Principles of Financial Computing

Prof. Yuh-Dauh Lyuu
Dept. Computer Science & Information Engineering
and
Department of Finance
National Taiwan University

Class Information

- Yuh-Dauh Lyuu. *Financial Engineering & Computation: Principles, Mathematics, Algorithms*. Cambridge University Press, 2002.
- Official Web page is

  `www.csie.ntu.edu.tw/~lyuu/finance1.html`

  – Homeworks and teaching assistants will be announced there.

Class Information (concluded)

- Check

  `www.csie.ntu.edu.tw/~lyuu/capitals.html`

  for some of the software.

- Please ask many questions in class.
  - The best way for me to remember you in a large class.[a]

[a] "[A] science concentrator [...] said that in his eighth semester of [Harvard] college, there was not a single science professor who could identify him by name." (*New York Times*, September 3, 2003.)

Useful Journals

- *Applied Mathematical Finance.*
- *Finance and Stochastics.*
- *Financial Analysts Journal.*
- *Journal of Banking & Finance.*
- *Journal of Computational Finance.*
- *Journal of Derivatives.*
- *Journal of Economic Dynamics & Control.*
- *Journal of Finance.*
- *Journal of Financial Economics.*

Useful Journals (continued)

- *Journal of Fixed Income.*
- *Journal of Futures Markets.*
- *Journal of Financial and Quantitative Analysis.*
- *Journal of Portfolio Management.*
- *Journal of Real Estate Finance and Economics.*
- *Management Science.*
- *Mathematical Finance.*

Useful Journals (concluded)

- *Quantitative Finance.*
- *Review of Financial Studies.*
- *Review of Derivatives Research.*
- *Risk Magazine.*
- *SIAM Journal on Financial Mathematics.*
- *Stochastics and Stochastics Reports.*

Introduction

You must go into finance, Amory.
— F. Scott Fitzgerald (1896–1940),
This Side of Paradise (1920)

The two most dangerous words in Wall Street
vocabulary are "financial engineering."
— Wilbur Ross (2007)

What This Course Is About

- Financial theories in pricing.
- Mathematical backgrounds.
- Derivative securities.
- Pricing models.
- Efficient algorithms in pricing financial instruments.
- Research problems.
- Help in finding your thesis directions.

What This Course Is *Not* About

- How to program.
- Basic calculus, probability, combinatorics, and algebra.
- Details of the financial markets.
- How to be rich.
- How the markets will perform tomorrow.
- Professional behavior.

The Modelers' Hippocratic Oath[a]

- I will remember that I didn't make the world, and it doesn't satisfy my equations.
- Though I will use models boldly to estimate value, I will not be overly impressed by mathematics.
- I will never sacrifice reality for elegance without explaining why I have done so.
- Nor will I give the people who use my model false comfort about its accuracy. Instead, I will make explicit its assumptions and oversights.
- I understand that my work may have enormous effects on society and the economy, many of them beyond my comprehension.

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[a] Emanuel Derman and Paul Wilmott, January 7, 2009.

Analysis of Algorithms

I can calculate the motions
of the heavenly bodies,
but not the madness of people.
— Isaac Newton (1642–1727)

It is unworthy of excellent men
to lose hours like slaves
in the labor of computation.
— Gottfried Wilhelm Leibniz (1646–1716)

Computability and Algorithms

- Algorithms are precise procedures that can be turned into computer programs.
- Uncomputable problems.
  - Does this program have infinite loops?
  - Is this program bug free?
- Computable problems.
  - Intractable problems.
  - Tractable problems.

Complexity

- A set of basic operations are assumed to take one unit of time $(+, -, \times, /, \log, x^y, e^x, \ldots)$.
- The total number of these operations is the total work done by an algorithm (its computational complexity).
- The space complexity is the amount of memory space used by an algorithm.
- Concentrate on the abstract complexity of an algorithm instead of its detailed implementation.
- Complexity is a good guide to an algorithm's *actual* running time.

Asymptotics

- Consider the search algorithm on p. 17.
- The worst-case complexity is n comparisons.
- There are operations besides comparison, to be sure.
- We care only about the asymptotic growth rate not the exact number of operations.
  - So the complexity of maintaining the loop is subsumed by the complexity of the body of the loop.
- The complexity is hence $O(n)$.

Algorithm for Searching an Element

```
for k = 1, 2, 3, ..., n do
    if x = A_k then
        return k;
    end if
end for
return not-found;
```


Common Complexities

- Let n stand for the "size" of the problem.
  - Number of elements, number of cash flows, number of time periods, etc.
- Linear time if the complexity is $O(n)$.
- Quadratic time if the complexity is $O(n^2)$.
- Cubic time if the complexity is $O(n^3)$.
- Superpolynomial if the complexity is higher than polynomials, say $2^{O(\sqrt{n})}$.[a]
- Exponential time if the complexity is $2^{O(n)}$.

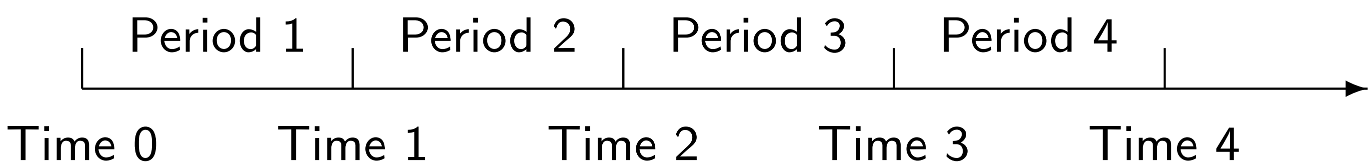
[a] E.g., Dai (R86526008, D8852600) and Lyuu (2007) and Lyuu and Wang (F95922018) (2011).

Basic Financial Mathematics

In the fifteenth century
mathematics was mainly concerned with
questions of commercial arithmetic and
the problems of the architect.
— Joseph Alois Schumpeter (1883–1950)

I'm more concerned about
the return of my money than
the return on my money.
— Will Rogers (1879–1935)

The Time Line



Time Value of Money[a]

$$\begin{aligned} \text{FV} &= \text{PV}(1+r)^n, \\ \text{PV} &= \text{FV} \times (1+r)^{-n}. \end{aligned}$$

- FV (future value).
- PV (present value).
- r: interest rate.

[a]Fibonacci (1170–1240) and Irving Fisher (1867–1947).

Periodic Compounding

- Suppose the annual interest rate r is compounded m times per annum.
- Then
$$1 \to \left(1+\frac{r}{m}\right) \to \left(1+\frac{r}{m}\right)^2 \to \left(1+\frac{r}{m}\right)^3 \to \cdots$$
- Hence, after n years,
$$\text{FV} = \text{PV}\left(1+\frac{r}{m}\right)^{nm}. \tag{1}$$

Common Compounding Methods

- Annual compounding: $m = 1$.
- Semiannual compounding: $m = 2$.
- Quarterly compounding: $m = 4$.
- Monthly compounding: $m = 12$.
- Weekly compounding: $m = 52$.
- Daily compounding: $m = 365$.

Easy Translations

- An annual interest rate of r compounded m times a year is "equivalent to" an interest rate of r/m per $1/m$ year.
- If a loan asks for a return of 1% per month, the annual interest rate will be 12% *with monthly compounding*.

Example

- Annual interest rate is 10% compounded twice per annum.
- Each dollar will grow to be

$$[\,1 + (0.1/2)\,]^2 = 1.1025$$

  one year from now.
- The rate is equivalent to an interest rate of 10.25% compounded once *per annum*,

$$1 + 0.1025 = 1.1025$$

Continuous Compounding[a]

- Let $m \to \infty$ so that
$$\left(1 + \frac{r}{m}\right)^m \to e^r$$
in Eq. (1) on p. 23.

- Then
$$\text{FV} = \text{PV} \times e^{rn},$$
where $e = 2.71828\ldots$.

[a] Jacob Bernoulli (1654–1705) in 1685.

Continuous Compounding (concluded)

- Continuous compounding is easier to work with.
- Suppose the annual interest rate is r_1 for n_1 years and r_2 for the following n_2 years.
- Then the FV of one dollar will be

$$e^{r_1 n_1 + r_2 n_2}$$

after $n_1 + n_2$ years.

The PV Formula

- The PV of the cash flow $C_1, C_2, \ldots, C_n$ at times $1, 2, \ldots, n$ is

$$\frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_n}{(1+y)^n}. \tag{2}$$

- This formula and its variations are the engine behind most of financial calculations.[a]
  - What is y?
  - What are C_i?
  - What is n?

[a]Cochrane (2005), "Asset pricing theory all stems from one simple concept [...]: price equals expected discounted payoff."

An Algorithm for Evaluating PV in Eq. (2)

```
1: x := 0;
2: for i = 1, 2, ..., n do
3:     x := x + C_i/(1 + y)^i;
4: end for
5: return x;
```

- The algorithm takes time proportional to $\sum_{i=1}^{n} i = O(n^2)$.
- Can improve it to $O(n)$ if you apply $a^b = e^{b \ln a}$ in step 3.[a]

[a] Recall that we count x^y as taking one unit of time.

Another Algorithm for Evaluating PV

$x := 0;$

$d := 1 + y;$

for $i = n, n-1, \ldots, 1$ **do**

$\quad x := (x + C_i)/d;$

end for

`return` $x;$

Horner's Rule: The Idea Behind p. 31

- This idea is

$$\left(\cdots\left(\left(\frac{C_n}{1+y}+C_{n-1}\right)\frac{1}{1+y}+C_{n-2}\right)\frac{1}{1+y}+\cdots\right)\frac{1}{1+y}.$$

  - Due to Horner (1786–1837) in 1819.

- The algorithm takes $O(n)$ time.

- It is the most efficient possible in terms of the absolute number of arithmetic operations.[a]

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[a] Borodin and Munro (1975).

Conversion between Compounding Methods

- Suppose r_1 is the annual rate with continuous compounding.
- Suppose r_2 is the equivalent rate compounded m times per annum.
- How are they related?

Conversion between Compounding Methods (concluded)

- Principle: Both interest rates must produce the same amount of money after one year.
- That is,

$$\left(1 + \frac{r_2}{m}\right)^m = e^{r_1}.$$

- Therefore,

$$\begin{aligned} r_1 &= m \ln\left(1 + \frac{r_2}{m}\right), \\ r_2 &= m\left(e^{r_1/m} - 1\right). \end{aligned}$$

Annuities[a]

- An annuity pays out the same C dollars at the end of each year for n years.
- With a rate of r, the FV at the end of the nth year is

$$\sum_{i=0}^{n-1} C(1+r)^i = C\,\frac{(1+r)^n - 1}{r}. \tag{3}$$

[a] Jan de Witt (1625–1672) in 1671 and Nicholas Bernoulli (1687–1759) in 1709.

General Annuities

- If m payments of C dollars each are received per year (the general annuity), then Eq. (3) becomes

$$C\,\frac{\left(1+\frac{r}{m}\right)^{nm}-1}{\frac{r}{m}}.$$

- The PV of a general annuity is

$$\sum_{i=1}^{nm} C\left(1+\frac{r}{m}\right)^{-i} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-nm}}{\frac{r}{m}}. \tag{4}$$

Amortization

- It is a method of repaying a loan through regular payments of interest *and* principal.
- The size of the loan (the original balance) is reduced by the principal part of each payment.
- The interest part of each payment pays the interest incurred on the remaining principal balance.
- As the principal gets paid down over the term of the loan, the interest part of the payment diminishes.

Example: Home Mortgage

- By paying down the principal consistently, the risk to the lender is lowered.
- When the borrower sells the house, the remaining principal is due the lender.
- Consider the equal-payment case, i.e., fixed-rate, level-payment, fully amortized mortgages.
  - They are called traditional mortgages in the U.S.

A Numerical Example

- Consider a 15-year, \$250,000 loan at 8.0% interest rate.
- Solving Eq. (4) on p. 36 with $\text{PV} = 250000$, $n = 15$, $m = 12$, and $r = 0.08$ gives a monthly payment of $C = 2389.13$.
- The amortization schedule is shown on p. 40.

Month	Payment	Interest	Principal	Remaining principal
				250,000.000
1	2,389.13	1,666.667	722.464	249,277.536
2	2,389.13	1,661.850	727.280	248,550.256
3	2,389.13	1,657.002	732.129	247,818.128
		...		
178	2,389.13	47.153	2,341.980	4,730.899
179	2,389.13	31.539	2,357.591	2,373.308
180	2,389.13	15.822	2,373.308	0.000
Total	430,043.438	180,043.438	250,000.000	

A Numerical Example (concluded)

In every month:

- The principal and interest parts add up to \$2,389.13.
- The remaining principal is reduced by the amount indicated under the Principal heading.[a]
- The interest is computed by multiplying the remaining balance of the previous month by 0.08/12.

[a] This column varies with r. Thanks to a lively class discussion on Feb 24, 2010.

Method 1 of Calculating the Remaining Principal

- A month's principal payment = monthly payment − (previous month's remaining principal) × (monthly interest rate).
- A month's remaining principal = previous month's remaining principal − principal payment calculated above.
- Generate the amortization schedule until you reach the particular month you are interested in.

Method 1 of Calculating the Remaining Principal (concluded)

- This method is relatively slow but is universal in its applicability.
- It can, for example, accommodate prepayment and variable interest rates.

Method 2 of Calculating the Remaining Principal

- Right after the kth payment, the remaining principal is the PV of the future $nm - k$ cash flows,

$$\sum_{i=1}^{nm-k} C \left(1 + \frac{r}{m}\right)^{-i} = C \, \frac{1 - \left(1 + \frac{r}{m}\right)^{-nm+k}}{\frac{r}{m}}. \tag{5}$$

- This method is faster but more limited in applications.

Yields

- The term yield denotes the return of investment.
- Two widely used yields are the bond equivalent yield (BEY) and the mortgage equivalent yield (MEY).
- Recall Eq. (1) on p. 23: $\text{FV} = \text{PV}\left(1 + \frac{r}{m}\right)^{nm}$.
- BEY corresponds to the r above that equates PV with FV when $m = 2$.
- MEY corresponds to the r above that equates PV with FV when $m = 12$.

Internal Rate of Return (IRR)

- It is the interest rate which equates an investment's PV with its price P,

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_n}{(1+y)^n}.$$

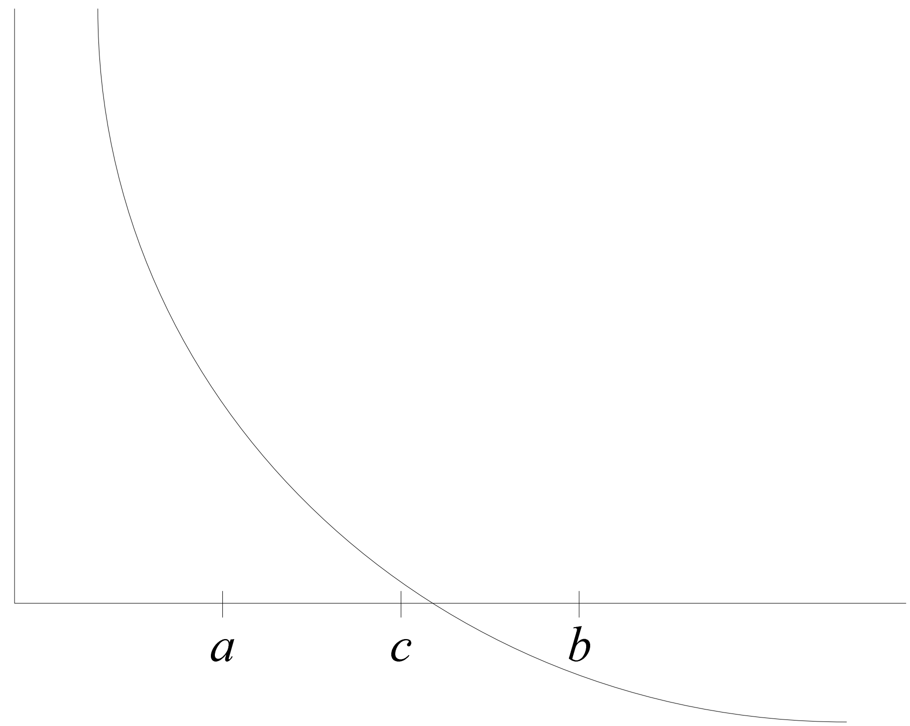
- The above formula is the foundation upon which pricing methodologies are built.

Numerical Methods for Yields

- Define
$$f(y) \equiv \sum_{t=1}^{n} \frac{C_t}{(1+y)^t} - P.$$
  - P is the market price.
- Solve $f(y) = 0$ for a real $y \geq -1$.
- The function $f(y)$ is monotonic in y if $C_t > 0$ for all t.
- A unique solution exists for a monotonic $f(y)$.

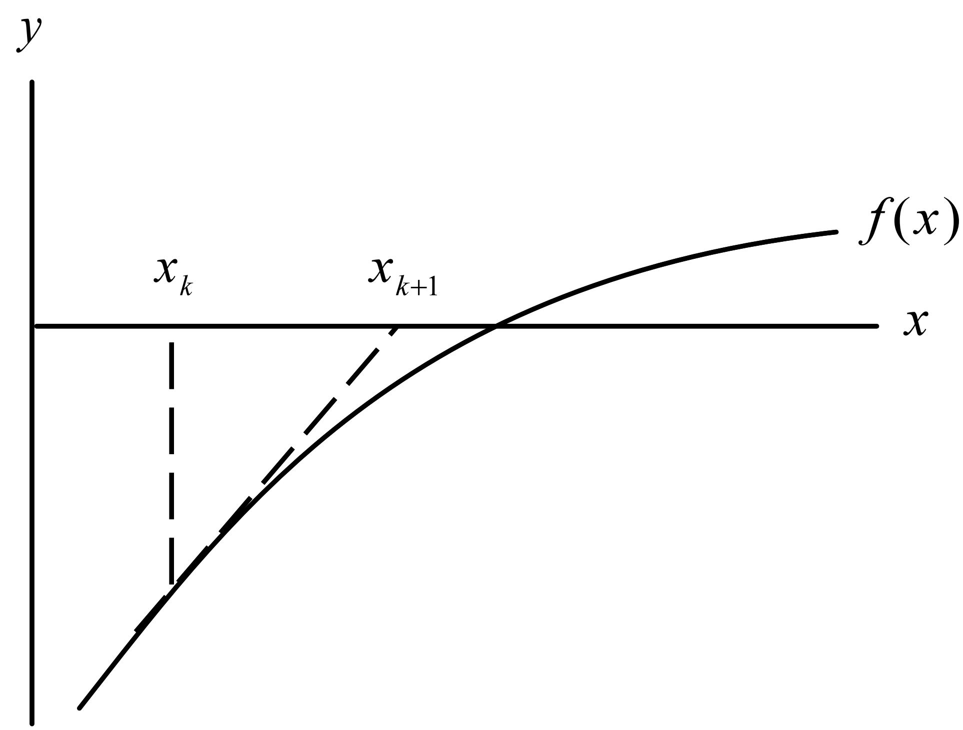
The Bisection Method

- Start with a and b where $a < b$ and $f(a)\, f(b) < 0$.
- Then $f(\xi)$ must be zero for some $\xi \in [\,a, b\,]$.
- If we evaluate f at the midpoint $c \equiv (a + b)/2$, either (1) $f(c) = 0$, (2) $f(a)\, f(c) < 0$, or (3) $f(c)\, f(b) < 0$.
- In the first case we are done, in the second case we continue the process with the new bracket $[\,a, c\,]$, and in the third case we continue with $[\,c, b\,]$.
- The bracket is halved in the latter two cases.
- After n steps, we will have confined ξ within a bracket of length $(b - a)/2^n$.


a
c
b

The Newton-Raphson Method

- It converges faster than the bisection method.
- Start with a first approximation x_0 to a root of $f(x) = 0$.
- Then
$$x_{k+1} \equiv x_k - \frac{f(x_k)}{f'(x_k)}.$$
- When computing yields,
$$f'(x) = -\sum_{t=1}^{n} \frac{tC_t}{(1+x)^{t+1}}.$$



y
f(x)
x_k
x_{k+1}
x

The Secant Method

- A variant of the Newton-Raphson method.
- Replace differentiation with difference.
- Start with two approximations x_0 and x_1.
- Then compute the $(k + 1)$st approximation with

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

The Secant Method (concluded)

- Its convergence rate is 1.618.
- This is slightly worse than the Newton-Raphson method's 2.
- But the secant method does not need to evaluate $f'(x_k)$ as needed by the Newton-Raphson method.
- This saves about 50% in computation efforts per iteration.
- The convergence rate of the bisection method is 1.

Solving Systems of Nonlinear Equations

- It is not easy to extend the bisection method to higher dimensions.
- But the Newton-Raphson method can be extended to higher dimensions.
- Let (x_k, y_k) be the kth approximation to the solution of the two simultaneous equations,

$$\begin{aligned} f(x, y) &= 0, \\ g(x, y) &= 0. \end{aligned}$$

Solving Systems of Nonlinear Equations (concluded)

- The $(k+1)$st approximation (x_{k+1}, y_{k+1}) satisfies the following linear equations,

$$\left[\begin{array}{cc} \frac{\partial f(x_k, y_k)}{\partial x} & \frac{\partial f(x_k, y_k)}{\partial y} \\ \frac{\partial g(x_k, y_k)}{\partial x} & \frac{\partial g(x_k, y_k)}{\partial y} \end{array}\right] \left[\begin{array}{c} \Delta x_{k+1} \\ \Delta y_{k+1} \end{array}\right] = -\left[\begin{array}{c} f(x_k, y_k) \\ g(x_k, y_k) \end{array}\right],$$

  where unknowns $\Delta x_{k+1} \equiv x_{k+1} - x_k$ and $\Delta y_{k+1} \equiv y_{k+1} - y_k$.

- The above has a unique solution for $(\Delta x_{k+1}, \Delta y_{k+1})$ when the 2×2 matrix is invertible.

- Set $(x_{k+1}, y_{k+1}) = (x_k + \Delta x_{k+1}, y_k + \Delta y_{k+1})$.

Zero-Coupon Bonds (Pure Discount Bonds)

- The price of a zero-coupon bond that pays F dollars in n periods is
$$F/(1+r)^n,$$
where r is the interest rate per period.

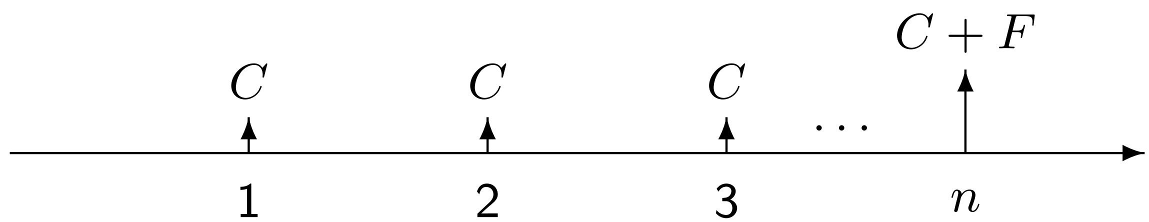
- Can meet future obligations without reinvestment risk.

Example

- The interest rate is 8% compounded semiannually.
- A zero-coupon bond that pays the par value 20 years from now will be priced at $1/(1.04)^{40}$, or 20.83%, of its par value.
- It will be quoted as 20.83.
- If the bond matures in 10 years instead of 20, its price would be 45.64.

Level-Coupon Bonds

- Coupon rate.
- Par value, paid at maturity.
- F denotes the par value, and C denotes the coupon.
- Cash flow:



- Coupon bonds can be thought of as a matching package of zero-coupon bonds, at least theoretically.[a]

[a] "You see, Daddy didn't bake the cake, and Daddy isn't the one who gets to eat it. But he gets to slice the cake and hand it out. And when he does, little golden crumbs fall off the cake. And Daddy gets to eat those," wrote Tom Wolfe (1931–) in *Bonfire of the Vanities* (1987).

Pricing Formula

$$
\begin{aligned}
P &= \sum_{i=1}^{n} \frac{C}{\left(1+\frac{r}{m}\right)^{i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}} \\
&= C\, \frac{1-\left(1+\frac{r}{m}\right)^{-n}}{\frac{r}{m}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}}. \qquad (6)
\end{aligned}
$$

- n: number of cash flows.
- m: number of payments per year.
- r: annual rate compounded m times per annum.
- Note $C = Fc/m$ when c is the annual coupon rate.
- Price P can be computed in $O(1)$ time.

Yields to Maturity

- It is the r that satisfies Eq. (6) on p. 59 with P being the bond price.
- For a 15% BEY, a 10-year bond with a coupon rate of 10% paid semiannually sells for

$$\begin{aligned} & 5 \times \frac{1 - [\,1 + (0.15/2)\,]^{-2\times 10}}{0.15/2} + \frac{100}{[\,1 + (0.15/2)\,]^{2\times 10}} \\ = \; & 74.5138 \end{aligned}$$

percent of par.

- So 15% is the yield to maturity if the bond is selling for 74.5138.

Price Behavior (1)

- Bond prices fall when interest rates rise, and vice versa.
- "Only 24 percent answered the question correctly."[a]

[a] CNN, December 21, 2001.

Price Behavior (2)

- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate is above the market interest rate;
  - at par (at its par value) when its coupon rate is equal to the market interest rate;
  - at a discount (below its par value) when its coupon rate is below the market interest rate.

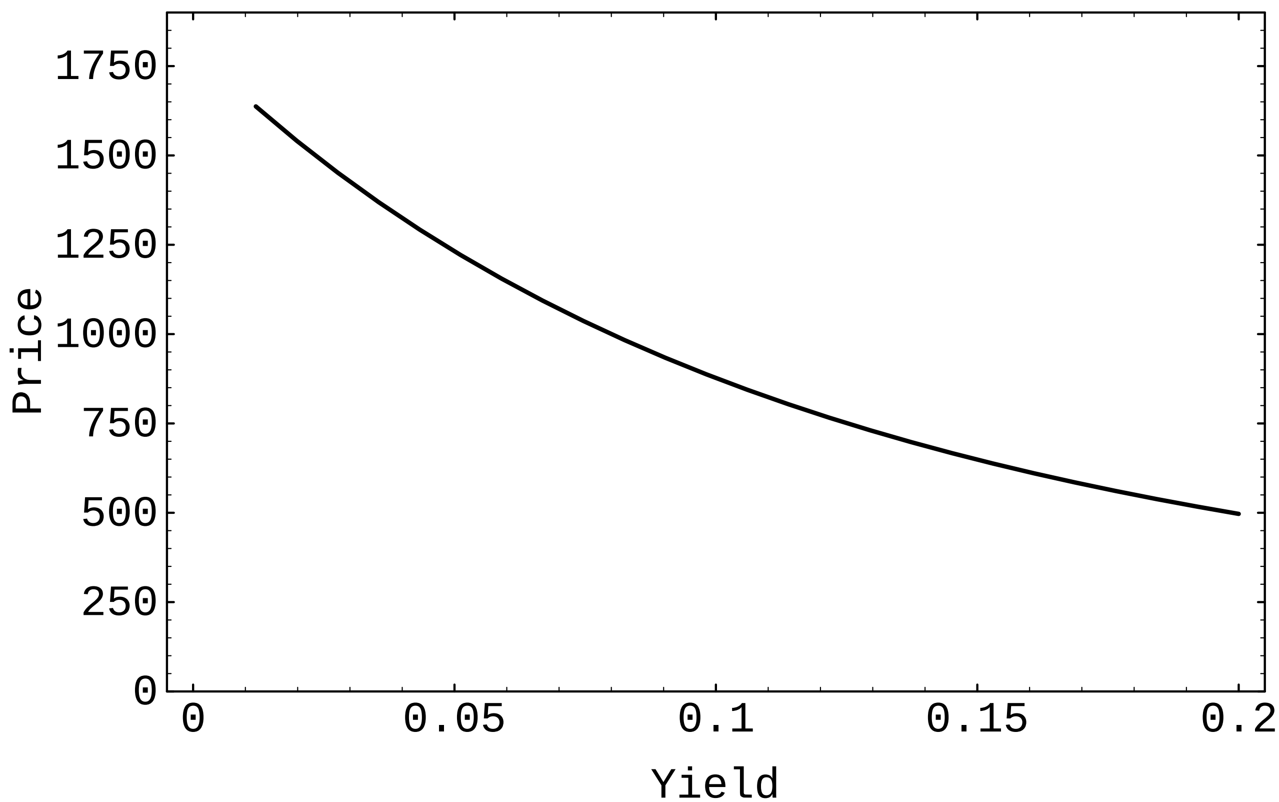
9% Coupon Bond

Yield (%)	Price (% of par)
7.5	113.37
8.0	108.65
8.5	104.19
9.0	100.00
9.5	96.04
10.0	92.31
10.5	88.79

Terminology

- Bonds selling at par are called par bonds.
- Bonds selling at a premium are called premium bonds.
- Bonds selling at a discount are called discount bonds.

Price Behavior (3): Convexity



Day Count Conventions: Actual/Actual

- The first "actual" refers to the actual number of days in a month.
- The second refers to the actual number of days in a coupon period.
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ to date $D_2 \equiv (y_2, m_2, d_2)$ is

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1)$$

- But if d_1 or d_2 is 31, we need to change it to 30 before applying the above formula.

Day Count Conventions: 30/360 (concluded)

- An equivalent formula without any adjustment is

$$\begin{aligned} & 360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1) \\ & + \max(30 - d_1, 0) + \min(d_2, 30). \end{aligned}$$

- Many variations regarding 31, Feb 28, and Feb 29.

Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

$$\omega \equiv \frac{\begin{array}{c}\text{number of days between the settlement}\\ \text{and the next coupon payment date}\end{array}}{\text{number of days in the coupon period}}. \tag{7}$$

- The price is now calculated by

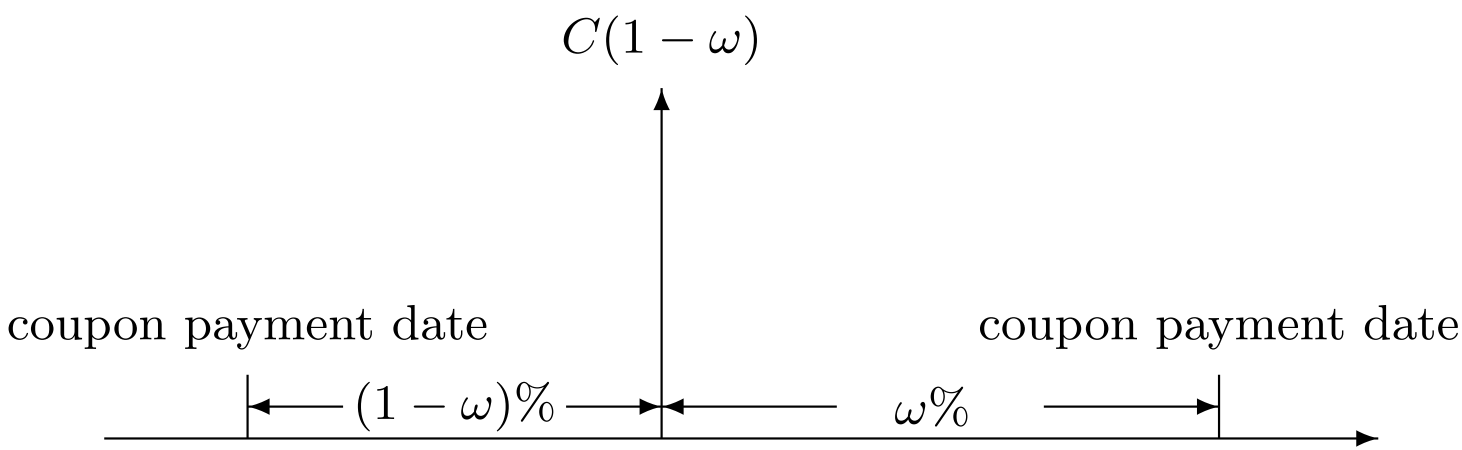
$$\mathrm{PV} = \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}. \tag{8}$$

Accrued Interest

- The buyer pays the invoice price (the quoted price *plus* the accrued interest).
- The accrued interest equals

$$C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega).$$

- The yield to maturity is the r satisfying Eq. (8) when P is the invoice price.
- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.



$C(1-\omega)$
coupon payment date
coupon payment date
$(1-\omega)\%$
$\omega\%$

Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.

Example ("30/360") (concluded)

- The accrued interest is $(10/2) \times \frac{180-60}{180} = 3.3333$ per \$100 of par value.
- The yield to maturity is 3%.
- This can be verified by Eq. (8) on p. 69 with
  - $\omega = 60/180$,
  - $m = 2$,
  - $C = 5$,
  - PV$= 111.2891 + 3.3333$,
  - $r = 0.03$.

Price Behavior (2) Revisited

- Before: A bond selling at par if the yield to maturity equals the coupon rate.
- But it assumed that the settlement date is on a coupon payment date.
- Now suppose the settlement date for a bond selling at par (i.e., the *quoted price* is equal to the par value) falls between two coupon payment dates.
- Then its yield to maturity is less than the coupon rate.
  - The short reason: Exponential growth is replaced by linear growth, hence "overpaying" the coupon.

