# *Bond Price Volatility*

“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy

# Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.
- Assume level-coupon bonds here.

## Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?
- Define price volatility by

$$-\frac{\frac{\partial P}{\partial y}}{P}.$$

# Price Volatility of Bonds

- The price volatility of a coupon bond is

$$-\frac{(C/y)\,n - \left(C/y^2\right)\left((1+y)^{n+1} - (1+y)\right) - nF}{(C/y)\left((1+y)^{n+1} - (1+y)\right) + F(1+y)}.$$

  - $F$ is the par value.
  - $C$ is the coupon payment per period.

- For bonds without embedded options,

$$-\frac{\frac{\partial P}{\partial y}}{P} > 0.$$

# Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' PVs divided by the asset's price.
- Formally,

$$\mathrm{MD} \equiv \frac{1}{P} \sum_{i=1}^{n} \frac{iC_i}{(1+y)^i}.$$

- The Macaulay duration, in periods, is equal to

$$\mathrm{MD} = -(1+y) \frac{\partial P}{\partial y} \frac{1}{P}. \tag{9}$$

## MD of Bonds

- The MD of a coupon bond is

$$\text{MD} = \frac{1}{P}\left[\sum_{i=1}^{n}\frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n}\right]. \tag{10}$$

- It can be simplified to

$$\text{MD} = \frac{c(1+y)\left[(1+y)^n - 1\right] + ny(y-c)}{cy\left[(1+y)^n - 1\right] + y^2},$$

  where $c$ is the period coupon rate.

- The MD of a zero-coupon bond equals $n$, its term to maturity.

- The MD of a coupon bond is less than $n$.

## Remarks

- Equations (9) on p. 80 and (10) on p. 81 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
  - That is, if the cash flow is independent of yields.
- To see this point, suppose the market yield declines.
- The MD will be lengthened.
- But for securities whose maturity actually decreases as a result, the MD (*as originally defined*) may decrease.

## How Not To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But it should be seen mainly as measuring *price volatility*.
- Many, if not most, duration-related terminology cannot be comprehended otherwise.

## Conversion

- For the MD to be year-based, modify Eq. (10) on p. 81 to

$$\frac{1}{P}\left[\sum_{i=1}^{n}\frac{i}{k}\,\frac{C}{\left(1+\frac{y}{k}\right)^{i}}+\frac{n}{k}\,\frac{F}{\left(1+\frac{y}{k}\right)^{n}}\right],$$

  where $y$ is the *annual* yield and $k$ is the compounding frequency per annum.

- Equation (9) on p. 80 also becomes

$$\text{MD}=-\left(1+\frac{y}{k}\right)\frac{\partial P}{\partial y}\,\frac{1}{P}.$$

- By definition, MD (in years) $=\dfrac{\text{MD (in periods)}}{k}$.

## Modified Duration

- Modified duration is defined as

$$\text{modified duration} \equiv -\frac{\partial P}{\partial y}\frac{1}{P} = \frac{\text{MD}}{(1+y)}. \tag{11}$$

- By Taylor expansion,

$$\text{percent price change} \approx -\text{modified duration} \times \text{yield change}.$$

## Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

## Modified Duration of a Portfolio

- The modified duration of a portfolio equals

$$\sum_i \omega_i D_i.$$

  - $D_i$ is the modified duration of the $i$th asset.
  - $\omega_i$ is the market value of that asset expressed as a percentage of the market value of the portfolio.

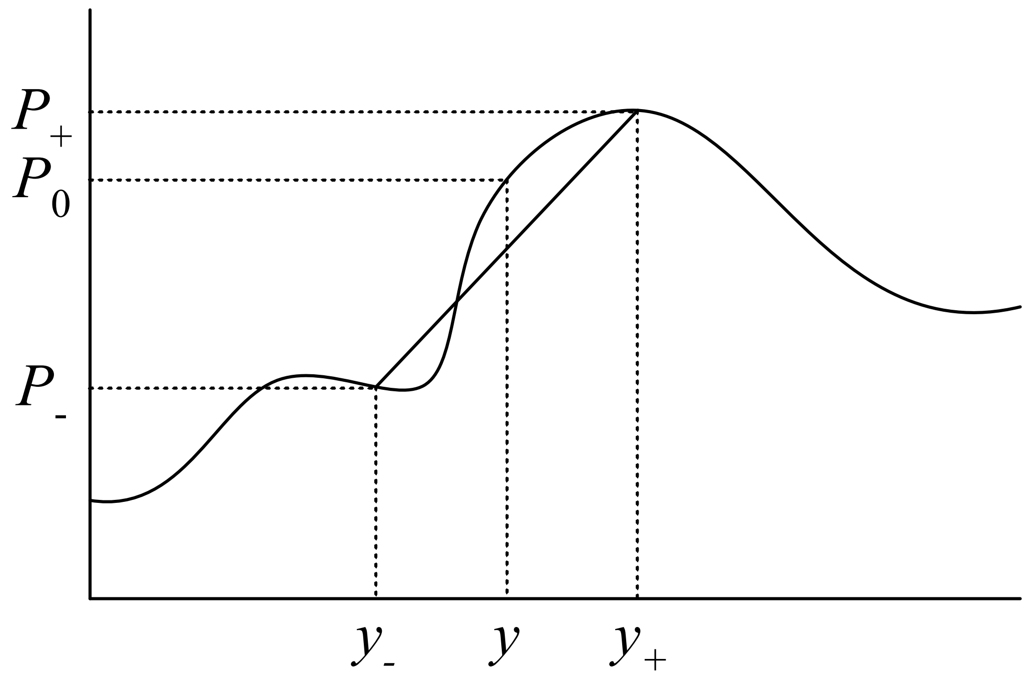
# Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

$$\frac{P_- - P_+}{P_0(y_+ - y_-)}.$$

  - $P_-$ is the price if the yield is decreased by $\Delta y$.
  - $P_+$ is the price if the yield is increased by $\Delta y$.
  - $P_0$ is the initial price, $y$ is the initial yield.
  - $\Delta y$ is small.
- See plot on p. 89.



$P_+$
$P_0$
$P_-$
$y_-$
$y$
$y_+$

## Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use

$$\frac{P_0 - P_+}{P_0 \, \Delta y}.$$

  - More economical but less accurate.

# The Practices

- Duration is usually expressed in percentage terms — call it $D_{\%}$ — for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by

$$-D_{\%} \times \Delta r$$

when the yield increases instantaneously by $\Delta r\%$.

  – Price will drop by 20% if $D_{\%} = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.

- In fact, $D_{\%}$ equals modified duration as originally defined (prove it!).

## Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as

$$\text{modified duration} \times \text{price } (\% \text{ of par}) = -\frac{\partial P}{\partial y}.$$

- The approximate *dollar* price change per \$100 of par value is

$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$

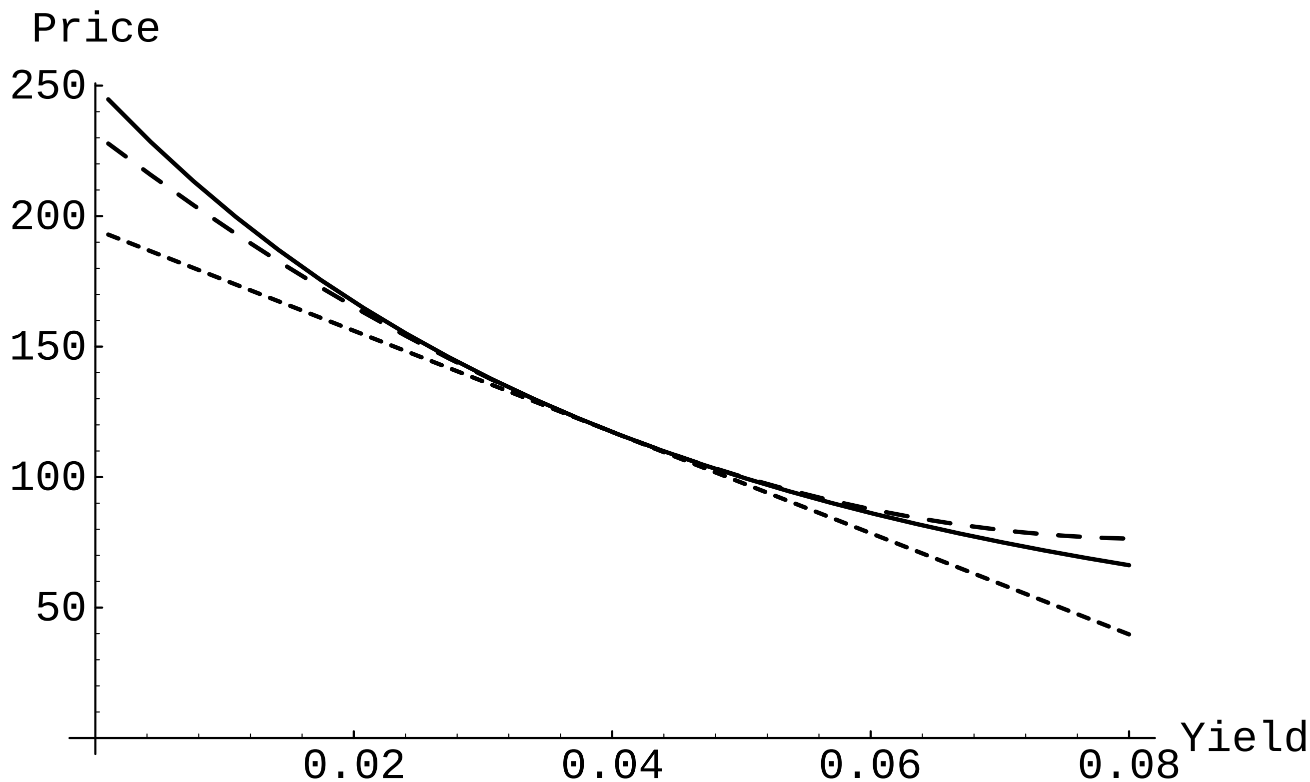
# Convexity

- Convexity is defined as

$$\text{convexity (in periods)} \equiv \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.$$

- The convexity of a coupon bond is positive (prove it!).
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).
- So between two bonds with the same duration, the one with a higher convexity is more valuable.[a]

[a] Do you spot a problem?


Price
250
200
150
100
50
0.02
0.04
0.06
0.08
Yield

## Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}$$

when there are $k$ periods per annum.

## Use of Convexity

- The approximation $\Delta P/P \approx -$ duration $\times$ yield change works for small yield changes.
- To improve upon it for larger yield changes, use

$$\begin{aligned}
\frac{\Delta P}{P} &\approx \frac{\partial P}{\partial y}\frac{1}{P}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\frac{1}{P}(\Delta y)^2 \\
&= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.
\end{aligned}$$

- Recall the figure on p. 94.

# The Practices

- Convexity is usually expressed in percentage terms — call it $C_{\%}$ — for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by $-D_{\%} \times \Delta r + C_{\%} \times (\Delta r)^2/2$ when the yield increases instantaneously by $\Delta r\%$.
  - Price will drop by 17% if $D_{\%} = 10$, $C_{\%} = 1.5$, and $\Delta r = 2$ because
  $$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$
- In fact, $C_{\%}$ equals convexity divided by 100 (prove it!).

# Effective Convexity

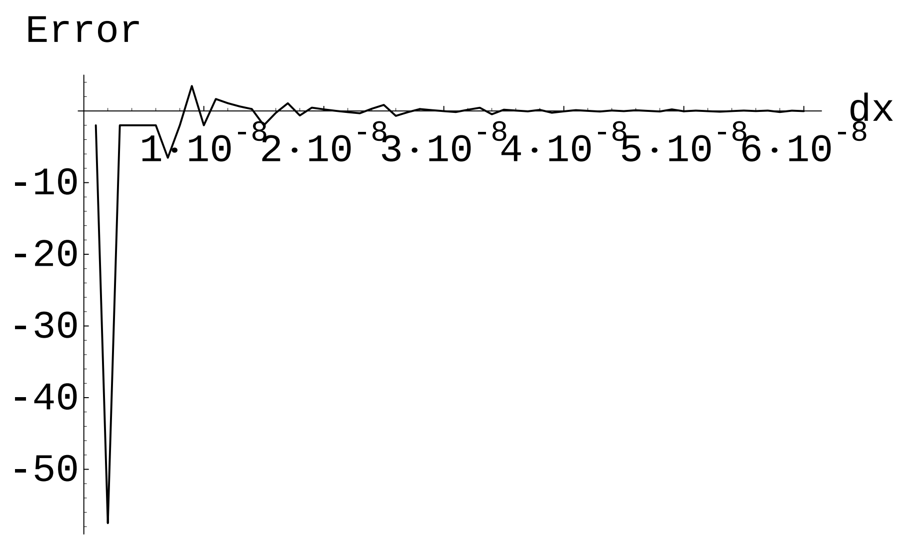
- The effective convexity is defined as

$$\frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2},$$

  - $P_-$ is the price if the yield is decreased by $\Delta y$.
  - $P_+$ is the price if the yield is increased by $\Delta y$.
  - $P_0$ is the initial price, $y$ is the initial yield.
  - $\Delta y$ is small.
- Effective convexity is most relevant when a bond's cash flow is interest rate sensitive.
- Numerically, choosing the right $\Delta y$ is a delicate matter.

## Approximate $d^2 f(x)^2/dx^2$ at $x = 1$, Where $f(x) = x^2$

The difference of $((1 + \Delta x)^2 + (1 - \Delta x)^2 - 2)/(\Delta x)^2$ and 2:



*Term Structure of Interest Rates*

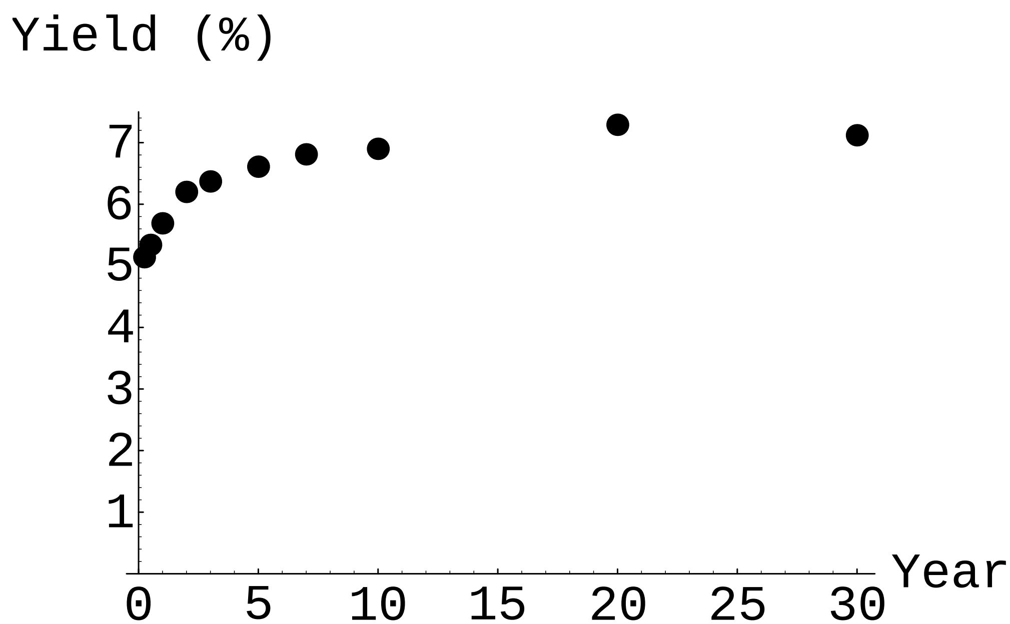
 

Why is it that the interest of money is lower,
when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don't lend it at interest.
Rather, give [it] to someone
from whom you won't get it back.
— Thomas Gospel 95

## Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds forms the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.


Yield (%)
7
6
5
4
3
2
1
0
5
10
15
20
25
30
Year

## Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots yields to maturity against maturity.
- A par yield curve is constructed from bonds trading near par.

## Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

## Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.
- The PV of one dollar $i$ periods from now is

$$[\,1 + S(i)\,]^{-i}.$$

- The one-period spot rate is called the short rate.
- Spot rate curve: Plot of spot rates against maturity.

# Problems with the PV Formula

- In the bond price formula,

$$\sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n},$$

  every cash flow is discounted at the same yield $y$.

- Consider two riskless bonds with different yields to maturity because of their different cash flow streams:

$$\sum_{i=1}^{n_1} \frac{C}{(1+y_1)^i} + \frac{F}{(1+y_1)^{n_1}},$$

$$\sum_{i=1}^{n_2} \frac{C}{(1+y_2)^i} + \frac{F}{(1+y_2)^{n_2}}.$$

## Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.
- But shouldn't they be discounted at the same rate?

# Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the $i$th bond paying $C_i$ dollars at time $i$.
- So a level-coupon bond has the price

$$P = \sum_{i=1}^{n} \frac{C}{[\,1 + S(i)\,]^i} + \frac{F}{[\,1 + S(n)\,]^n}. \tag{12}$$

- This pricing method incorporates information from the term structure.
- Discount each cash flow at the corresponding spot rate.

# Discount Factors

- In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

  – Above, $d(i) \equiv [\, 1 + S(i) \,]^{-i}$, $i = 1, 2, \ldots, n$, are called discount factors.
  – $d(i)$ is the PV of one dollar $i$ periods from now.

- The discount factors are often interpolated to form a continuous function called the discount function.

# Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.
- Compute $S(2)$ from the two-period coupon bond price $P$ by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[\,1 + S(2)\,]^2}.$$

# Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price $P$ of the $n$-period coupon bond and $S(1), S(2), \ldots, S(n-1)$.
- Then $S(n)$ can be computed from Eq. (12) on p. 109, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[\,1 + S(i)\,]^i} + \frac{F}{[\,1 + S(n)\,]^n}.$$

- The running time is $O(n)$ (see text).
- The procedure is called bootstrapping.

## Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.[a]

[a] Any economic justifications?

# Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.
- Were this bond riskless, it would fetch

$$P^* = \sum_{t=1}^{n} \frac{C_t}{[\,1 + S(t)\,]^t}.$$

- Since risk must be compensated, $P < P^*$.
- Yield spread is the difference between the IRR of the risky bond and that of a riskless bond with comparable maturity.

## Static Spread

- The static spread is the amount $s$ by which the spot rate curve has to shift in parallel to price the risky bond:

$$P = \sum_{t=1}^{n} \frac{C_t}{[\,1 + s + S(t)\,]^t}.$$

- Unlike the yield spread, the static spread incorporates information from the term structure.

# Of Spot Rate Curve and Yield Curve

- $y_k$: yield to maturity for the $k$-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

## Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But this is only a trend not a mathematical truth.[a]

[a] See a counterexample in the text.

# Forward Rates

- The yield curve contains information regarding future interest rates currently "expected" by the market.
- Invest \$1 for $j$ periods to end up with $[1+S(j)]^j$ dollars at time $j$.
  - The maturity strategy.
- Invest \$1 in bonds for $i$ periods and at time $i$ invest the proceeds in bonds for another $j-i$ periods where $j>i$.
- Will have $[1+S(i)]^i[1+S(i,j)]^{j-i}$ dollars at time $j$.
  - $S(i,j)$: $(j-i)$-period spot rate $i$ periods from now.
  - The rollover strategy.

# Forward Rates (concluded)

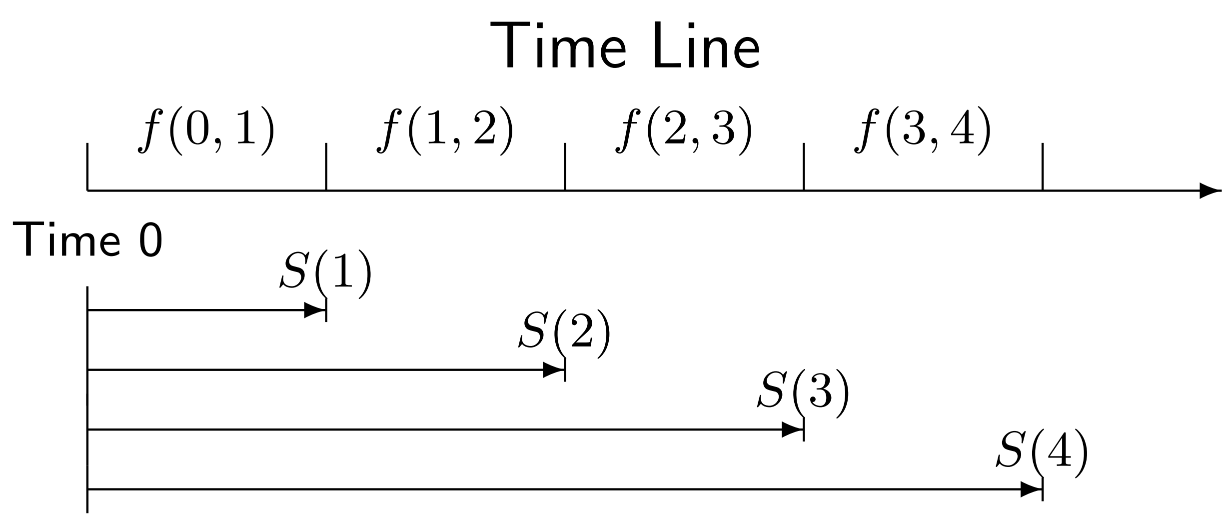
- When $S(i,j)$ equals

$$f(i,j) \equiv \left[ \frac{(1+S(j))^j}{(1+S(i))^i} \right]^{1/(j-i)} - 1, \tag{13}$$

  we will end up with $[\,1+S(j)\,]^j$ dollars again.

- By definition, $f(0,j) = S(j)$.

- $f(i,j)$ is called the (implied) forward rates.
  - More precisely, the $(j-i)$-period forward rate $i$ periods from now.


Time Line
$f(0,1)$
$f(1,2)$
$f(2,3)$
$f(3,4)$
Time 0
$S(1)$
$S(2)$
$S(3)$
$S(4)$

# Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate two investment strategies.
- $f(i, i+1)$ are instantaneous forward rates or one-period forward rates.

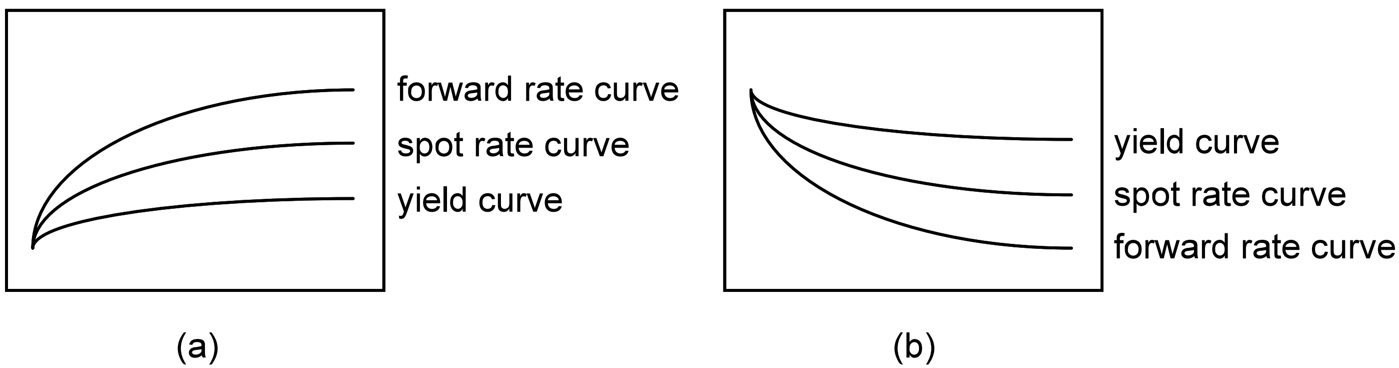
## Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i, j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i, j) < S(j) < \cdots < S(i).$$


forward rate curve
spot rate curve
yield curve
(a)
yield curve
spot rate curve
forward rate curve
(b)

## Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of \$1 at time $n$ can be derived in two ways.
- Buy $n$-period zero-coupon bonds and receive

$$[\,1+S(n)\,]^n.$$

- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is

$$[\,1+S(1)\,][\,1+f(1,2)\,]\cdots[\,1+f(n-1,n)\,].$$

# Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curves (concluded)

- Since they are identical,

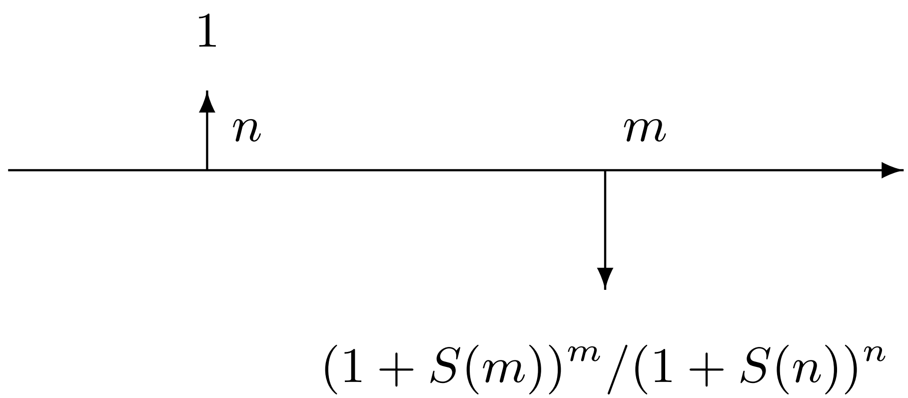
$$S(n) = \{[\,1+S(1)\,][\,1+f(1,2)\,] \cdots [\,1+f(n-1,n)\,]\}^{1/n} - 1. \tag{14}$$

- Hence, the forward rates, specifically the one-period forward rates, determine the spot rate curve.
- Other equivalencies can be derived similarly, such as

$$f(T,T+1) = \frac{d(T)}{d(T+1)} - 1.$$

# Locking in the Forward Rate $f(n, m)$

- Buy one $n$-period zero-coupon bond for $1/(1 + S(n))^n$.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ $m$-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow $1/(1 + S(n))^n$.
- At time $n$ there will be a cash inflow of \$1.
- At time $m$ there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.
- This implies the rate $f(n, m)$ between times $n$ and $m$.


1
n
m
(1 + S(m))^m/(1 + S(n))^n

# Forward Contracts

- We had generated the cash flow of a financial instrument called forward contract.
- Agreed upon today, it enables one to
  - Borrow money at time $n$ in the future, and
  - Repay the loan at time $m > n$ with an interest rate equal to the forward rate

$$f(n, m).$$

- Can the spot rate curve be an arbitrary curve?[a]

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[a] Contributed by Mr. Dai, Tian-Shyr (`R86526008`, `D88526006`) in 1998.

# Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

# Spot and Forward Rates under Continuous Compounding (concluded)

- The formula for the forward rate:

$$f(i,j) = \frac{jS(j) - iS(i)}{j-i}.$$

- The one-period forward rate:

$$f(j,j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

-

$$f(T) \equiv \lim_{\Delta T \to 0} f(T, T+\Delta T) = S(T) + T\,\frac{\partial S}{\partial T}.$$

- $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$.

## Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a,b) = E[\, S(a,b) \,]. \tag{15}$$

- It does not imply that the forward rate is an accurate predictor for the future spot rate.
- It implies the maturity strategy and the rollover strategy produce the same result at the horizon on the average.

## Unbiased Expectations Theory and Spot Rate Curve

- It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from Eq. (13) on p. 119.
  - So $E[\, S(j, j+1) \,] > S(j+1) > \cdots > S(1)$ if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

## More Implications

- The theory has been rejected by most empirical studies with the possible exception of the period prior to 1915.
- Since the term structure has been upward sloping about 80% of the time, the theory would imply that investors have expected interest rates to rise 80% of the time.
- Riskless bonds, regardless of their different maturities, are expected to earn the same return on the average.
- That would mean investors are indifferent to risk.

# A "Bad" Expectations Theory

- The expected returns on all possible riskless bond strategies are equal for *all* holding periods.
- So

$$(1 + S(2))^2 = (1 + S(1))\, E[\, 1 + S(1, 2)\, ] \qquad (16)$$

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

- After rearrangement,

$$\frac{1}{E[\, 1 + S(1, 2)\, ]} = \frac{1 + S(1)}{(1 + S(2))^2}.$$

# A "Bad" Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond and sells it after one period.
  - The expected return is $E[\,(1 + S(1,2))^{-1}\,]/(1 + S(2))^{-2}$.
  - Strategy two buys a one-period bond with a return of $1 + S(1)$.
- The theory says the returns are equal:

$$\frac{1 + S(1)}{(1 + S(2))^2} = E\left[\,\frac{1}{1 + S(1,2)}\,\right].$$

# A "Bad" Expectations Theory (concluded)

- Combine this with Eq. (16) on p. 134 to obtain

$$E\left[\frac{1}{1+S(1,2)}\right]=\frac{1}{E[\,1+S(1,2)\,]}.$$

- But this is impossible save for a certain economy.
  - Jensen's inequality states that $E[\,g(X)\,] > g(E[\,X\,])$ for any nondegenerate random variable $X$ and strictly convex function $g$ (i.e., $g''(x) > 0$).
  - Use $g(x) \equiv (1+x)^{-1}$ to prove our point.

# Local Expectations Theory

- The expected rate of return of any bond over *a single period* equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1 + S(1) \quad \text{for } \textit{all } n > 1.$$

- This theory is the basis of many interest rate models.

# Duration Revisited

- To handle more general types of spot rate curve changes, define a vector $[\, c_1, c_2, \ldots, c_n \,]$ that characterizes the perceived type of change.
  - Parallel shift: $[\, 1, 1, \ldots, 1 \,]$.
  - Twist: $[\, 1, 1, \ldots, 1, -1, \ldots, -1 \,]$.
  - $\cdots$
- Let $P(y) \equiv \sum_i C_i / (1 + S(i) + y c_i)^i$ be the price associated with the cash flow $C_1, C_2, \ldots$.
- Define duration as

$$-\frac{\partial P(y)/P(0)}{\partial y} \bigg|_{y=0}.$$

*Fundamental Statistical Concepts*

There are three kinds of lies:
lies, damn lies, and statistics.
— Benjamin Disraeli (1804–1881)

One death is a tragedy,
but a million deaths are a statistic.
— Josef Stalin (1879–1953)

# Moments

- The variance of a random variable $X$ is defined as

$$\mathrm{Var}[\,X\,] \equiv E\left[\,(X - E[\,X\,])^2\,\right].$$

- The covariance between random variables $X$ and $Y$ is

$$\mathrm{Cov}[\,X, Y\,] \equiv E\,[\,(X - \mu_X)(Y - \mu_Y)\,],$$

  where $\mu_X$ and $\mu_Y$ are the means of $X$ and $Y$, respectively.

- Random variables $X$ and $Y$ are uncorrelated if

$$\mathrm{Cov}[\,X, Y\,] = 0.$$

# Correlation

- The standard deviation of $X$ is the square root of the variance,

$$\sigma_X \equiv \sqrt{\mathrm{Var}[\,X\,]}\,.$$

- The correlation (or correlation coefficient) between $X$ and $Y$ is

$$\rho_{X,Y} \equiv \frac{\mathrm{Cov}[\,X,Y\,]}{\sigma_X \sigma_Y},$$

provided both have nonzero standard deviations.[a]

[a]Paul Wilmott, "the correlations between financial quantities are notoriously unstable."

## Variance of Sum

- Variance of a weighted sum of random variables equals

$$\mathrm{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \,\mathrm{Cov}[\, X_i, X_j \,].$$

- It becomes

$$\sum_{i=1}^{n} a_i^2 \,\mathrm{Var}[\, X_i \,]$$

when $X_i$ are uncorrelated.

# Conditional Expectation

- "$X \mid I$" denotes $X$ conditional on the information set $I$.
- The information set can be another random variable's value or the past values of $X$, say.
- The conditional expectation $E[\, X \mid I \,]$ is the expected value of $X$ conditional on $I$; it is a random variable.
- The law of iterated conditional expectations:

$$E[\, X \,] = E[\, E[\, X \mid I \,]\,].$$

- If $I_2$ contains at least as much information as $I_1$, then

$$E[\, X \mid I_1 \,] = E[\, E[\, X \mid I_2 \,] \mid I_1 \,]. \tag{17}$$

# The Normal Distribution

- A random variable $X$ has the normal distribution with mean $\mu$ and variance $\sigma^2$ if its probability density function is
$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}.$$
- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the distribution function
$$\text{Prob}[\, X \leq z\,] = N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx.$$

# Moment Generating Function

- The moment generating function of random variable $X$ is
$$\theta_X(t) \equiv E[\, e^{tX} \,].$$
- The moment generating function of $X \sim N(\mu, \sigma^2)$ is
$$\theta_X(t) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]. \tag{18}$$

# The Multivariate Normal Distribution

- If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then

$$\sum_i X_i \sim N\left(\sum_i \mu_i, \sum_i \sigma_i^2\right).$$

- Let $X_i \sim N(\mu_i, \sigma_i^2)$, which may not be independent.
- Suppose

$$\sum_{i=1}^{n} t_i X_i \sim N\left(\sum_{i=1}^{n} t_i\, \mu_i, \sum_{i=1}^{n}\sum_{j=1}^{n} t_i t_j \operatorname{Cov}[\, X_i, X_j\, ]\right)$$

for every linear combination $\sum_{i=1}^{n} t_i X_i$.[a]

- $X_i$ are said to have a multivariate normal distribution.

[a] Corrected by Mr. Huang, Guo-Hua (`R98922107`) on March 10, 2010.

## Generation of Univariate Normal Distributions

- Let $X$ be uniformly distributed over $(0, 1]$ so that $\text{Prob}[X \leq x] = x$ for $0 < x \leq 1$.
- Repeatedly draw two samples $x_1$ and $x_2$ from $X$ until

$$\omega \equiv (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1.$$

- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where

$$c \equiv \sqrt{-2(\ln \omega)/\omega}\,.$$

# A Dirty Trick and a Right Attitude

- Let $\xi_i$ are independent and uniformly distributed over $(0, 1)$.
- A simple method to generate the standard normal variable is to calculate[a]

$$\sum_{i=1}^{12} \xi_i - 6.$$

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[a] Jäckel (2002), "this is not a highly accurate approximation and should only be used to establish ballpark estimates."

# A Dirty Trick and a Right Attitude (concluded)

- Always blame your random number generator last.[a]
- Instead, check your programs first.

[a] "The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings." William Shakespeare (1564–1616), *Julius Caesar*.

## Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation $\rho$ can be generated.
- Let $X_1$ and $X_2$ be independent standard normal variables.
- Set

$$
\begin{aligned}
U &\equiv aX_1, \\
V &\equiv \rho U + \sqrt{1-\rho^2}\, aX_2.
\end{aligned}
$$

- $U$ and $V$ are the desired random variables with $\text{Var}[\,U\,] = \text{Var}[\,V\,] = a^2$ and $\text{Cov}[\,U, V\,] = \rho a^2$.

# The Lognormal Distribution

- A random variable $Y$ is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \equiv e^X$.
- The mean and variance of $Y$ are

$$\mu_Y = e^{\mu+\sigma^2/2} \text{ and } \sigma_Y^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right), \tag{19}$$

respectively.

  – They follow from $E[\,Y^n\,] = e^{n\mu+n^2\sigma^2/2}$.