$$
\begin{array}{cccccccccc}
 & & & & x+2\sqrt{\Delta t} & & & & & f(x+2\sqrt{\Delta t}) \\
 & & & \nearrow & & & & & \nearrow & \\
 & & x+\sqrt{\Delta t} & & & & & f(x+\sqrt{\Delta t}) & & \\
 & \nearrow & & \searrow & & & \nearrow & & \searrow & \\
x & & & & x & f(x) & & & & f(x) \\
 & \searrow & & \nearrow & & & \searrow & & \nearrow & \\
 & & x-\sqrt{\Delta t} & & & & & f(x-\sqrt{\Delta t}) & & \\
 & & & \searrow & & & & & \searrow & \\
 & & & & x-2\sqrt{\Delta t} & & & & & f(x-2\sqrt{\Delta t})
\end{array}
$$

Binomial CIR (concluded)

- The probability of an up move at each node r is
$$p(r) \equiv \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}. \tag{113}$$
  - $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.
- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Numerical Examples

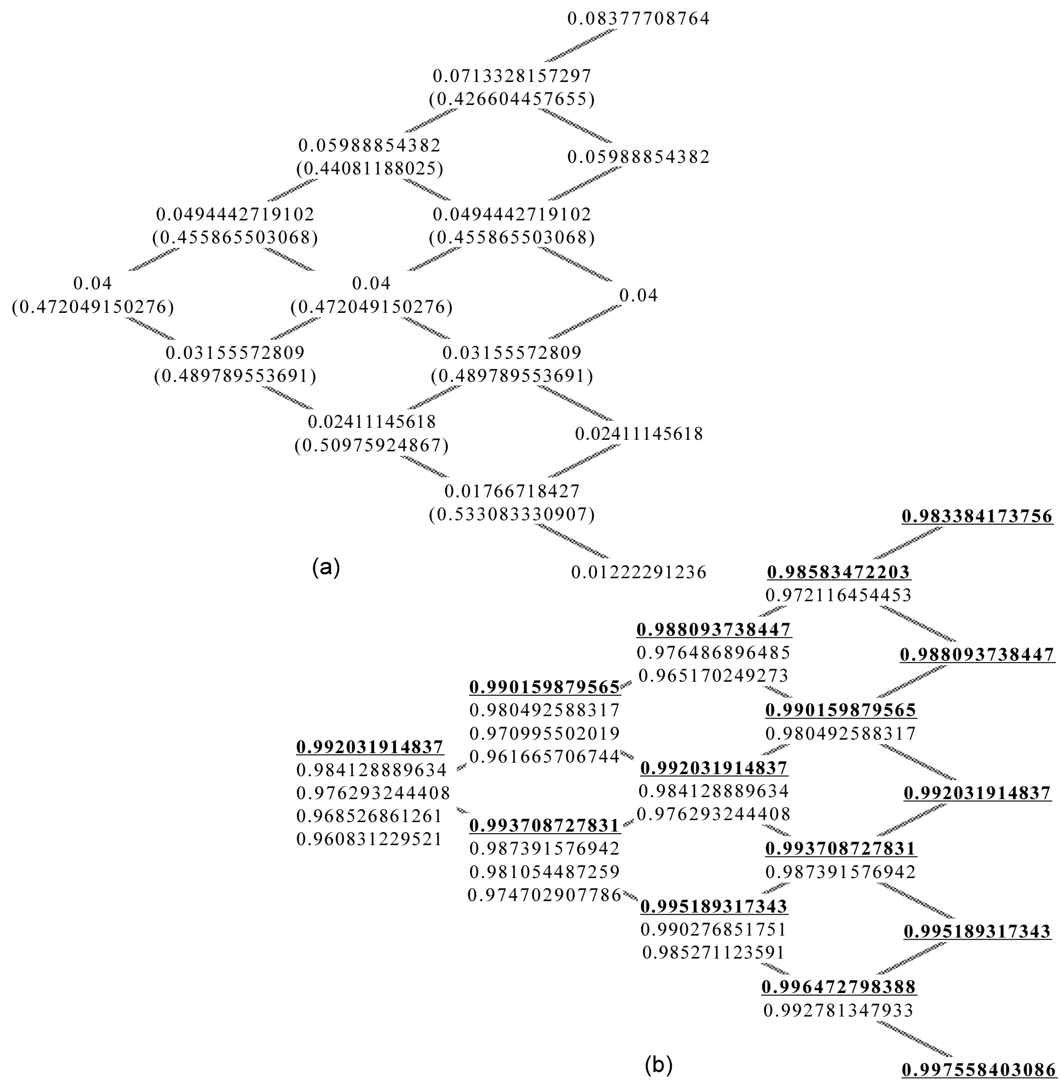
- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.

- See p. 944(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.



0.08377708764
0.0713328157297
(0.426604457655)
0.05988854382
(0.44081188025)
0.05988854382
0.0494442719102
(0.455865503068)
0.0494442719102
(0.455865503068)
0.04
(0.472049150276)
0.04
(0.472049150276)
0.04
0.03155572809
(0.489789553691)
0.03155572809
(0.489789553691)
0.02411145618
(0.50975924867)
0.02411145618
0.01766718427
(0.533083330907)
0.01222291236
(a)
0.983384173756
0.98583472203
0.972116454453
0.988093738447
0.976486896485
0.965170249273
0.988093738447
0.990159879565
0.980492588317
0.970995502019
0.961665706744
0.990159879565
0.980492588317
0.992031914837
0.984128889634
0.976293244408
0.968526861261
0.960831229521
0.992031914837
0.984128889634
0.976293244408
0.992031914837
0.993708727831
0.987391576942
0.981054487259
0.974702907786
0.993708727831
0.987391576942
0.995189317343
0.990276851751
0.985271123591
0.995189317343
0.996472798388
0.992781347933
(b)
0.997558403086

Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$.

Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process $dy = \alpha(y, t)\, dt + \sigma(y, t)\, dW$.
- Make sure the binomial model's drift and diffusion converge to the above process by setting the probability of an up move to

$$\frac{\alpha(y, t)\, \Delta t + y - y_{\mathrm{d}}}{y_{\mathrm{u}} - y_{\mathrm{d}}}.$$

- Here $y_{\mathrm{u}} \equiv y + \sigma(y, t)\sqrt{\Delta t}$ and $y_{\mathrm{d}} \equiv y - \sigma(y, t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.
- The displacements are identical, at $\sigma(y, t)\sqrt{\Delta t}$.

[a] Nelson and Ramaswamy (1990).

A General Method (continued)

- But the binomial tree may not combine:

$$\sigma(y,t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}},t)\sqrt{\Delta t} \neq -\sigma(y,t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}},t)\sqrt{\Delta t}$$

  in general.

- When $\sigma(y,t)$ is a constant independent of y, equality holds and the tree combines.

- To achieve this, define the transformation

$$x(y,t) \equiv \int^{y} \sigma(z,t)^{-1}\,dz.$$

- Then x follows $dx = m(y,t)\,dt + dW$ for some $m(y,t)$ (see text).

A General Method (continued)

- The key is that the diffusion term is now a constant, and the binomial tree for x combines.
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.
- Note that $y_{\mathrm{u}}(x,t) \equiv y(x+\sqrt{\Delta t}, t+\Delta t)$ and $y_{\mathrm{d}}(x,t) \equiv y(x-\sqrt{\Delta t}, t+\Delta t)$.

A General Method (concluded)

- The transformation is
$$\int^r (\sigma\sqrt{z})^{-1}\, dz = 2\sqrt{r}/\sigma$$
for the CIR model.
- The transformation is
$$\int^S (\sigma z)^{-1}\, dz = (1/\sigma)\ln S$$
for the Black-Scholes model.
- The familiar binomial option pricing model in fact discretizes $\ln S$ not S.

Model Calibration

- In the time-series approach, the time series of short rates is used to estimate the parameters of the process.
- This approach may help in validating the proposed interest rate process.
- But it alone cannot be used to estimate the risk premium parameter λ.
- The model prices based on the estimated parameters may also deviate a lot from those in the market.

Model Calibration (concluded)

- The cross-sectional approach uses a cross section of bond prices observed at the same time.
- The parameters are to be such that the model prices closely match those in the market.
- After this procedure, the calibrated model can be used to price interest rate derivatives.
- Unlike the time-series approach, the cross-sectional approach is unable to separate out the interest rate risk premium from the model parameters.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two-factor ones.

Options on Coupon Bonds[a]

- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time T on a bond with par value \$1.
- Let X denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all i.
- The payoff for the option is

$$\max\left(\sum_{i=1}^{n} c_i P(r(T), T, t_i) - X, 0\right).$$

[a] Jamshidian (1989).

Options on Coupon Bonds (continued)

- At time T, there is a unique value r^* for $r(T)$ that renders the coupon bond's price equal the strike price X.
- This r^* can be obtained by solving $X = \sum_{i=1}^{n} c_i P(r, T, t_i)$ numerically for r.
- The solution is also unique for one-factor models whose bond price is a monotonically decreasing function of r.
- Let $X_i \equiv P(r^*, T, t_i)$, the value at time T of a zero-coupon bond with par value \$1 and maturing at time t_i if $r(T) = r^*$.

Options on Coupon Bonds (concluded)

- Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$.
- As $X = \sum_i c_i X_i$, the option's payoff equals

$$\begin{aligned} & \max\left(\sum_{i=1}^{n} c_i P(r(T), T, t_i) - \sum_i c_i X_i, 0\right) \\ = & \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0). \end{aligned}$$

- Thus the call is a package of n options on the underlying zero-coupon bond.
- Why can't we do the same thing for Asian options?[a]

[a]Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.

No-Arbitrage Term Structure Models

How much of the structure of our theories
really tells us about things in nature,
and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)

Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.

No-Arbitrage Models[a]

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

---

[a] Ho and Lee (1986).

No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.

The Ho-Lee Model[a]

- The short rates at any given time are evenly spaced.
- Let p denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

[a] Ho and Lee (1986).

$$
\begin{array}{cccccc}
 & & & & & \nearrow \\
 & & & & r_3 & \\
 & & & \nearrow & & \searrow \\
 & & r_2 & & & \\
 & \nearrow & & \searrow & & \nearrow \\
r_1 & & & & r_3 + v_3 & \\
 & \searrow & & \nearrow & & \searrow \\
 & & r_2 + v_2 & & & \\
 & & & \searrow & & \nearrow \\
 & & & & r_3 + 2v_3 & \\
 & & & & & \searrow
\end{array}
$$

The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t+1), P(t, t+2), \ldots$ at time t identified with the root of the tree.
- Let the discount factors in the next period be

$$
\begin{array}{ll}
P_{\mathrm{d}}(t+1, t+2), P_{\mathrm{d}}(t+1, t+3), \ldots & \text{if short rate moves down} \\
P_{\mathrm{u}}(t+1, t+2), P_{\mathrm{u}}(t+1, t+3), \ldots & \text{if short rate moves up}
\end{array}
$$

- By backward induction, it is not hard to see that for $n \geq 2$,

$$
P_{\mathrm{u}}(t+1, t+n) = P_{\mathrm{d}}(t+1, t+n)\, e^{-(v_2 + \cdots + v_n)} \tag{114}
$$

(see text).

The Ho-Lee Model (continued)

- It is also not hard to check that the n-period zero-coupon bond has yields

$$\begin{aligned} y_{\mathrm{d}}(n) &\equiv -\frac{\ln P_{\mathrm{d}}(t+1,t+n)}{n-1} \\ y_{\mathrm{u}}(n) &\equiv -\frac{\ln P_{\mathrm{u}}(t+1,t+n)}{n-1} = y_{\mathrm{d}}(n) + \frac{v_2 + \cdots + v_n}{n-1} \end{aligned}$$

- The volatility of the yield to maturity for this bond is therefore

$$\begin{aligned} \kappa_n &\equiv \sqrt{p y_{\mathrm{u}}(n)^2 + (1-p)\, y_{\mathrm{d}}(n)^2 - [\, p y_{\mathrm{u}}(n) + (1-p)\, y_{\mathrm{d}}(n)\,]^2} \\ &= \sqrt{p(1-p)}\; (y_{\mathrm{u}}(n) - y_{\mathrm{d}}(n)) \\ &= \sqrt{p(1-p)}\; \frac{v_2 + \cdots + v_n}{n-1}. \end{aligned}$$

The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)}\, v_2. \tag{115}$$

- The variance of the short rate therefore equals $p(1-p)(r_{\mathrm{u}} - r_{\mathrm{d}})^2$, where r_{u} and r_{d} are the two successor rates.[a]

[a] Contrast this with the lognormal model.

The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of $\kappa_2, \kappa_3, \ldots$.
  - It is independent of the r_i.
- It is easy to compute the v_is from the volatility structure, and vice versa.
- The r_is can be computed by forward induction.
- The volatility structure is supplied by the market.

The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy

$$P(t, t+n) = (p P_{\mathrm{u}}(t+1, t+n) + (1-p)\, P_{\mathrm{d}}(t+1, t+n))\, P(t, t+1)$$

- Combine the above with Eq. (114) on p. 966 and assume $p = 1/2$ to obtain[a]

$$P_{\mathrm{d}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \, \frac{2 \times \exp[\, v_2 + \cdots + v_n \,]}{1 + \exp[\, v_2 + \cdots + v_n \,]}, \tag{116}$$

$$P_{\mathrm{u}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \, \frac{2}{1 + \exp[\, v_2 + \cdots + v_n \,]}. \tag{116$'$}$$

[a] In the limit, only the volatility matters.

The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.
- Suppose all v_i equal some constant v and $\delta \equiv e^v > 0$.
- Then

$$
\begin{aligned}
P_{\mathrm{d}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}}, \\
P_{\mathrm{u}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1+\delta^{n-1}}.
\end{aligned}
$$

- Short rate volatility σ equals $v/2$ by Eq. (115) on p. 968.
- Price derivatives by taking expectations under the risk-neutral probability.

The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an n-period zero-coupon bond is

$$r(t, t+n) \equiv \ln \left(\frac{P(t+1, t+n)}{P(t, t+n)} \right).$$

- Its value is either $\ln \frac{P_{\mathrm{d}}(t+1,t+n)}{P(t,t+n)}$ or $\ln \frac{P_{\mathrm{u}}(t+1,t+n)}{P(t,t+n)}$.
- Thus the variance of return is

$$\mathrm{Var}[\, r(t, t+n)\,] = p(1-p)((n-1)\, v)^2 = (n-1)^2 \sigma^2.$$

The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t, t+n)$ and $r(t, t+m)$ is $(n-1)(m-1)\,\sigma^2$ (see text).
- As a result, the correlation between any two one-period rates of return is unity.
- Strong correlation between rates is inherent in all one-factor Markovian models.

The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is

$$dr = \theta(t)\, dt + \sigma\, dW.$$

- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying, i.e., $dr = \theta(t)\, dt + \sigma(t)\, dW$.
- This corresponds to the discrete-time model in which v_i are not all identical.

The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.

Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model's state variables (factors) not its parameters.
- Model *parameters*, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.

Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model's intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.

The Black-Derman-Toy Model[a]

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 817ff (repeated on next page).
- The volatility structure is given by the market.
- From it, the short rate volatilities (thus v_i) are determined together with r_i.

[a]Black, Derman, and Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).

$$
\begin{tikzcd}[row sep=small]
 & & & r_4 \\
 & & r_3 \arrow[ur] \arrow[dr] & \\
 & r_2 \arrow[ur] \arrow[dr] & & r_4 v_4 \\
r_1 \arrow[ur] \arrow[dr] & & r_3 v_3 \arrow[ur] \arrow[dr] & \\
 & r_2 v_2 \arrow[ur] \arrow[dr] & & r_4 v_4^2 \\
 & & r_3 v_3^2 \arrow[ur] \arrow[dr] & \\
 & & & r_4 v_4^3
\end{tikzcd}
$$

The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes v_i are given a priori.
  - A related model of Salomon Brothers takes v_i to be constants.
- Lognormal models preclude negative short rates.

The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the i-period zero-coupon bond be denoted by κ_i.
- P_{u} is the price of the i-period zero-coupon bond one period from now if the short rate makes an up move.

The BDT Model: Volatility Structure (concluded)

- P_{d} is the price of the i-period zero-coupon bond one period from now if the short rate makes a down move.
- Corresponding to these two prices are the following yields to maturity,

$$\begin{aligned} y_{\mathrm{u}} &\equiv P_{\mathrm{u}}^{-1/(i-1)} - 1, \\ y_{\mathrm{d}} &\equiv P_{\mathrm{d}}^{-1/(i-1)} - 1. \end{aligned}$$

- The yield volatility is defined as $\kappa_i \equiv (1/2) \ln(y_{\mathrm{u}}/y_{\mathrm{d}})$.

The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

$$(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1}).$$

  - They define the binomial tree up to period $i - 1$.
- We now proceed to calculate r_i and v_i to extend the tree to period i.

The BDT Model: Calibration (continued)

- Assume the price of the i-period zero can move to P_{u} or P_{d} one period from now.
- Let y denote the current i-period spot rate, which is known.
- In a risk-neutral economy,

$$\frac{P_{\mathrm{u}} + P_{\mathrm{d}}}{2(1 + r_1)} = \frac{1}{(1 + y)^i}. \tag{117}$$

- Obviously, P_{u} and P_{d} are functions of the unknown r_i and v_i.

The BDT Model: Calibration (continued)

- Viewed from now, the future $(i-1)$-period spot rate at time one is uncertain.
- Recall that y_{u} and y_{d} represent the spot rates at the up node and the down node, respectively (p. 982).
- With κ^2 denoting their variance, we have

$$\kappa_i = \frac{1}{2} \ln \left(\frac{{P_{\mathrm{u}}}^{-1/(i-1)} - 1}{{P_{\mathrm{d}}}^{-1/(i-1)} - 1} \right). \tag{118}$$

The BDT Model: Calibration (continued)

- We will employ forward induction to derive a quadratic-time calibration algorithm.[a]
- Recall that forward induction inductively figures out, by moving forward in time, how much \$1 at a node contributes to the price (review p. 843(a)).
- This number is called the state price and is the price of the claim that pays \$1 at that node and zero elsewhere.

[a] Chen (`R84526007`) and Lyuu (1997); Lyuu (1999).

The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period i be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i-1$ be $P_1, P_2, \ldots, P_i$, corresponding to rates $r, rv, \ldots, rv^{i-1}$, respectively.
- One dollar at time i has a present value of

$$f(r, v) \equiv \frac{P_1}{1+r} + \frac{P_2}{1+rv} + \frac{P_3}{1+rv^2} + \cdots + \frac{P_i}{1+rv^{i-1}}.$$

The BDT Model: Calibration (continued)

- The yield volatility is

$$g(r,v) \equiv \frac{1}{2} \ln \left(\frac{\left(\frac{P_{\mathrm{u},1}}{1+rv} + \frac{P_{\mathrm{u},2}}{1+rv^2} + \cdots + \frac{P_{\mathrm{u},i-1}}{1+rv^{i-1}} \right)^{-1/(i-1)} - 1}{\left(\frac{P_{\mathrm{d},1}}{1+r} + \frac{P_{\mathrm{d},2}}{1+rv} + \cdots + \frac{P_{\mathrm{d},i-1}}{1+rv^{i-2}} \right)^{-1/(i-1)} - 1} \right).$$

- Above, $P_{\mathrm{u},1}, P_{\mathrm{u},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the up node (like $r_2 v_2$ on p. 979).
- And $P_{\mathrm{d},1}, P_{\mathrm{d},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the down node (like r_2 on p. 979).

The BDT Model: Calibration (concluded)

- Now solve
$$f(r, v) = \frac{1}{(1+y)^i} \quad \text{and} \quad g(r, v) = \kappa_i$$
for $r = r_i$ and $v = v_i$.
- This $O(n^2)$-time algorithm appears in the text.

The BDT Model: Continuous-Time Limit

- The continuous-time limit of the BDT model is

$$d \ln r = \left(\theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t)\, dW.$$

- The short rate volatility clearly should be a declining function of time for the model to display mean reversion.
  - That makes $\sigma'(t) < 0$.
- In particular, constant volatility will not attain mean reversion.

The Black-Karasinski Model[a]

- The BK model stipulates that the short rate follows

$$d \ln r = \kappa(t)(\theta(t) - \ln r)\, dt + \sigma(t)\, dW.$$

- This explicitly mean-reverting model depends on time through $\kappa(\cdot)$, $\theta(\cdot)$, and $\sigma(\cdot)$.
- The BK model hence has one more degree of freedom than the BDT model.
- The speed of mean reversion $\kappa(t)$ and the short rate volatility $\sigma(t)$ are independent.

[a]Black and Karasinski (1991).

The Black-Karasinski Model: Discrete Time

- The discrete-time version of the BK model has the same representation as the BDT model.
- To maintain a combining binomial tree, however, requires some manipulations.
- The next plot illustrates the ideas in which

$$\begin{aligned} t_2 &\equiv t_1 + \Delta t_1, \\ t_3 &\equiv t_2 + \Delta t_2. \end{aligned}$$

$$
\begin{array}{ccccc}
 & & & \nearrow & \\
 & & \ln r_{\mathrm{d}}(t_2) & & \\
 & \nearrow & & \searrow & \\
\ln r(t_1) & & & & \ln r_{\mathrm{du}}(t_3) = \ln r_{\mathrm{ud}}(t_3) \\
 & \searrow & & \nearrow & \\
 & & \ln r_{\mathrm{u}}(t_2) & & \\
 & & & \searrow &
\end{array}
$$

The Black-Karasinski Model: Discrete Time (continued)

- Note that

$$\begin{aligned}\ln r_{\mathrm{d}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 - \sigma(t_1)\sqrt{\Delta t_1}\,,\\ \ln r_{\mathrm{u}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 + \sigma(t_1)\sqrt{\Delta t_1}\,.\end{aligned}$$

- To ensure that an up move followed by a down move coincides with a down move followed by an up move, impose

$$\begin{aligned}&\ln r_{\mathrm{d}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{d}}(t_2))\,\Delta t_2 + \sigma(t_2)\sqrt{\Delta t_2}\,,\\ = \;&\ln r_{\mathrm{u}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{u}}(t_2))\,\Delta t_2 - \sigma(t_2)\sqrt{\Delta t_2}\,.\end{aligned}$$

The Black-Karasinski Model: Discrete Time (concluded)

- They imply

$$\kappa(t_2) = \frac{1 - (\sigma(t_2)/\sigma(t_1))\sqrt{\Delta t_2/\Delta t_1}}{\Delta t_2}. \tag{119}$$

- So from Δt_1, we can calculate the Δt_2 that satisfies the combining condition and then iterate.
  - $t_0 \to \Delta t_0 \to t_1 \to \Delta t_1 \to t_2 \to \Delta t_2 \to \cdots \to T$ (roughly).

Problems with Lognormal Models in General

- Lognormal models such as BDT and BK share the problem that $E^{\pi}[\,M(t)\,] = \infty$ for any finite t if they the continuously compounded rate.
- Hence periodic compounding should be used.
- Another issue is computational.
- Lognormal models usually do not give analytical solutions to even basic fixed-income securities.
- As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.

Problems with Lognormal Models in General (concluded)

- This problem can be somewhat mitigated by adopting different time steps: Use a fine time step up to the maturity of the short-dated derivative and a coarse time step beyond the maturity.[a]
- A down side of this procedure is that it has to be carried out for each derivative.
- Finally, empirically, interest rates do not follow the lognormal distribution.

[a] Hull and White (1993).

The Extended Vasicek Model[a]

- Hull and White proposed models that extend the Vasicek model and the CIR model.
- They are called the extended Vasicek model and the extended CIR model.
- The extended Vasicek model adds time dependence to the original Vasicek model,

$$dr = (\theta(t) - a(t)\,r)\,dt + \sigma(t)\,dW.$$

- Like the Ho-Lee model, this is a normal model, and the inclusion of $\theta(t)$ allows for an exact fit to the current spot rate curve.

[a]Hull and White (1990).

The Extended Vasicek Model (concluded)

- Function $\sigma(t)$ defines the short rate volatility, and $a(t)$ determines the shape of the volatility structure.
- Under this model, many European-style securities can be evaluated analytically, and efficient numerical procedures can be developed for American-style securities.

The Hull-White Model

- The Hull-White model is the following special case,

$$dr = (\theta(t) - ar)\, dt + \sigma\, dW.$$

- When the current term structure is matched,[a]

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

[a] Hull and White (1993).

The Extended CIR Model

- In the extended CIR model the short rate follows

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\sqrt{r}\, dW.$$

- The functions $\theta(t)$, $a(t)$, and $\sigma(t)$ are implied from market observables.
- With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.

The Hull-White Model: Calibration[a]

- We describe a trinomial forward induction scheme to calibrate the Hull-White model given a and σ.
- As with the Ho-Lee model, the set of achievable short rates is evenly spaced.
- Let r_0 be the annualized, continuously compounded short rate at time zero.
- Every short rate on the tree takes on a value $r_0 + j\Delta r$ for some integer j.

[a] Hull and White (1993).

The Hull-White Model: Calibration (continued)

- Time increments on the tree are also equally spaced at Δt apart.
- Hence nodes are located at times $i\Delta t$ for $i = 0, 1, 2, \ldots$.
- We shall refer to the node on the tree with $t_i \equiv i\Delta t$ and $r_j \equiv r_0 + j\Delta r$ as the (i, j) node.
- The short rate at node (i, j), which equals r_j, is effective for the time period $[\, t_i, t_{i+1})$.

The Hull-White Model: Calibration (continued)

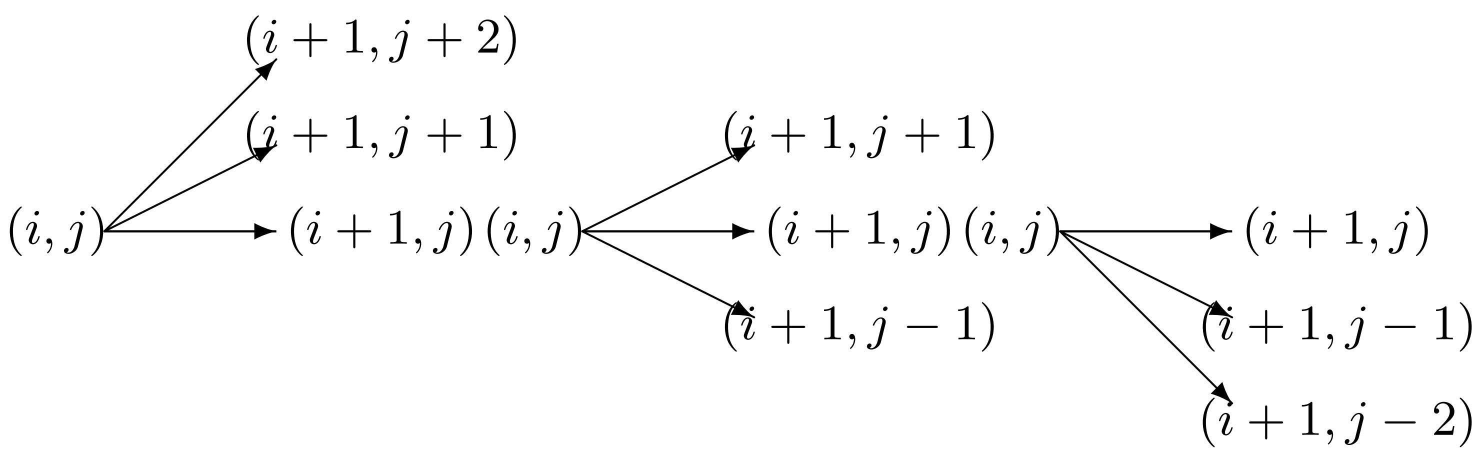
- Use

$$\mu_{i,j} \equiv \theta(t_i) - a r_j \tag{120}$$

to denote the drift rate, or the expected change, of the short rate as seen from node (i, j).

- The three distinct possibilities for node (i, j) with three branches incident from it are displayed on p. 1005.
- The interest rate movement described by the middle branch may be an increase of Δr, no change, or a decrease of Δr.

The Hull-White Model: Calibration (continued)



The Hull-White Model: Calibration (continued)

- The upper and the lower branches bracket the middle branch.
- Define

$$\begin{aligned}
p_1(i,j) &\equiv \text{the probability of following the upper branch from node } (i,j)\\
p_2(i,j) &\equiv \text{the probability of following the middle branch from node } (i,j)\\
p_3(i,j) &\equiv \text{the probability of following the lower branch from node } (i,j)
\end{aligned}$$

- The root of the tree is set to the current short rate r_0.
- Inductively, the drift $\mu_{i,j}$ at node (i,j) is a function of $\theta(t_i)$.

The Hull-White Model: Calibration (continued)

- Once $\theta(t_i)$ is available, $\mu_{i,j}$ can be derived via Eq. (120) on p. 1004.
- This in turn determines the branching scheme at every node (i, j) for each j, as we will see shortly.
- The value of $\theta(t_i)$ must thus be made consistent with the spot rate $r(0, t_{i+2})$.

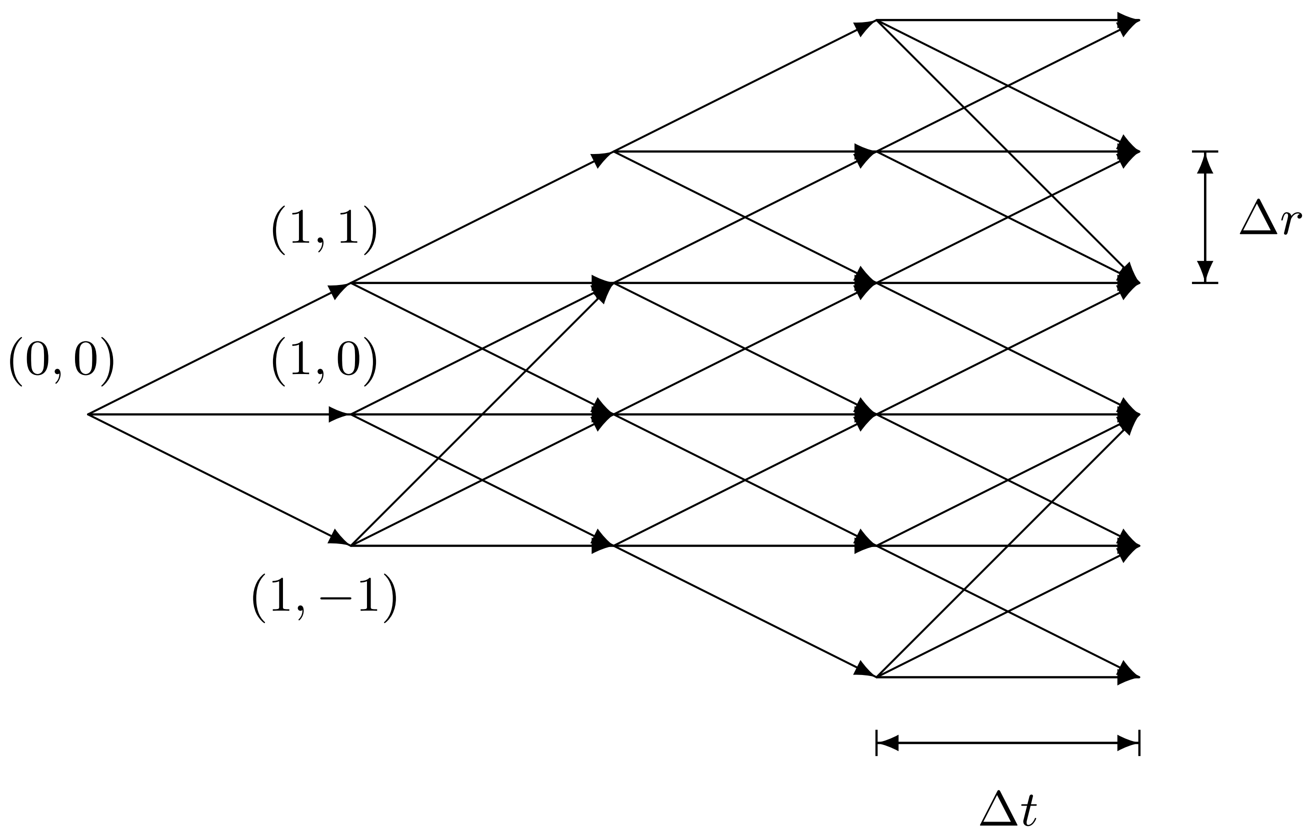
The Hull-White Model: Calibration (continued)

- The branches emanating from node (i, j) with their accompanying probabilities[a] must be chosen to be consistent with $\mu_{i,j}$ and σ.
- This is accomplished by letting the middle node be as close as possible to the current value of the short rate plus the drift.
- Let k be the number among $\{ j-1, j, j+1 \}$ that makes the short rate reached by the middle branch, r_k, closest to $r_j + \mu_{i,j} \Delta t$.

[a] $p_1(i, j)$, $p_2(i, j)$, and $p_3(i, j)$.

The Hull-White Model: Calibration (continued)

- Then the three nodes following node (i, j) are nodes $(i + 1, k + 1)$, $(i + 1, k)$, and $(i + 1, k - 1)$.
- The resulting tree may have the geometry depicted on p. 1010.
- The resulting tree combines because of the constant jump sizes to reach k.



(0, 0)
(1, 1)
(1, 0)
(1, −1)
Δr
Δt

The Hull-White Model: Calibration (continued)

- The probabilities for moving along these branches are functions of $\mu_{i,j}$, σ, j, and k:

$$p_1(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} + \frac{\eta}{2\Delta r} \tag{121}$$

$$p_2(i,j) = 1 - \frac{\sigma^2 \Delta t + \eta^2}{(\Delta r)^2} \tag{121$'$}$$

$$p_3(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} - \frac{\eta}{2\Delta r} \tag{121$''$}$$

where $\eta \equiv \mu_{i,j} \Delta t + (j-k)\, \Delta r$.

The Hull-White Model: Calibration (continued)

- As trinomial tree algorithms are but explicit methods in disguise, certain relations must hold for Δr and Δt to guarantee stability.
- It can be shown that their values must satisfy

$$\frac{\sigma\sqrt{3\Delta t}}{2} \leq \Delta r \leq 2\sigma\sqrt{\Delta t}$$

for the probabilities to lie between zero and one.

  - For example, Δr can be set to $\sigma\sqrt{3\Delta t}$.[a]

[a] Hull and White (1988).

The Hull-White Model: Calibration (continued)

- Now it only remains to determine $\theta(t_i)$.
- At this point at time t_i, $r(0, t_1)$, $r(0, t_2)$, $\ldots$, $r(0, t_{i+1})$ have already been matched.
- Let $Q(i, j)$ denote the value of the state contingent claim that pays one dollar at node (i, j) and zero otherwise.
- By construction, the state prices $Q(i, j)$ for all j are known by now.
- We begin with state price $Q(0, 0) = 1$.

The Hull-White Model: Calibration (continued)

- Let $\hat{r}(i)$ refer to the short rate value at time t_i.
- The value at time zero of a zero-coupon bond maturing at time t_{i+2} is then

$$
\begin{aligned}
& e^{-r(0,t_{i+2})(i+2)\,\Delta t} \\
= \; & \sum_j Q(i,j)\, e^{-r_j \Delta t}\, E^{\pi} \left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right]. \quad (122)
\end{aligned}
$$

- The right-hand side represents the value of \$1 obtained by holding a zero-coupon bond until time t_{i+1} and then reinvesting the proceeds at that time at the prevailing short rate $\hat{r}(i+1)$, which is stochastic.

The Hull-White Model: Calibration (continued)

- The expectation (122) can be approximated by

$$E^{\pi}\left[\, e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \approx e^{-r_j \Delta t}\left(1 - \mu_{i,j}(\Delta t)^2 + \frac{\sigma^2(\Delta t)^3}{2}\right). \quad (123)$$

- Substitute Eq. (123) into Eq. (122) and replace $\mu_{i,j}$ with $\theta(t_i) - a r_j$ to obtain

$$\theta(t_i) \approx \frac{\sum_j Q(i,j)\, e^{-2r_j \Delta t}\left(1 + a r_j (\Delta t)^2 + \sigma^2 (\Delta t)^3/2\right) - e^{-r(0,t_{i+2})(i+2)\,\Delta t}}{(\Delta t)^2 \sum_j Q(i,j)\, e^{-2r_j \Delta t}}.$$

The Hull-White Model: Calibration (continued)

- For the Hull-White model, the expectation in Eq. (123) on p. 1015 is actually known analytically by Eq. (18) on p. 151:

$$E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] = e^{-r_j \Delta t + (-\theta(t_i) + a r_j + \sigma^2 \Delta t/2)(\Delta t)^2}.$$

- Therefore, alternatively,

$$\theta(t_i) = \frac{r(0, t_{i+2})(i+2)}{\Delta t} + \frac{\sigma^2 \Delta t}{2} + \frac{\ln \sum_j Q(i,j)\, e^{-2 r_j \Delta t + a r_j (\Delta t)^2}}{(\Delta t)^2}.$$

The Hull-White Model: Calibration (concluded)

- With $\theta(t_i)$ in hand, we can compute $\mu_{i,j}$, the probabilities, and finally the state prices at time t_{i+1}:

$$Q(i+1, j) = \sum_{(i, j^*) \text{ is connected to } (i+1, j) \text{ with probability } p_{j^*}} p_{j^*} e^{-r_{j^*} \Delta t} Q(i, j^*).$$

- There are at most 5 choices for j^*.
- The total running time is $O(n^2)$.
- The space requirement is $O(n)$ (why?).

Comments on the Hull-White Model

- One can try different values of a and σ for each option or have an a value common to all options but use a different σ value for each option.
- Either approach can match all the option prices exactly.
- If the demand is for a single set of parameters that replicate all option prices, the Hull-White model can be calibrated to all the observed option prices by choosing a and σ that minimize the mean-squared pricing error.[a]

[a]Hull and White (1995).