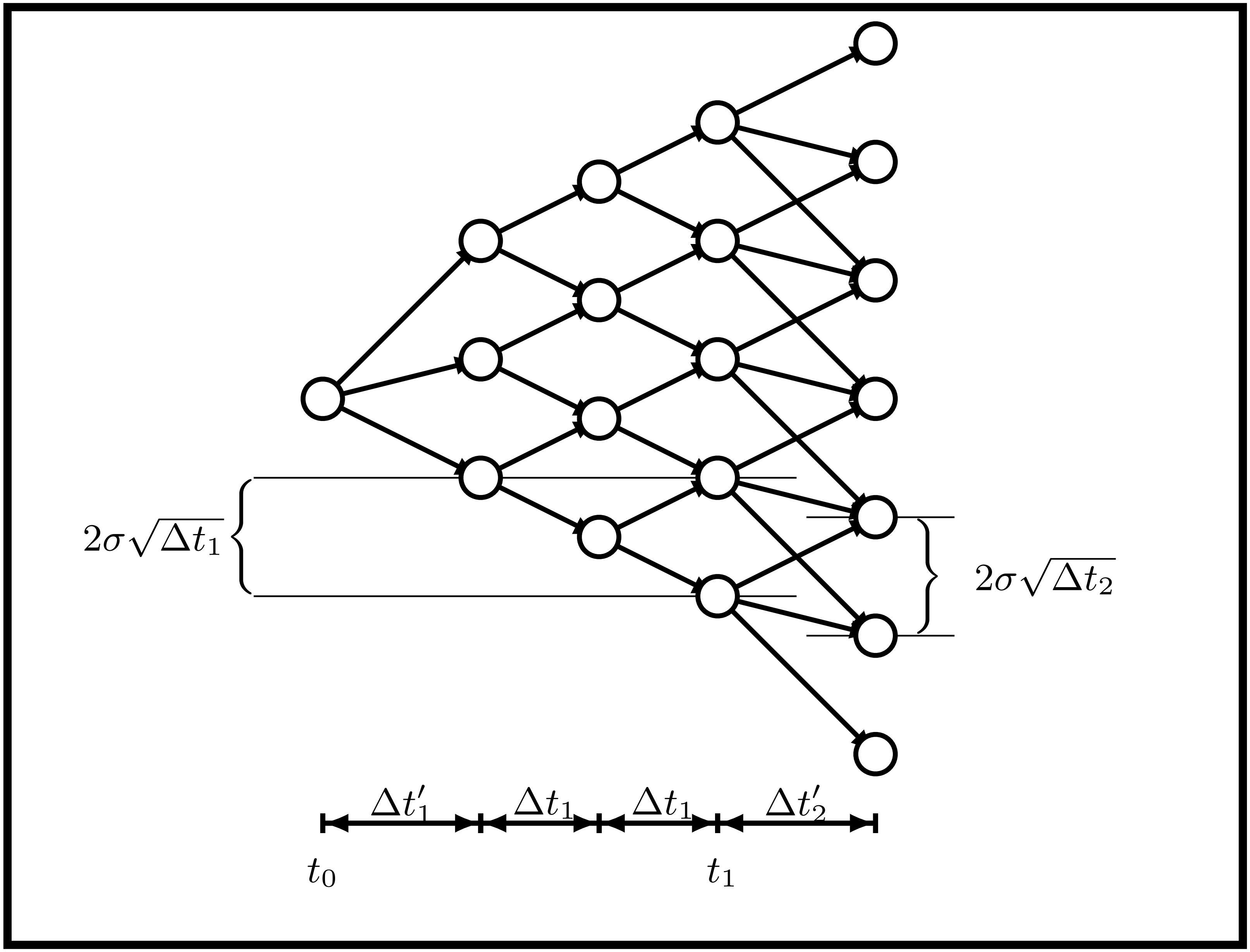
The thin line denotes the double-barrier option prices computed by the trinomial tree against the running time in seconds (such as point A). The thick line denotes those computed by the binomial-trinomial tree (such as point B).

Pricing Discrete Barrier Options

- Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.
- They are more common than the continuously monitored versions.
- The main difficulty with pricing discrete barrier options lies in matching the monitored times.
- Here is why.
- Suppose each period has a duration of Δt and the $\ell > 1$ monitored times are $t_0 = 0, t_1, t_2, \dots, t_\ell = T$.

Pricing Discrete Barrier Options (continued)

- It is unlikely that *all* monitored times coincide with the end of a period on the tree, meaning Δt divides t_i for all i.
- The binomial-trinomial tree can handle discrete options with ease, however.
- Simply build a binomial-trinomial tree from time 0 to time t_1, followed by one from time t_1 to time t_2, and so on until time t_ℓ.
- See p. 625.



$2\sigma\sqrt{\Delta t_1}$
$2\sigma\sqrt{\Delta t_2}$
$\Delta t_1'$
Δt_1
Δt_1
$\Delta t_2'$
t_0
t_1

Pricing Discrete Barrier Options (concluded)

- This procedure works even if each t_i is associated with a distinct barrier or if each window $[\,t_i, t_{i+1})$ has its own continuously monitored barrier or double barriers.

Options on a Stock That Pays Known Dividends

- Many ad hoc assumptions have been postulated for option pricing with known dividends.[a]
  1. The one we saw earlier models the stock price minus the present value of the anticipated dividends as following geometric Brownian motion.
  2. One can also model the stock price plus the forward values of the dividends as following geometric Brownian motion.

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[a] Frishling (2002).

Options on a Stock That Pays Known Dividends (continued)

- The most realistic model assumes the stock price decreases by the amount of the dividend paid at the ex-dividend date.
- The stock price follows geometric Brownian motion between adjacent ex-dividend dates.
- But this model results in binomial trees that grow exponentially (recall p. 269).
- The binomial-trinomial tree can often avoid the exponential explosion for the known-dividends case.

Options on a Stock That Pays Known Dividends (continued)

- Suppose that the known dividend is D dollars and the ex-dividend date is at time t.
- So there are $m \equiv t/\Delta t$ periods between time 0 and the ex-dividend date.
- To avoid negative stock prices, we need to make sure the lowest stock price at time t is at least D, i.e., $Se^{-(t/\Delta t)\sigma\sqrt{\Delta t}} \geq D$.
  - Equivalently,

$$\Delta t \geq \left[\frac{t\sigma}{\ln(S/D)} \right]^2.$$

Options on a Stock That Pays Known Dividends (continued)

- Build a binomial tree from time 0 to time t as before.
- Subtract D from all the stock prices on the tree at time t to represent the price drop on the ex-dividend date.
- Assume the top node's price equals S'.
  - As usual, its two successor nodes will have prices $S'u$ and $S'u^{-1}$.
- The remaining nodes' successor nodes will have prices

$$S'u^{-3}, S'u^{-5}, S'u^{-7}, \dots ,$$

same as the binomial tree.

Options on a Stock That Pays Known Dividends (concluded)

- For each node at time t below the top node, we build the trinomial connection.
- Note that the binomial-trinomial structure remains valid in the special case when $\Delta t' = \Delta t$ on p. 607.
- Hence the construction can be completed.
- From time $t + \Delta t$ onward, the standard binomial tree will be used until the maturity date or the next ex-dividend date when the procedure can be repeated.
- The resulting tree is called the stair tree.[a]

[a] Dai (`R86526008`, `D8852600`) and Lyuu (2004).

Other Applications of Binomial-Trinomial Trees

- Pricing guaranteed minimum withdrawal benefits.[a]
- Option pricing with stochastic volatilities.[b]
- Efficient Parisian option pricing.[c]
- Option pricing with time-varying volatilities and time-varying barriers.[d]
- Defaultable bond pricing.[e]

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[a] Wu (R96723058) (2009).
[b] Huang (R97922073) (2010).
[c] Huang (R97922081) (2010).
[d] Chou (R97944012) (2010).
[e] Dai (R86526008, D8852600), Lyuu, and Wang (F95922018) (2009, 2010).