

# *Option Basics*

The shift toward options as  
the center of gravity of finance [...]  
— Merton H. Miller (1923–2000)

## Calls and Puts

- A call gives its holder the right to buy a number of the underlying asset by paying a strike price.
- A put gives its holder the right to sell a number of the underlying asset for the strike price.
- How to price options?<sup>a</sup>

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<sup>a</sup>It can be traced to Aristotle's (384 B.C.–322 B.C.) *Politics*, if not earlier.

## Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- An option can be exercised prior to the expiration date: early exercise.

## American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option.

## Convenient Conventions

- $C$ : call value.
- $P$ : put value.
- $X$ : strike price.
- $S$ : stock price.
- $D$ : dividend.

## Payoff, Mathematically Speaking

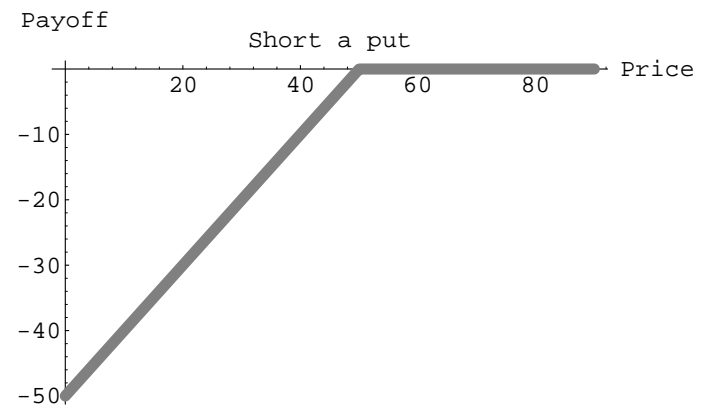
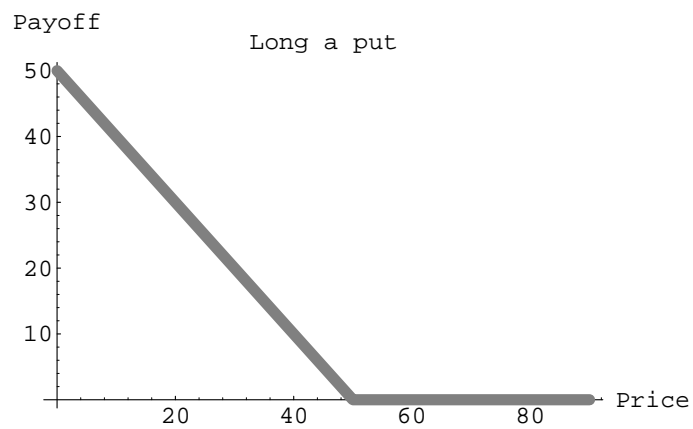
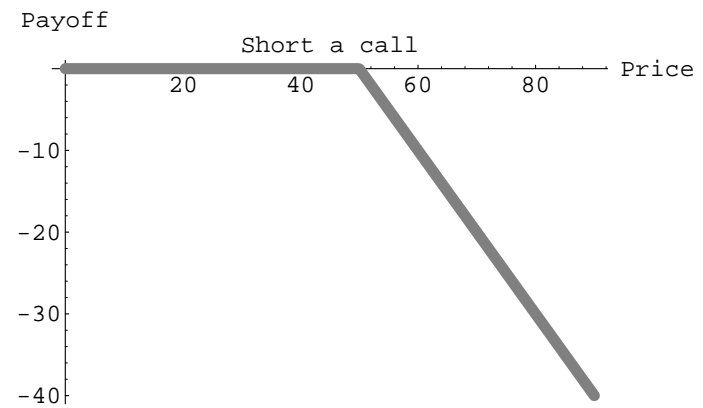
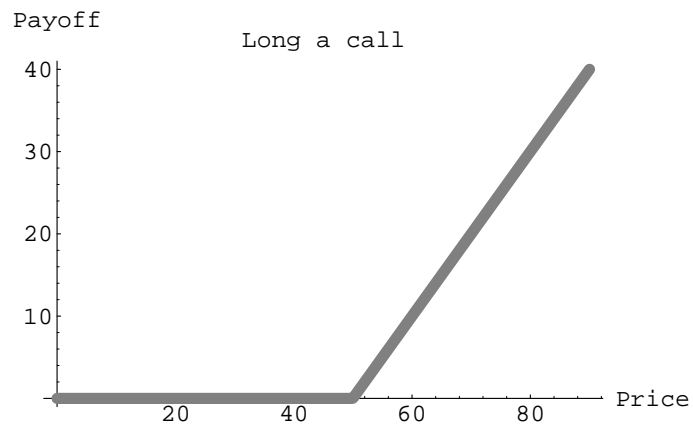
- The payoff of a call at expiration is

$$C = \max(0, S - X).$$

- The payoff of a put at expiration is

$$P = \max(0, X - S).$$

- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.





## Payoff, Mathematically Speaking (continued)

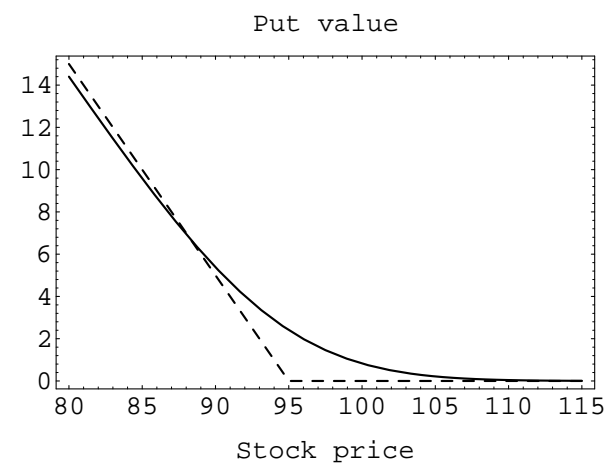
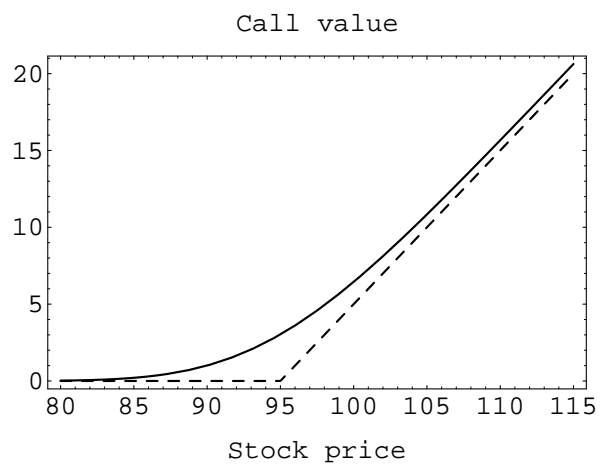
- At any time  $t$  before the expiration date, we call  $\max(0, S_t - X)$  the intrinsic value of a call.
- At any time  $t$  before the expiration date, we call  $\max(0, X - S_t)$  the intrinsic value of a put.

## Payoff, Mathematically Speaking (concluded)

- A call is in the money if  $S > X$ , at the money if  $S = X$ , and out of the money if  $S < X$ .
- A put is in the money if  $S < X$ , at the money if  $S = X$ , and out of the money if  $S > X$ .
- Options that are in the money at expiration should be exercised.<sup>a</sup>
- Finding an option's value at any time before expiration is a major intellectual breakthrough.

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<sup>a</sup>11% of option holders let in-the-money options expire worthless.



## Cash Dividends

- Exchange-traded stock options are not cash dividend-protected (or simply protected).
  - The option contract is not adjusted for cash dividends.
- The stock price falls by an amount roughly equal to the amount of the cash dividend as it goes ex-dividend.
- Cash dividends are detrimental for calls.
- The opposite is true for puts.

## Stock Splits and Stock Dividends

- Options are adjusted for stock splits.
- After an  $n$ -for- $m$  stock split, the strike price is only  $m/n$  times its previous value, and the number of shares covered by one contract becomes  $n/m$  times its previous value.
- Exchange-traded stock options are adjusted for stock dividends.
- Options are assumed to be unprotected.

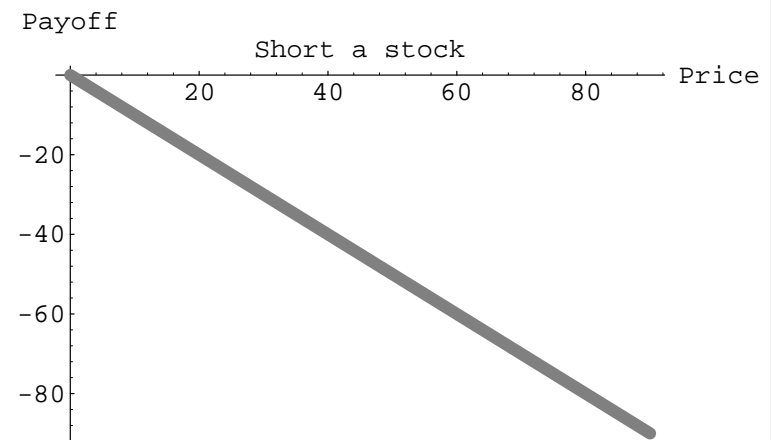
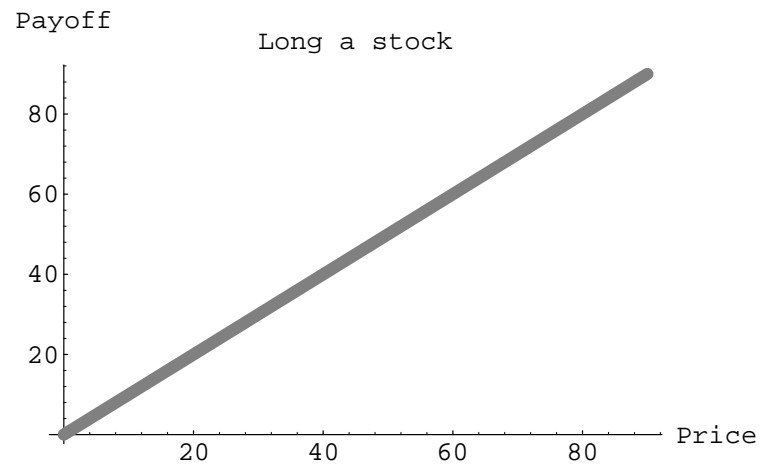
## Example

- Consider an option to buy 100 shares of a company for \$50 per share.
- A 2-for-1 split changes the term to a strike price of \$25 per share for 200 shares.

## Short Selling

- Short selling (or simply shorting) involves selling an asset that is *not* owned with the intention of buying it back later.
  - If you short 1,000 XYZ shares, the broker borrows them from another client to sell them in the market.
  - This action generates proceeds for the investor.
  - The investor can close out the short position by buying 1,000 XYZ shares.
  - Clearly, the investor profits if the stock price falls.
- Not all assets can be shorted.

## Payoff of Stock



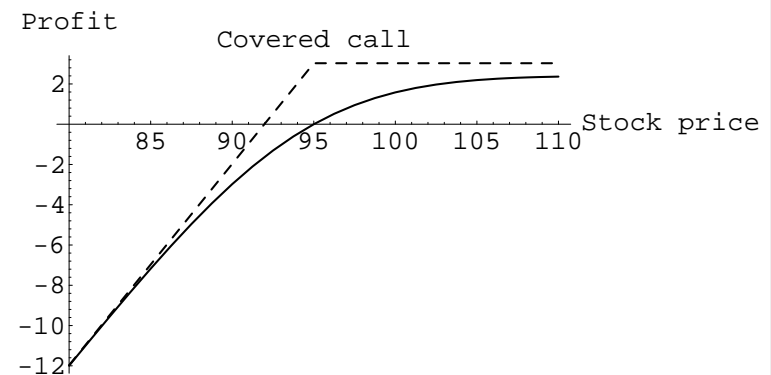
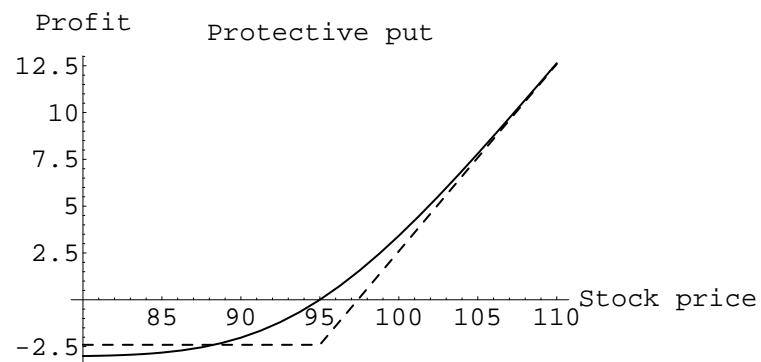


## Covered Position: Hedge

- A hedge combines an option with its underlying stock in such a way that one protects the other against loss.
- Protective put: A long position in stock with a long put.
- Covered call: A long position in stock with a short call.<sup>a</sup>
- Both strategies break even only if the stock price rises, so they are bullish.

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<sup>a</sup>A short position has a payoff opposite in sign to that of a long position.



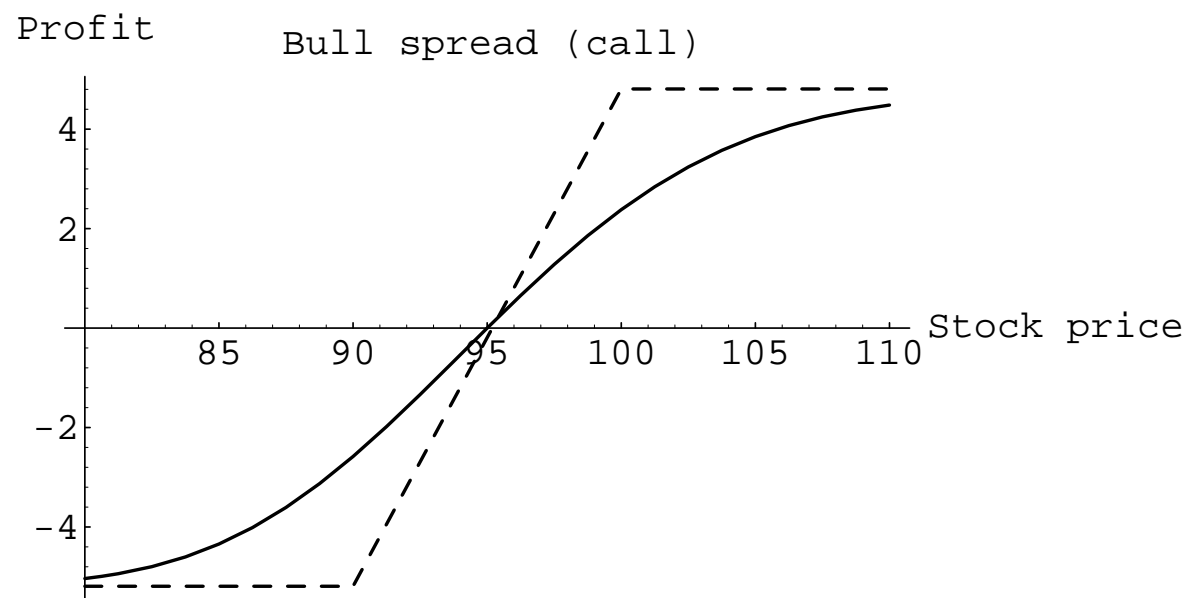
Solid lines are profits of the portfolio one month before maturity assuming the portfolio is set up when  $S = 95$  then.

## Covered Position: Spread

- A spread consists of options of the same type and on the same underlying asset but with different strike prices or expiration dates.
- We use  $X_L$ ,  $X_M$ , and  $X_H$  to denote the strike prices with  $X_L < X_M < X_H$ .

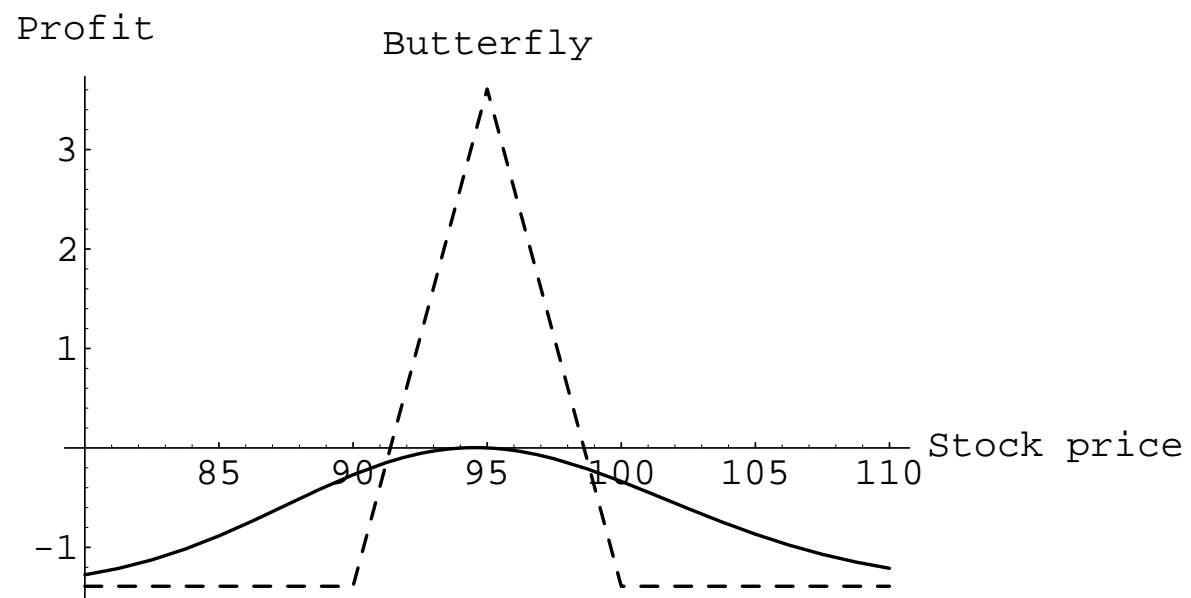
## Covered Position: Spread (continued)

- A bull call spread consists of a long  $X_L$  call and a short  $X_H$  call with the same expiration date.
  - The initial investment is  $C_L - C_H$ .
  - The maximum profit is  $(X_H - X_L) - (C_L - C_H)$ .
  - The maximum loss is  $C_L - C_H$ .



## Covered Position: Spread (continued)

- Writing an  $X_H$  put and buying an  $X_L$  put with identical expiration date creates the bull put spread.
- A bear spread amounts to selling a bull spread.
- It profits from declining stock prices.
- Three calls or three puts with different strike prices and the same expiration date create a butterfly spread.
  - The spread is long one  $X_L$  call, long one  $X_H$  call, and short two  $X_M$  calls.



## Covered Position: Spread (concluded)

- A butterfly spread pays off a positive amount at expiration only if the asset price falls between  $X_L$  and  $X_H$ .
- A butterfly spread with a small  $X_H - X_L$  approximates a state contingent claim,<sup>a</sup> which pays \$1 only when a particular price results.
- The price of a state contingent claim is called a state price.

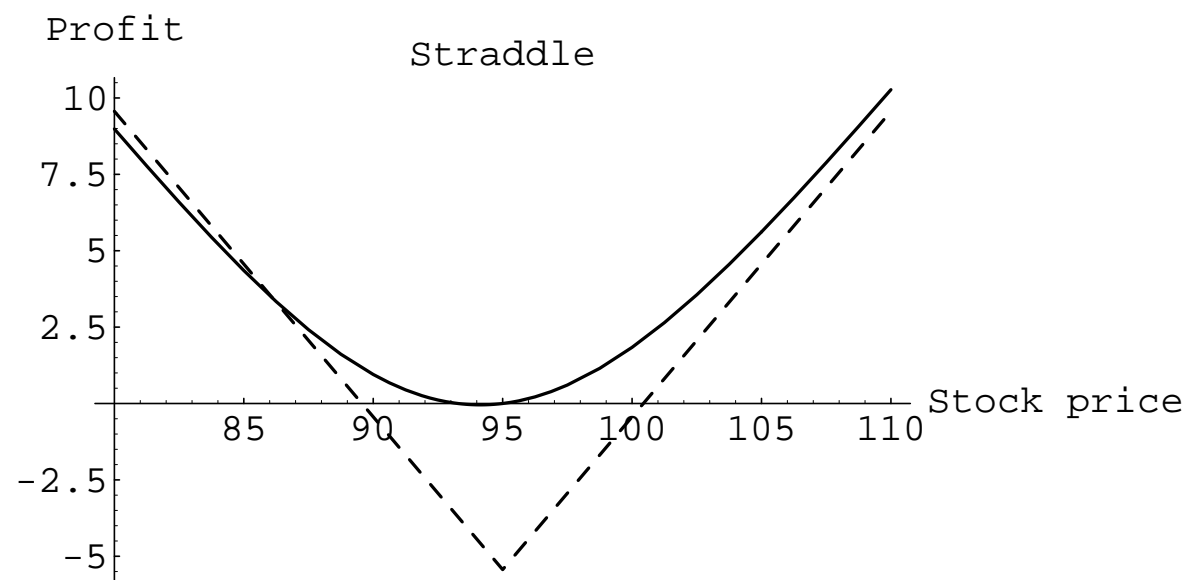
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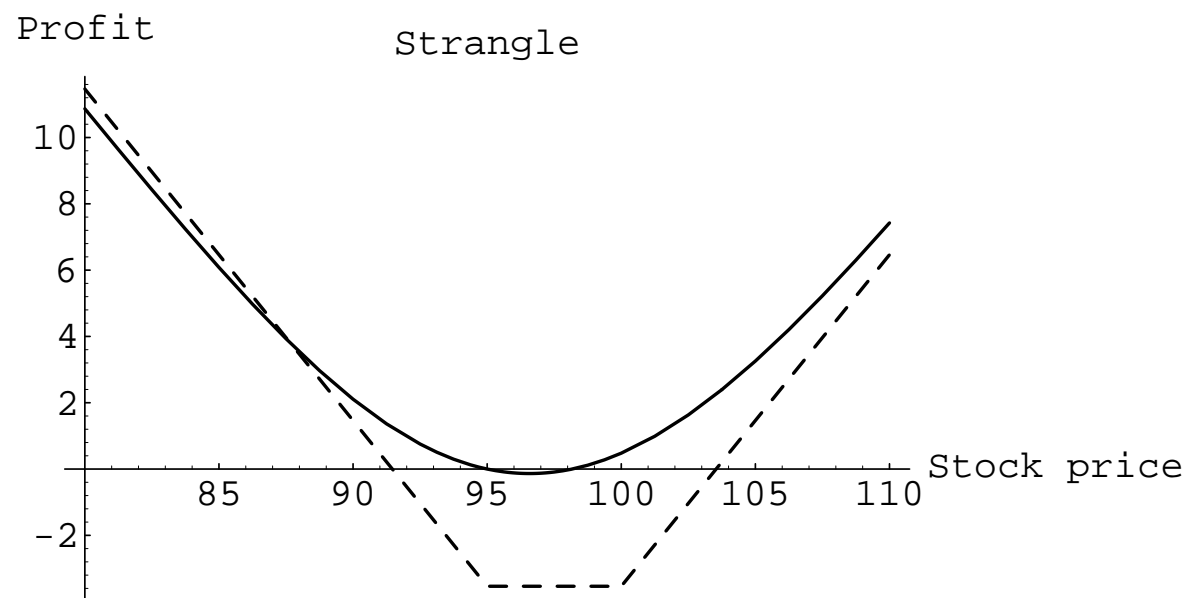
<sup>a</sup>Alternatively, Arrow security.



## Covered Position: Combination

- A combination consists of options of different types on the same underlying asset, and they are either all bought or all written.
- Straddle: A long call and a long put with the same strike price and expiration date.
  - Since it profits from high volatility, a person who buys a straddle is said to be long volatility.
  - Selling a straddle benefits from low volatility.
- Strangle: Identical to a straddle except that the call's strike price is higher than the put's.





# *Arbitrage in Option Pricing*

All general laws are attended with inconveniences,  
when applied to particular cases.  
— David Hume (1711–1776)

## Arbitrage

- The no-arbitrage principle says there is no free lunch.
- It supplies the argument for option pricing.
- A riskless arbitrage opportunity is one that, without any initial investment, generates nonnegative returns under all circumstances and positive returns under some.
- In an efficient market, such opportunities do not exist (for long).
- The portfolio dominance principle: Portfolio A should be more valuable than B if A's payoff is at least as good under all circumstances and better under some.

## A Corollary

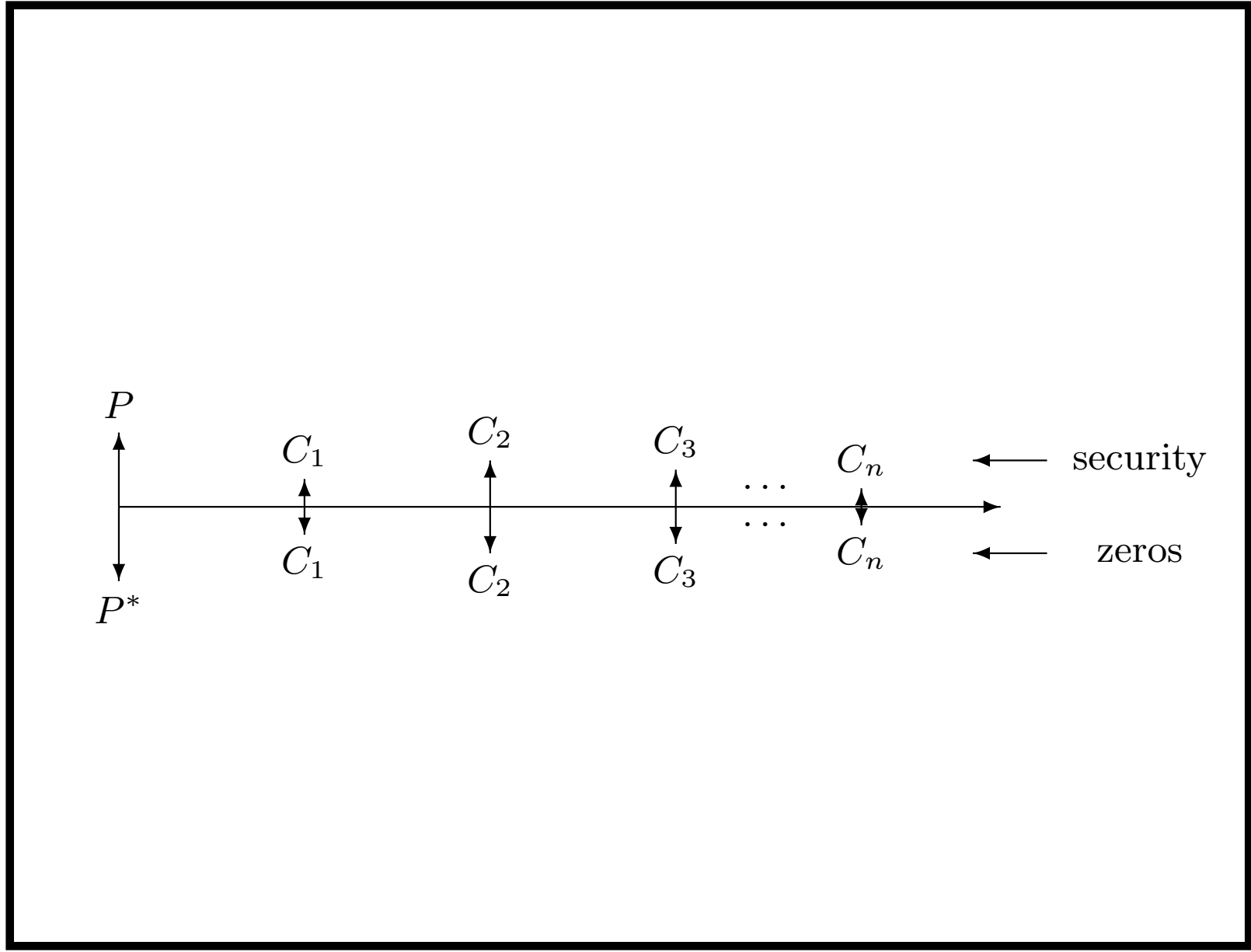
- A portfolio yielding a zero return in every possible scenario must have a zero PV.
  - Short the portfolio if its PV is positive.
  - Buy it if its PV is negative.
  - In both cases, a free lunch is created.

## The PV Formula Justified

**Theorem 1**  $P = \sum_{i=1}^n C_i d(i)$  for a certain cash flow  $C_1, C_2, \dots, C_n$ .

- Suppose the price  $P^* < P$ .
- Short the zeros that match the security's  $n$  cash flows.
- The proceeds are  $P$  dollars.
- Then use  $P^*$  of the proceeds to buy the security.
- The cash inflows of the security will offset exactly the obligations of the zeros.
- A riskless profit of  $P - P^*$  dollars has been realized now.
- If  $P^* > P$ , just reverse the trades.





## Two More Examples

- An American option cannot be worth less than the intrinsic value.
  - Suppose the opposite is true.
  - Now, buy the option, promptly exercise it, and close the stock position.
  - The cost of buying the option is less than the payoff, which is the intrinsic value.<sup>a</sup>
  - So there is an arbitrage profit.

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<sup>a</sup> $\max(0, S_t - X)$  or  $\max(0, X - S_t)$ .

## Two More Examples (concluded)

- A put or a call must have a nonnegative value.
  - Otherwise, one can buy it for a positive cash flow now and end up with a nonnegative amount at expiration.

## Relative Option Prices

- These relations hold regardless of the model for stock prices.
- Assume, among other things, that there are no transactions costs or margin requirements, borrowing and lending are available at the riskless interest rate, interest rates are nonnegative, and there are no arbitrage opportunities.
- Let the current time be time zero.
- $PV(x)$  stands for the PV of  $x$  dollars at expiration.
- Hence  $PV(x) = xd(\tau)$  where  $\tau$  is the time to expiration.

## Put-Call Parity (Castelli, 1877)

$$C = P + S - PV(X). \quad (20)$$

- Consider the portfolio of one short European call, one long European put, one share of stock, and a loan of  $PV(X)$ .
- All options are assumed to carry the same strike price  $X$  and time to expiration,  $\tau$ .
- The initial cash flow is therefore

$$C - P - S + PV(X).$$

## The Proof (continued)

- At expiration, if the stock price  $S_\tau \leq X$ , the put will be worth  $X - S_\tau$  and the call will expire worthless.
- After the loan, now  $X$ , is repaid, the net future cash flow is zero:

$$0 + (X - S_\tau) + S_\tau - X = 0.$$

- On the other hand, if  $S_\tau > X$ , the call will be worth  $S_\tau - X$  and the put will expire worthless.
- After the loan, now  $X$ , is repaid, the net future cash flow is again zero:

$$-(S_\tau - X) + 0 + S_\tau - X = 0.$$

## The Proof (concluded)

- The net future cash flow is zero in either case.
- The no-arbitrage principle implies that the initial investment to set up the portfolio must be nil as well.

## Consequences of Put-Call Parity

- There is only one kind of European option.
  - The other can be replicated from it in combination with stock and riskless lending or borrowing.
  - Combinations such as this create synthetic securities.
- $S = C - P + PV(X)$  says a stock is equivalent to a portfolio containing a long call, a short put, and lending  $PV(X)$ .
- $C - P = S - PV(X)$  implies a long call and a short put amount to a long position in stock and borrowing the PV of the strike price (buying stock on margin).



## Intrinsic Value

**Lemma 2** *An American call or a European call on a non-dividend-paying stock is never worth less than its intrinsic value.*

- The put-call parity implies
$$C = (S - X) + (X - \text{PV}(X)) + P \geq S - X.$$
- Recall  $C \geq 0$ .
- It follows that  $C \geq \max(S - X, 0)$ , the intrinsic value.
- An American call also cannot be worth less than its intrinsic value (p. 186).

## Intrinsic Value (concluded)

A European put on a non-dividend-paying stock may be worth less than its intrinsic value (p. 163).

**Lemma 3** *For European puts,  $P \geq \max(\text{PV}(X) - S, 0)$ .*

- Prove it with the put-call parity.
- Can explain the right figure on p. 163 why  $P < X - S$  when  $S$  is small.

## Early Exercise of American Calls

European calls and American calls are identical when the underlying stock pays no dividends.

**Theorem 4 (Merton (1973))** *An American call on a non-dividend-paying stock should not be exercised before expiration.*

- By an exercise in text,  $C \geq \max(S - PV(X), 0)$ .
- If the call is exercised, the value is the smaller  $S - X$ .

## Remarks

- The above theorem does not mean American calls should be kept until maturity.
- What it does imply is that when early exercise is being considered, a *better* alternative is to sell it.
- Early exercise may become optimal for American calls on a dividend-paying stock.
  - Stock price declines as the stock goes ex-dividend.

## Early Exercise of American Calls: Dividend Case

Surprisingly, an American call should be exercised only at a few dates.

**Theorem 5** *An American call will only be exercised at expiration or just before an ex-dividend date.*

In contrast, it might be optimal to exercise an American put even if the underlying stock does not pay dividends.

## A General Result

**Theorem 6 (Cox and Rubinstein (1985))** *Any piecewise linear payoff function can be replicated using a portfolio of calls and puts.*

## Convexity of Option Prices

**Lemma 7** *For three otherwise identical calls or puts with strike prices  $X_1 < X_2 < X_3$ ,*

$$C_{X_2} \leq \omega C_{X_1} + (1 - \omega) C_{X_3}$$

$$P_{X_2} \leq \omega P_{X_1} + (1 - \omega) P_{X_3}$$

*Here*

$$\omega \equiv (X_3 - X_2)/(X_3 - X_1).$$

*(Equivalently,  $X_2 = \omega X_1 + (1 - \omega) X_3$ .)*

## The Intuition behind Lemma 7<sup>a</sup>

- Consider  $\omega C_{X_1} + (1 - \omega) C_{X_3} - C_{X_2}$ .
- This is a butterfly spread (p. 174).
- It has a nonnegative value as

$$\omega \max(S - X_1, 0) + (1 - \omega) \max(S - X_3, 0) - \max(S - X_2, 0) \geq 0.$$

- Therefore,  $\omega C_{X_1} + (1 - \omega) C_{X_3} - C_{X_2} \geq 0$ .
- In the limit,  $\partial^2 C / \partial X^2 \geq 0$ .

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<sup>a</sup>Contributed by Mr. Cheng, Jen-Chieh (B96703032) on March 17, 2010.



## Option on a Portfolio vs. Portfolio of Options

- Consider a portfolio of non-dividend-paying assets with weights  $\omega_i$ .
- Let  $C_i$  denote the price of a European call on asset  $i$  with strike price  $X_i$ .
- All options expire on the same date.
- An option on a portfolio is cheaper than a portfolio of options.

**Theorem 8** *The call on the portfolio with a strike price  $X \equiv \sum_i \omega_i X_i$  has a value at most  $\sum_i \omega_i C_i$ .*

The same result holds for European puts.

# *Option Pricing Models*

If the world of sense does not fit mathematics,  
so much the worse for the world of sense.  
— Bertrand Russell (1872–1970)

Black insisted that anything one could do  
with a mouse could be done better  
with macro redefinitions  
of particular keys on the keyboard.  
— Emanuel Derman,  
*My Life as a Quant* (2004)

## The Setting

- The no-arbitrage principle is insufficient to pin down the exact option value.
- Need a model of probabilistic behavior of stock prices.
- One major obstacle is that it seems a risk-adjusted interest rate is needed to discount the option's payoff.
- Breakthrough came in 1973 when Black (1938–1995) and Scholes with help from Merton published their celebrated option pricing model.<sup>a</sup>
  - Known as the Black-Scholes option pricing model.

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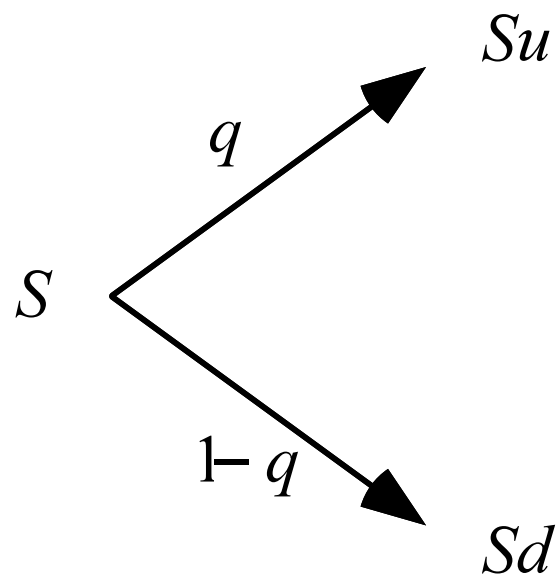
<sup>a</sup>The results were obtained as early as June 1969.

## Terms and Approach

- $C$ : call value.
- $P$ : put value.
- $X$ : strike price
- $S$ : stock price
- $\hat{r} > 0$ : the continuously compounded riskless rate per period.
- $R \equiv e^{\hat{r}}$ : gross return.
- Start from the discrete-time binomial model.

## Binomial Option Pricing Model (BOPM)

- Time is discrete and measured in periods.
- If the current stock price is  $S$ , it can go to  $Su$  with probability  $q$  and  $Sd$  with probability  $1 - q$ , where  $0 < q < 1$  and  $d < u$ .
  - In fact,  $d < R < u$  must hold to rule out arbitrage.
- Six pieces of information will suffice to determine the option value based on arbitrage considerations:  $S$ ,  $u$ ,  $d$ ,  $X$ ,  $\hat{r}$ , and the number of periods to expiration.



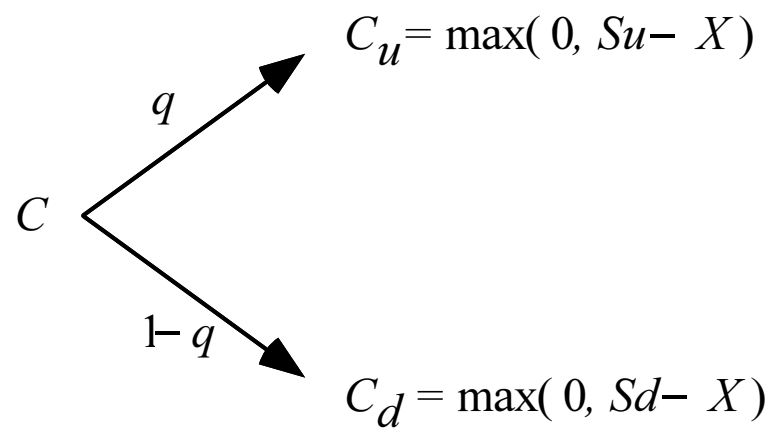
## Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- $C_u$  is the call price at time one if the stock price moves to  $Su$ .
- $C_d$  is the call price at time one if the stock price moves to  $Sd$ .
- Clearly,

$$C_u = \max(0, Su - X),$$

$$C_d = \max(0, Sd - X).$$





### Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Set up a portfolio of  $h$  shares of stock and  $B$  dollars in riskless bonds.
  - This costs  $hS + B$ .
  - We call  $h$  the hedge ratio or delta.
- The value of this portfolio at time one is either  $hSu + RB$  or  $hSd + RB$ .
- Choose  $h$  and  $B$  such that the portfolio replicates the payoff of the call,

$$hSu + RB = C_u,$$

$$hSd + RB = C_d.$$

### Call on a Non-Dividend-Paying Stock: Single Period (concluded)

- Solve the above equations to obtain

$$h = \frac{C_u - C_d}{S_u - S_d} \geq 0, \quad (21)$$

$$B = \frac{uC_d - dC_u}{(u - d)R}. \quad (22)$$

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio,<sup>a</sup>

$$C = hS + B.$$

- As  $uC_d - dC_u < 0$ , the equivalent portfolio is a levered long position in stocks.

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<sup>a</sup>Or the replicating portfolio, as it replicates the option.

## American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$ .
  - When  $hS + B \geq S - X$ , the call should not be exercised immediately.
  - When  $hS + B < S - X$ , the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 4 (p. 195).
- So  $C = hS + B$ .

## Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is  $(P_u - P_d)/(Su - Sd) \leq 0$ , where

$$P_u = \max(0, X - Su),$$

$$P_d = \max(0, X - Sd).$$

- Let  $B = \frac{uP_d - dP_u}{(u-d)R}$ .
- The European put is worth  $hS + B$ .
- The American put is worth  $\max(hS + B, X - S)$ .
  - Early exercise is always possible with American puts.

## Risk

- Surprisingly, the option value is independent of  $q$ .<sup>a</sup>
- Hence it is independent of the expected gross return of the stock,  $qSu + (1 - q)Sd$ .
- It therefore does not directly depend on investors' risk preferences.
- The option value depends on the sizes of price changes,  $u$  and  $d$ , which the investors must agree upon.
- Note that the set of possible stock prices is the same whatever  $q$  is.

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<sup>a</sup>More precisely, not directly dependent on  $q$ . Thanks to a lively class discussion on March 16, 2011.

## Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$

- Rewrite it as

$$hS + B = \frac{pC_u + (1-p)C_d}{R},$$

where

$$p \equiv \frac{R-d}{u-d}.$$

- As  $0 < p < 1$ , it may be interpreted as a probability.

## Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate  $\hat{r}$  under  $p$  as  $pSu + (1 - p)Sd = RS$ .
- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason,  $p$  is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the FV is the riskless rate in a risk-neutral economy.



## Binomial Distribution

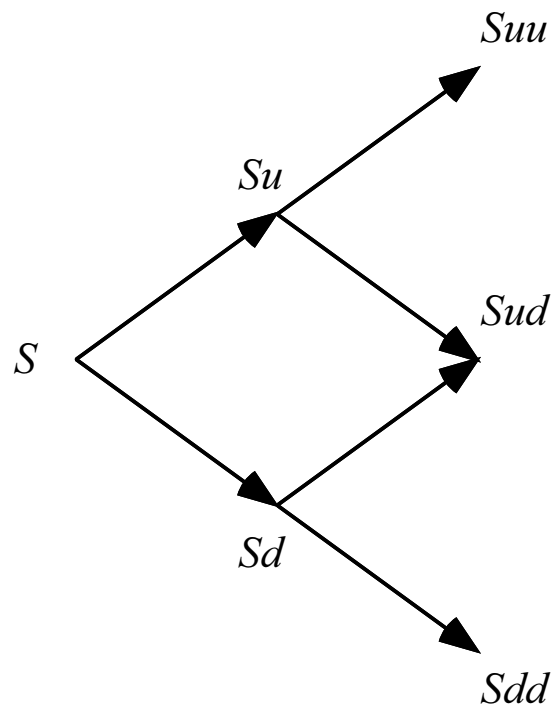
- Denote the binomial distribution with parameters  $n$  and  $p$  by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1 - p)^{n-j} = \frac{n!}{j! (n - j)!} p^j (1 - p)^{n-j}.$$

- $n! = 1 \times 2 \times \cdots \times n$ .
- Convention:  $0! = 1$ .
- Suppose you toss a coin  $n$  times with  $p$  being the probability of getting heads.
- Then  $b(j; n, p)$  is the probability of getting  $j$  heads.

## Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on three possible prices at time two:  $S_{uu}$ ,  $S_{ud}$ , and  $S_{dd}$ .
  - There are 4 paths.
  - But the tree combines.
- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.



## Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

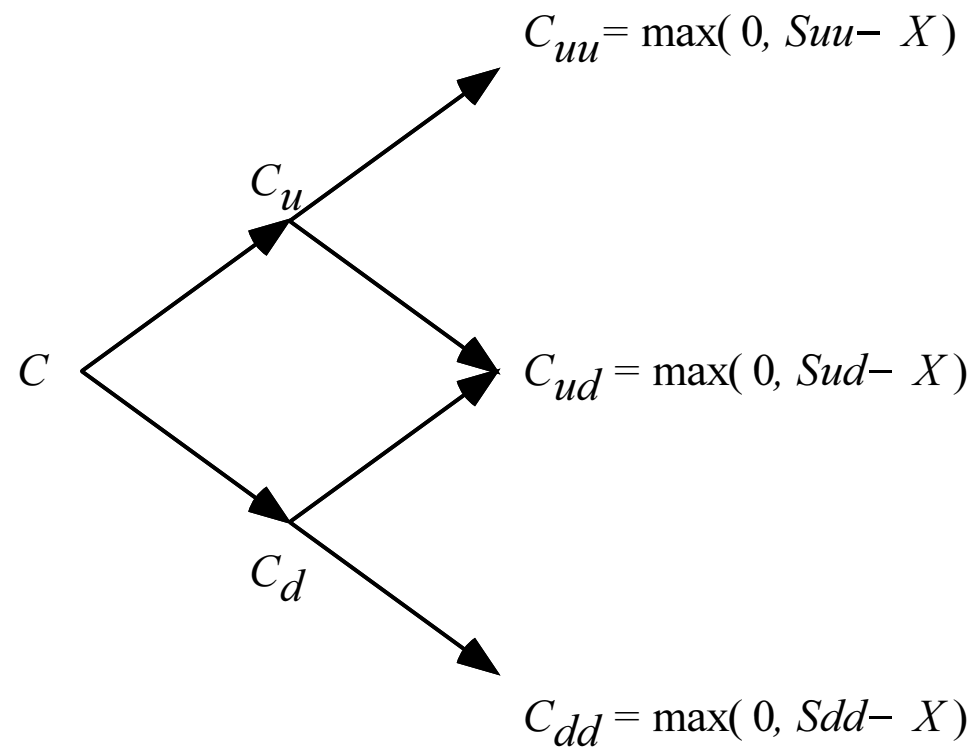
- Let  $C_{uu}$  be the call's value at time two if the stock price is  $S_{uu}$ .
- Thus,

$$C_{uu} = \max(0, S_{uu} - X).$$

- $C_{ud}$  and  $C_{dd}$  can be calculated analogously,

$$C_{ud} = \max(0, S_{ud} - X),$$

$$C_{dd} = \max(0, S_{dd} - X).$$



## Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time 1 can be obtained by applying the same logic:

$$\begin{aligned}C_u &= \frac{pC_{uu} + (1-p)C_{ud}}{R}, \\C_d &= \frac{pC_{ud} + (1-p)C_{dd}}{R}.\end{aligned}\tag{23}$$

- Deltas can be derived from Eq. (21) on p. 211.
- For example, the delta at  $C_u$  is

$$\frac{C_{uu} - C_{ud}}{S_{uu} - S_{ud}}.$$

## Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute

$$\frac{pC_u + (1 - p) C_d}{R}$$

as the option price.

- The values of delta  $h$  and  $B$  can be derived from Eqs. (21)–(22) on p. 211.

## Early Exercise

- Since the call will not be exercised at time 1 even if it is American,  $C_u \geq Su - X$  and  $C_d \geq Sd - X$ .
- Therefore,

$$\begin{aligned} hS + B &= \frac{pC_u + (1-p)C_d}{R} \geq \frac{[pu + (1-p)d]S - X}{R} \\ &= S - \frac{X}{R} > S - X. \end{aligned}$$

– The call again will not be exercised at present.<sup>a</sup>

- So

$$C = hS + B = \frac{pC_u + (1-p)C_d}{R}.$$

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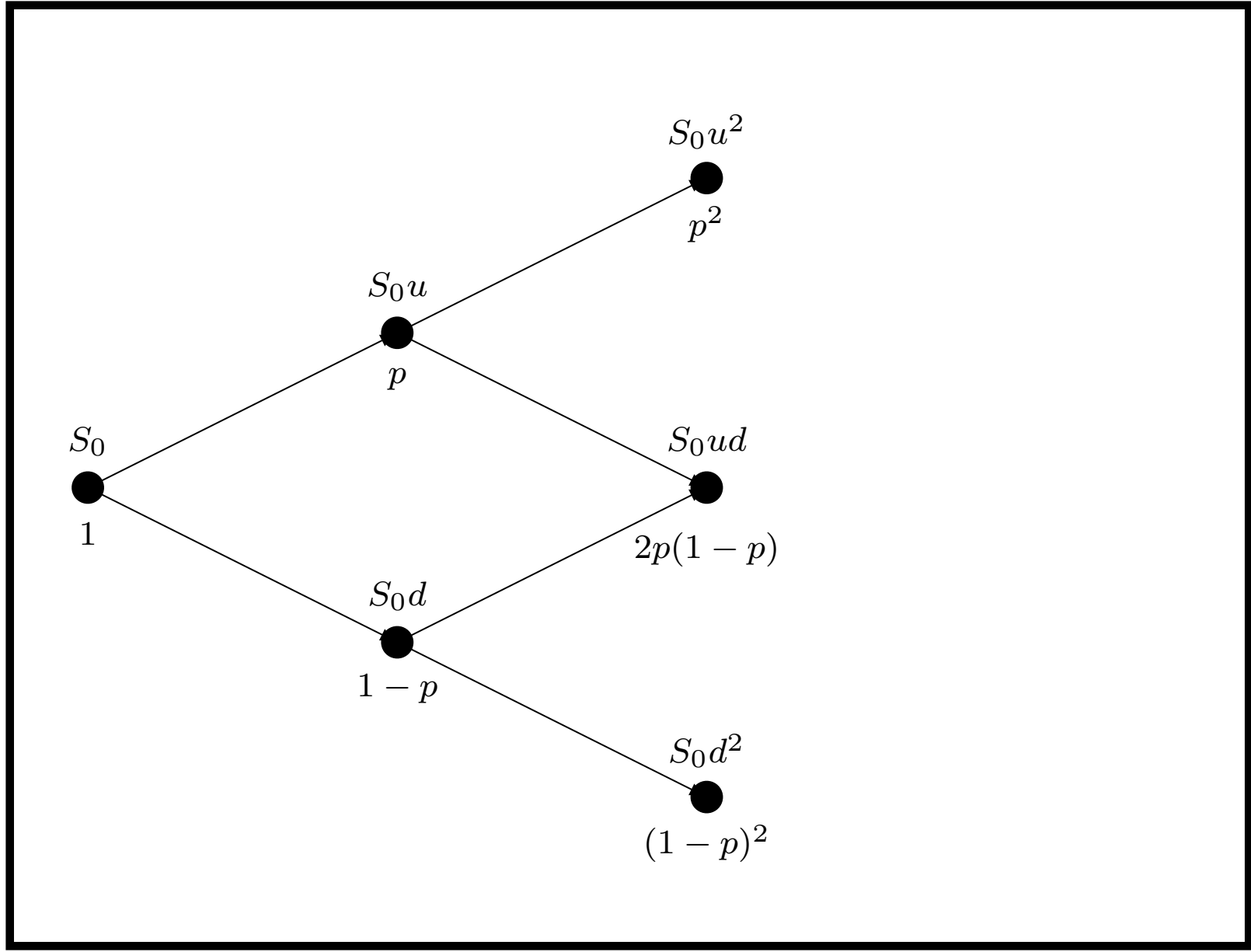
<sup>a</sup>Consistent with Theorem 4 (p. 195).



## Backward Induction of Zermelo (1871–1953)

- The above expression calculates  $C$  from the two successor nodes  $C_u$  and  $C_d$  and none beyond.
- The same computation happened at  $C_u$  and  $C_d$ , too, as demonstrated in Eq. (23) on p. 222.
- This recursive procedure is called backward induction.
- $C$  equals

$$\begin{aligned} & [p^2 C_{uu} + 2p(1-p) C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\ = & [p^2 \max(0, Su^2 - X) + 2p(1-p) \max(0, Sud - X) \\ & + (1-p)^2 \max(0, Sd^2 - X)]/R^2. \end{aligned}$$



## Backward Induction (concluded)

- In the  $n$ -period case,

$$C = \frac{\sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \times \max(0, Su^j d^{n-j} - X)}{R^n}.$$

- The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- Similarly,

$$P = \frac{\sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \times \max(0, X - Su^j d^{n-j})}{R^n}.$$

## Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function  $\mathcal{D}$ , its value is

$$e^{-\hat{r}n} E^{\pi}[\mathcal{D}].$$

- $E^{\pi}$  means the expectation is taken under the risk-neutral probability.
- The “equivalence” between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.

## Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- The maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.
  - Changes in value are due entirely to capital gains.

## Can You Figure Out $u, d$ without Knowing $q$ ?<sup>a</sup>

- Yes, you can, under BOPM.
- Let us observe the time series of past stock prices, e.g.,

$$\begin{array}{c} u \text{ is available} \\ \overbrace{S, Su,} \quad Su^2, \underbrace{Su^3, Su^3d, \dots}_{d \text{ is available}} \end{array}$$

- So with sufficiently long history, you will figure out  $u$  and  $d$  without knowing  $q$ .

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<sup>a</sup>Contributed by Mr. Hsu, Jia-Shuo (D97945003) on March 11, 2009.

## The Binomial Option Pricing Formula

- The stock prices at time  $n$  are

$$Su^n, Su^{n-1}d, \dots, Sd^n.$$

- Let  $a$  be the minimum number of upward price moves for the call to finish in the money.
- So  $a$  is the smallest nonnegative integer such that

$$Su^a d^{n-a} \geq X,$$

or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

## The Binomial Option Pricing Formula (concluded)

- Hence,

$$\begin{aligned} C &= \frac{\sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} (Su^j d^{n-j} - X)}{R^n} \quad (24) \\ &= S \sum_{j=a}^n \binom{n}{j} \frac{(pu)^j [(1-p)d]^{n-j}}{R^n} \\ &\quad - \frac{X}{R^n} \sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} \\ &= S \sum_{j=a}^n b(j; n, pu/R) - Xe^{-\hat{r}n} \sum_{j=a}^n b(j; n, p). \end{aligned}$$

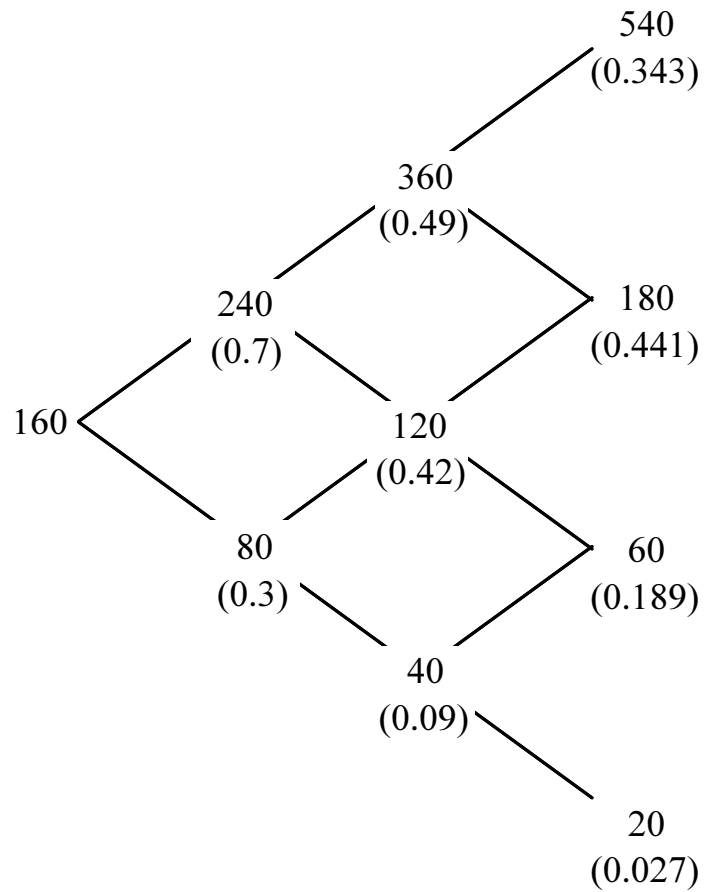


## Numerical Examples

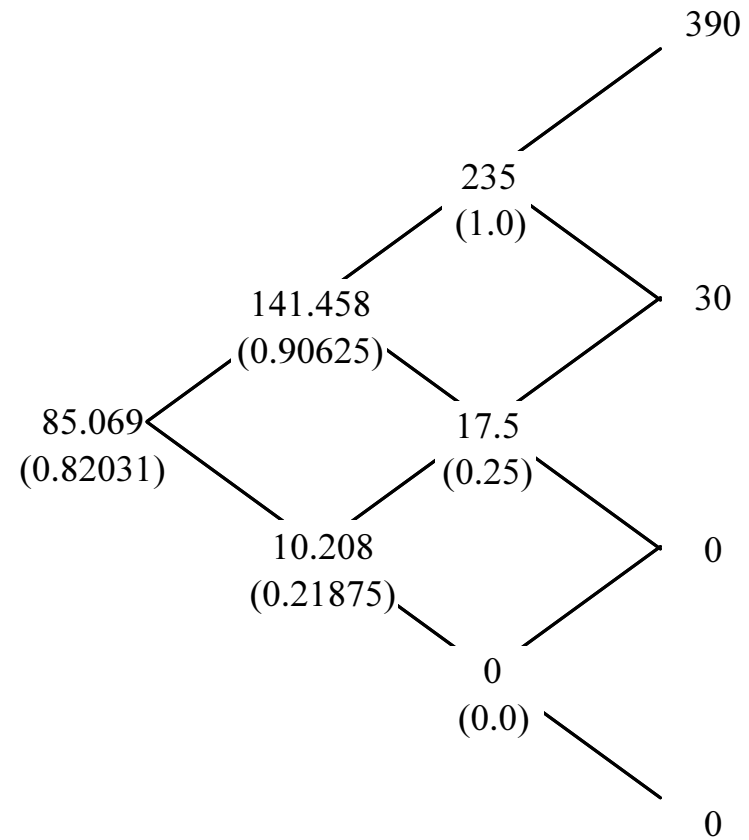
- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$  and  $d = 0.5$ .
- $r = 18.232\%$  per period ( $R = e^{0.18232} = 1.2$ ).
  - Hence  $p = (R - d)/(u - d) = 0.7$ .
- Consider a European call on this stock with  $X = 150$  and  $n = 3$ .
- The call value is \$85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$

Binomial process for the stock price  
(probabilities in parentheses)



Binomial process for the call price  
(hedge ratios in parentheses)



## Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90 and invest \$85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow  $0.82031 \times 160 - 85.069 = 46.1806$  dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars,}$$

is the arbitrage profit as we will see.

## Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy

$$0.90625 - 0.82031 = 0.08594$$

more shares at the cost of  $0.08594 \times 240 = 20.6256$   
dollars financed by borrowing.

- Debt now totals  $20.6256 + 46.1806 \times 1.2 = 76.04232$   
dollars.

## Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell  $0.90625 - 0.25 = 0.65625$  shares.
- This generates an income of  $0.65625 \times 120 = 78.75$  dollars.
- Use this income to reduce the debt to

$$76.04232 \times 1.2 - 78.75 = 12.5$$

dollars.

## Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to \$180.
- The call we wrote finishes in the money.
- For a loss of  $180 - 150 = 30$  dollars, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to  $12.5 \times 1.2 + 30 = 45$  dollars.
- It is repaid by selling the 0.25 shares of stock for  $0.25 \times 180 = 45$  dollars.

## Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

$$0.25 \times 60 = 15$$

dollars.

- Use it to repay the debt of  $12.5 \times 1.2 = 15$  dollars.