Ito's Lemma (continued)

Theorem 18 (Alternative Ito's Lemma) *Let*
$W_1, W_2, \ldots, W_m$ *be Wiener processes and*
$X \equiv (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose*
$f : R^m \to R$ *is twice continuously differentiable and* X_i *is*
an Ito process with $dX_i = a_i \, dt + b_i \, dW_i$. *Then* $df(X)$ *is the*
following Ito process,

$$df(X) = \sum_{i=1}^{m} f_i(X) \, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X) \, dX_i \, dX_k.$$

Ito's Lemma (concluded)

- The multiplication table for Theorem 18 is

$\times$	dW_i	dt
dW_k	$\rho_{ik}\, dt$	0
dt	0	0

- Here, ρ_{ik} denotes the correlation between dW_i and dW_k.

Geometric Brownian Motion

- Consider the geometric Brownian motion process
  $Y(t) \equiv e^{X(t)}$

  - $X(t)$ is a (μ, σ) Brownian motion.

  - Hence $dX = \mu \, dt + \sigma \, dW$ by Eq. (47) on p. 464.

- As $\partial Y / \partial X = Y$ and $\partial^2 Y / \partial X^2 = Y$, Ito's formula (52) on p. 495 implies

$$
\begin{aligned}
dY &= Y \, dX + (1/2) \, Y \, (dX)^2 \\
&= Y \, (\mu \, dt + \sigma \, dW) + (1/2) \, Y \, (\mu \, dt + \sigma \, dW)^2 \\
&= Y \, (\mu \, dt + \sigma \, dW) + (1/2) \, Y \sigma^2 \, dt.
\end{aligned}
$$

Geometric Brownian Motion (concluded)

- Hence
$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right) dt + \sigma \, dW.$$

- The annualized instantaneous rate of return is $\mu + \sigma^2/2$ not μ.

Product of Geometric Brownian Motion Processes

- Let

$$
\begin{aligned}
dY/Y &= a\,dt + b\,dW_Y, \\
dZ/Z &= f\,dt + g\,dW_Z.
\end{aligned}
$$

- Consider the Ito process $U \equiv YZ$.

- Apply Ito's lemma (Theorem 18 on p. 501):

$$
\begin{aligned}
dU &= Z\,dY + Y\,dZ + dY\,dZ \\
&= ZY(a\,dt + b\,dW_Y) + YZ(f\,dt + g\,dW_Z) \\
&\quad + YZ(a\,dt + b\,dW_Y)(f\,dt + g\,dW_Z) \\
&= U(a + f + bg\rho)\,dt + Ub\,dW_Y + Ug\,dW_Z.
\end{aligned}
$$

Product of Geometric Brownian Motion Processes (continued)

- The product of two (or more) correlated geometric Brownian motion processes thus remains geometric Brownian motion.

- Note that

$$
\begin{aligned}
Y &= \exp\left[\left(a - b^2/2\right) dt + b\, dW_Y\right], \\
Z &= \exp\left[\left(f - g^2/2\right) dt + g\, dW_Z\right], \\
U &= \exp\left[\left(a + f - \left(b^2 + g^2\right)/2\right) dt + b\, dW_Y + g\, dW_Z\right].
\end{aligned}
$$

Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.

- This holds even if Y and Z are correlated.

- Finally, $\ln Y$ and $\ln Z$ have correlation ρ.

Quotients of Geometric Brownian Motion Processes

- Suppose Y and Z are drawn from p. 505.

- Let $U \equiv Y/Z$.

- We now show that[a]

$$\frac{dU}{U} = (a - f + g^2 - bg\rho)\,dt + b\,dW_Y - g\,dW_Z.$$
(54)

- Keep in mind that dW_Y and dW_Z have correlation ρ.

---

[a]Exercise 14.3.6 of the textbook is erroneous.

Quotients of Geometric Brownian Motion Processes (concluded)

- The multidimensional Ito's lemma (Theorem 18 on p. 501) can be employed to show that

$$
\begin{aligned}
& dU \\
= \; & (1/Z)\, dY - (Y/Z^2)\, dZ - (1/Z^2)\, dY\, dZ + (Y/Z^3)\,(dZ)^2 \\
= \; & (1/Z)(aY\, dt + bY\, dW_Y) - (Y/Z^2)(fZ\, dt + gZ\, dW_Z) \\
& -(1/Z^2)(bgYZ\rho\, dt) + (Y/Z^3)(g^2 Z^2\, dt) \\
= \; & U(a\, dt + b\, dW_Y) - U(f\, dt + g\, dW_Z) \\
& -U(bg\rho\, dt) + U(g^2\, dt) \\
= \; & U(a - f + g^2 - bg\rho)\, dt + Ub\, dW_Y - Ug\, dW_Z.
\end{aligned}
$$

Forward Price

- Suppose S follows

$$\frac{dS}{S} = \mu \, dt + \sigma \, dW.$$

- Consider $F(S,t) \equiv S e^{y(T-t)}$.

- Observe that

$$
\begin{aligned}
\frac{\partial F}{\partial S} &= e^{y(T-t)}, \\
\frac{\partial F}{\partial t} &= -y S e^{y(T-t)}.
\end{aligned}
$$

Forward Prices (concluded)

- Then

$$
\begin{aligned}
dF &= e^{y(T-t)}\, dS - y S e^{y(T-t)}\, dt \\
&= S e^{y(T-t)} \left(\mu\, dt + \sigma\, dW \right) - y S e^{y(T-t)}\, dt \\
&= F(\mu - y)\, dt + F\sigma\, dW
\end{aligned}
$$

by Eq. (53) on p. 500.

- Thus F follows

$$
\frac{dF}{F} = (\mu - y)\, dt + \sigma\, dW.
$$

- This result has applications in forward and futures contracts.[a]

---

[a]This is also consistent with p. 455.

Ornstein-Uhlenbeck Process

- The Ornstein-Uhlenbeck process:

$$dX = -\kappa X\, dt + \sigma\, dW,$$

where $\kappa, \sigma \geq 0$.

- It is known that

$$
\begin{aligned}
E[\,X(t)\,] &= e^{-\kappa(t-t_0)}\, E[\,x_0\,], \\[2mm]
\mathrm{Var}[\,X(t)\,] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-t_0)}\right) + e^{-2\kappa(t-t_0)}\,\mathrm{Var}[\,x_0\,], \\[2mm]
\mathrm{Cov}[\,X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa(t-s)}\left[\,1 - e^{-2\kappa(s-t_0)}\,\right] \\
&\quad + e^{-\kappa(t+s-2t_0)}\,\mathrm{Var}[\,x_0\,],
\end{aligned}
$$

for $t_0 \leq s \leq t$ and $X(t_0) = x_0$.

Ornstein-Uhlenbeck Process (continued)

- $X(t)$ is normally distributed if x_0 is a constant or normally distributed.

- X is said to be a normal process.

- $E[x_0] = x_0$ and $\mathrm{Var}[x_0] = 0$ if x_0 is a constant.

- The Ornstein-Uhlenbeck process has the following mean reversion property.

  - When $X > 0$, X is pulled toward zero.

  - When $X < 0$, it is pulled toward zero again.

Ornstein-Uhlenbeck Process (continued)

- Another version:

$$dX = \kappa(\mu - X)\,dt + \sigma\,dW,$$

  where $\sigma \geq 0$.

- Given $X(t_0) = x_0$, a constant, it is known that

$$
\begin{aligned}
E[\,X(t)\,] &= \mu + (x_0 - \mu)\,e^{-\kappa(t-t_0)}, &&(55)\\
\mathrm{Var}[\,X(t)\,] &= \frac{\sigma^2}{2\kappa}\left[\,1 - e^{-2\kappa(t-t_0)}\,\right],
\end{aligned}
$$

  for $t_0 \leq t$.

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly μ and $\sigma/\sqrt{2\kappa}$, respectively.

- For large t, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$.

- The process is mean-reverting.

  - X tends to move toward μ.

  - Useful for modeling term structure, stock price volatility, and stock price return.

Square-Root Process

- Suppose X is an Ornstein-Uhlenbeck process.

- Ito's lemma says $V \equiv X^2$ has the differential,

$$
\begin{aligned}
dV &= 2X\,dX + (dX)^2 \\
&= 2\sqrt{V}\,(-\kappa\sqrt{V}\,dt + \sigma\,dW) + \sigma^2\,dt \\
&= \left(-2\kappa V + \sigma^2\right)dt + 2\sigma\sqrt{V}\,dW,
\end{aligned}
$$

a square-root process.

Square-Root Process (continued)

- In general, the square-root process has the stochastic differential equation,

$$dX = \kappa(\mu - X)\,dt + \sigma\sqrt{X}\,dW,$$

  where $\kappa, \sigma \geq 0$ and the initial value of X is a nonnegative constant.

- Like the Ornstein-Uhlenbeck process, it possesses mean reversion: X tends to move toward μ, but the volatility is proportional to $\sqrt{X}$ instead of a constant.

Square-Root Process (continued)

- When X hits zero and $\mu \geq 0$, the probability is one that it will not move below zero.

    - Zero is a reflecting boundary.

- Hence, the square-root process is a good candidate for modeling interest rate movements.[a]

- The Ornstein-Uhlenbeck process, in contrast, allows negative interest rates.

- The two processes are related .

_____

[a]Cox, Ingersoll, and Ross (1985).

Square-Root Process (concluded)

- The random variable $2cX(t)$ follows the noncentral chi-square distribution,[a]

$$\chi\left(\frac{4\kappa\mu}{\sigma^2}, 2cX(0)\, e^{-\kappa t}\right),$$

  where $c \equiv (2\kappa/\sigma^2)(1 - e^{-\kappa t})^{-1}$.

- Given $X(0) = x_0$, a constant,

$$E[\,X(t)\,] = x_0 e^{-\kappa t} + \mu\left(1 - e^{-\kappa t}\right),$$

$$\mathrm{Var}[\,X(t)\,] = x_0\,\frac{\sigma^2}{\kappa}\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \mu\,\frac{\sigma^2}{2\kappa}\left(1 - e^{-\kappa t}\right)^2,$$

  for $t \geq 0$.

---

[a]William Feller (1906–1970) in 1951.

Continuous-Time Derivatives Pricing

I have hardly met a mathematician
who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Fischer [Black] is the only real genius
I've ever met in finance. Other people,
like Robert Merton or Stephen Ross,
are just very smart and quick,
but they think like me.
Fischer came from someplace else entirely.
— John C. Cox, quoted in Mehrling (2005)

Toward the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation.

- The key step is recognizing that the same random process drives both securities.

- As their prices are perfectly correlated, we figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.

- The removal of uncertainty forces the portfolio's return to be the riskless rate.

Assumptions

- The stock price follows $dS = \mu S\, dt + \sigma S\, dW$.

- There are no dividends.

- Trading is continuous, and short selling is allowed.

- There are no transactions costs or taxes.

- All securities are infinitely divisible.

- The term structure of riskless rates is flat at r.

- There is unlimited riskless borrowing and lending.

- t is the current time, T is the expiration time, and $\tau \equiv T - t$.

Black-Scholes Differential Equation

- Let C be the price of a derivative on S.

- From Ito's lemma (p. 497),

$$dC = \left(\mu S \, \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \, \sigma^2 S^2 \, \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \, \frac{\partial C}{\partial S} \, dW.$$

  − The same W drives both C and S.

- Short one derivative and long $\partial C / \partial S$ shares of stock (call it Π).

- By construction,

$$\Pi = -C + S(\partial C / \partial S).$$

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is[a]

$$d\Pi = -dC + \frac{\partial C}{\partial S}\, dS.$$

- Substitute the formulas for dC and dS into the partial differential equation to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r\Pi\, dt$.

---

[a]Mathematically speaking, it is not quite right (Bergman, 1982).

Black-Scholes Differential Equation (concluded)

- So

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt = r \left(C - S \frac{\partial C}{\partial S} \right) dt.$$

- Equate the terms to finally obtain

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$

- When there is a dividend yield q,

$$\frac{\partial C}{\partial t} + (r - q) S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rC. \qquad (56)$$

- Identity (56) leads to an alternative way of computing Θ numerically from Δ and Γ.

- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rC.$$

  – A definite relation thus exists between Γ and Θ.

[Black] got the equation [in 1969] but then was unable to solve it. Had he been a better physicist he would have recognized it as a form of the familiar heat exchange equation, and applied the known solution. Had he been a better mathematician, he could have solved the equation from first principles. Certainly Merton would have known exactly what to do with the equation had he ever seen it.

— Perry Mehrling (2005)

PDEs for Asian Options

- Add the new variable $A(t) \equiv \int_0^t S(u)\, du$.

- Then the value V of the Asian option satisfies this two-dimensional PDE:[a]

$$\frac{\partial V}{\partial t} + rS\, \frac{\partial V}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 V}{\partial S^2} + S\, \frac{\partial V}{\partial A} = rV.$$

- The terminal conditions are

$$
\begin{aligned}
V(T, S, A) &= \max\left(\frac{A}{T} - X, 0\right) &&\text{for call,} \\
V(T, S, A) &= \max\left(X - \frac{A}{T}, 0\right) &&\text{for put.}
\end{aligned}
$$

---

[a]Kemna and Vorst (1990).

PDEs for Asian Options (continued)

- The two-dimensional PDE produces algorithms similar to that on pp. 348ff.

- But one-dimensional PDEs are available for Asian options.[a]

- For example, Večeř (2001) derives the following PDE for Asian calls:

$$\frac{\partial u}{\partial t} + r \left(1 - \frac{t}{T} - z \right) \frac{\partial u}{\partial z} + \frac{\left(1 - \frac{t}{T} - z \right)^2 \sigma^2}{2} \frac{\partial^2 u}{\partial z^2} = 0$$

  with the terminal condition $u(T, z) = \max(z, 0)$.

---

[a]Rogers and Shi (1995); Večeř (2001); Dubois and Lelièvre (2005).

PDEs for Asian Options (concluded)

- For Asian puts:

$$\frac{\partial u}{\partial t} + r\left(\frac{t}{T} - 1 - z\right)\frac{\partial u}{\partial z} + \frac{\left(\frac{t}{T} - 1 - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

  with the same terminal condition.

- One-dimensional PDEs lead to highly efficient numerical methods.

Heston's Stochastic-Volatility Model[a]

- Heston assumes the stock price follows

$$
\begin{aligned}
\frac{dS}{S} &= (\mu - q)\, dt + \sqrt{V}\, dW_1, &\quad (57)\\
dV &= \kappa(\theta - V)\, dt + \sigma\sqrt{V}\, dW_2. &\quad (58)
\end{aligned}
$$

  - V is the instantaneous variance, which follows a square-root process.
  - dW_1 and dW_2 have correlation ρ.
  - The riskless rate r is constant.

- It may be the most popular continuous-time stochastic-volatility model.

---

[a]Heston (1993).

Heston's Stochastic-Volatility Model (continued)

- Heston assumes the market price of risk is $b_2\sqrt{V}$.

- So $\mu = r + b_2 V$.

- Define

$$
\begin{aligned}
dW_1^* &= dW_1 + b_2\sqrt{V}\,dt, \\
dW_2^* &= dW_2 + \rho b_2\sqrt{V}\,dt, \\
\kappa^* &= \kappa + \rho b_2 \sigma, \\
\theta^* &= \frac{\theta\kappa}{\kappa + \rho b_2 \sigma}.
\end{aligned}
$$

- dW_1^* and dW_2^* have correlation ρ.

- Under the risk-neutral probability measure Q, both W_1^* and W_2^* are Wiener processes.

Heston's Stochastic-Volatility Model (continued)

- Heston's model becomes, under probability measure Q,

$$
\begin{aligned}
\frac{dS}{S} &= (r - q)\, dt + \sqrt{V}\, dW_1^*, \\
dV &= \kappa^*(\theta^* - V)\, dt + \sigma\sqrt{V}\, dW_2^*.
\end{aligned}
$$

Heston's Stochastic-Volatility Model (continued)

- Define

$$
\begin{aligned}
\phi(u,\tau) \;=\;& \exp\Big\{\, \imath u(\ln S + (r-q)\,\tau) \\
& +\theta^*\kappa^*\sigma^{-2}\left[(\kappa^* - \rho\sigma u\imath - d)\,\tau - 2\ln\frac{1 - ge^{-d\tau}}{1 - g}\right] \\
& + \frac{v\sigma^{-2}(\kappa^* - \rho\sigma u\imath - d)\left(1 - e^{-d\tau}\right)}{1 - ge^{-d\tau}} \Big\}, \\
d \;=\;& \sqrt{(\rho\sigma u\imath - \kappa^*)^2 - \sigma^2(-\imath u - u^2)}\,, \\
g \;=\;& (\kappa^* - \rho\sigma u\imath - d)/(\kappa^* - \rho\sigma u\imath + d).
\end{aligned}
$$

Heston's Stochastic-Volatility Model (concluded)

The formulas are[a]

$$
\begin{aligned}
C &= S \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left(\frac{X^{-\imath u} \phi(u - \imath, \tau)}{\imath u S e^{r\tau}} \right) du \right] \\
&\quad - X e^{-r\tau} \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left(\frac{X^{-\imath u} \phi(u, \tau)}{\imath u} \right) du \right], \\
P &= X e^{-r\tau} \left[\frac{1}{2} - \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left(\frac{X^{-\imath u} \phi(u, \tau)}{\imath u} \right) du \right], \\
&\quad - S \left[\frac{1}{2} - \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left(\frac{X^{-\imath u} \phi(u - \imath, \tau)}{\imath u S e^{r\tau}} \right) du \right],
\end{aligned}
$$

where $\imath = \sqrt{-1}$ and $\mathrm{Re}(x)$ denotes the real part of the complex number x.

---

[a] Contributed by Mr. Chen, Chun-Ying (D95723006) on August 17, 2008 and Mr. Liou, Yan-Fu (R92723060) on August 26, 2008.

Stochastic-Volatility Models and Further Extensions[a]

- How to explain the October 1987 crash?

- Stochastic-volatility models require an implausibly high-volatility level prior to *and* after the crash.

- Merton (1976) proposed jump models.

- Discontinuous jump models *in the asset price* can alleviate the problem somewhat.

---

[a]Eraker (2004).

Stochastic-Volatility Models and Further Extensions (continued)

- But if the jump intensity is a constant, it cannot explain the tendency of large movements to cluster over time.

- This assumption also has no impacts on option prices.

- Jump-diffusion models combine both.
  - E.g., add a jump process to Eq. (57) on p. 532.

Stochastic-Volatility Models and Further Extensions (concluded)

- But they still do not adequately describe the systematic variations in option prices.[a]

- Jumps in volatility are alternatives.[b]

  - E.g., add correlated jump processes to Eqs. (57) and Eq. (58) on p. 532.

- Such models allow high level of volatility caused by a jump to volatility.[c]

---

[a]Bates (2000) and Pan (2002).
[b]Duffie, Pan, and Singleton (2000).
[c]Eraker, Johnnes, and Polson (2000).

Hedging

When Professors Scholes and Merton and I
invested in warrants,
Professor Merton lost the most money.
And I lost the least.
— Fischer Black (1938–1995)

Delta Hedge

- The delta (hedge ratio) of a derivative f is defined as $\Delta \equiv \partial f / \partial S$.

- Thus $\Delta f \approx \Delta \times \Delta S$ for relatively small changes in the stock price, ΔS.

- A delta-neutral portfolio is hedged in the sense that it is immunized against small changes in the stock price.

- A trading strategy that dynamically maintains a delta-neutral portfolio is called delta hedge.

Delta Hedge (concluded)

- Delta changes with the stock price.

- A delta hedge needs to be rebalanced periodically in order to maintain delta neutrality.

- In the limit where the portfolio is adjusted continuously, perfect hedge is achieved and the strategy becomes self-financing.

Implementing Delta Hedge

- We want to hedge N *short* derivatives.

- Assume the stock pays no dividends.

- The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus B borrowed dollars such that

$$-N \times f + N \times \Delta \times S - B = 0.$$

- At next rebalancing point when the delta is Δ', buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares with a total borrowing of $B' = N \times \Delta' \times S' - N \times f'$.

- Delta hedge is the discrete-time analog of the continuous-time limit and will rarely be self-financing.

Example

- A hedger is *short* 10,000 European calls.

- $\sigma = 30\%$ and $r = 6\%$.

- This call's expiration is four weeks away, its strike price is \$50, and each call has a current value of $f = 1.76791$.

- As an option covers 100 shares of stock, $N = 1,000,000$.

- The trader adjusts the portfolio weekly.

- The calls are replicated well if the cumulative cost of trading *stock* is close to the call premium's FV.[a]

---

[a]This example takes the replication viewpoint.

Example (continued)

- As $\Delta = 0.538560$, $N \times \Delta = 538,560$ shares are purchased for a total cost of $538{,}560 \times 50 = 26{,}928{,}000$ dollars to make the portfolio delta-neutral.

- The trader finances the purchase by borrowing

$$B = N \times \Delta \times S - N \times f = 25{,}160{,}090$$

  dollars net.[a]

- The portfolio has zero net value now.

---

[a]This takes the hedging viewpoint — an alternative. See an exercise in the text.

Example (continued)

- At 3 weeks to expiration, the stock price rises to $51.

- The new call value is $f' = 2.10580$.

- So the portfolio is worth

$$-N \times f' + 538{,}560 \times 51 - Be^{0.06/52} = 171{,}622$$

before rebalancing.

Example (continued)

- A delta hedge does not replicate the calls perfectly; it is not self-financing as $171,622 can be withdrawn.

- The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.

- In fact, the tracking error *over one rebalancing act* is positive about 68% of the time, but its expected value is essentially zero.[a]

- It is furthermore proportional to vega.

---

[a]Boyle and Emanuel (1980).

Example (continued)

- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.

- With a higher delta $\Delta' = 0.640355$, the trader buys $N \times (\Delta' - \Delta) = 101,795$ shares for \$5,191,545.

- The number of shares is increased to $N \times \Delta' = 640,355$.

Example (continued)

- The cumulative cost is

$$26{,}928{,}000 \times e^{0.06/52} + 5{,}191{,}545 = 32{,}150{,}634.$$

- The portfolio is again delta-neutral.

		Option value	Delta	Change in delta	No. shares bought	Cost of shares	Cumulative cost
τ	S	f	Δ		$N \times (5)$	$(1) \times (6)$	FV(8')+(7)
	(1)	(2)	(3)	(5)	(6)	(7)	(8)
4	50	1.7679	0.53856	—	538,560	26,928,000	26,928,000
3	51	2.1058	0.64036	0.10180	101,795	5,191,545	32,150,634
2	53	3.3509	0.85578	0.21542	215,425	11,417,525	43,605,277
1	52	2.2427	0.83983	−0.01595	−15,955	−829,660	42,825,960
0	54	4.0000	1.00000	0.16017	160,175	8,649,450	51,524,853

The total number of shares is 1,000,000 at expiration
(trading takes place at expiration, too).

Example (concluded)

- At expiration, the trader has 1,000,000 shares.

- They are exercised against by the in-the-money calls for $50,000,000.

- The trader is left with an obligation of

$$51{,}524{,}853 - 50{,}000{,}000 = 1{,}524{,}853,$$

  which represents the replication cost.

- Compared with the FV of the call premium,

$$1{,}767{,}910 \times e^{0.06 \times 4/52} = 1{,}776{,}088,$$

  the net gain is $1{,}776{,}088 - 1{,}524{,}853 = 251{,}235.$

Tracking Error Revisited

- Define the dollar gamma as $S^2\Gamma$.

- The change in value of a delta-hedged *long* option position after a duration of Δt is proportional to the dollar gamma.

- It is about

$$(1/2)S^2\Gamma[\,(\Delta S/S)^2 - \sigma^2\Delta t\,].$$

  – $(\Delta S/S)^2$ is called the daily realized variance.

Tracking Error Revisited (continued)

- Let the rebalancing times be $t_1, t_2, \ldots, t_n$.

- Let $\Delta S_i = S_{i+1} - S_i$.

- The total tracking error at expiration is about

$$\sum_{i=0}^{n-1} e^{r(T-t_i)} \frac{S_i^2 \Gamma_i}{2} \left[\left(\frac{\Delta S_i}{S_i} \right)^2 - \sigma^2 \Delta t \right],$$

- The tracking error is path dependent.

Tracking Error Revisited (concluded)[a]

- The tracking error ϵ_n *over n rebalancing acts* (such as 251,235 on p. 552) has about the same probability of being positive as being negative.

- Subject to certain regularity conditions, the root-mean-square tracking error $\sqrt{E[\,\epsilon_n^2\,]}$ is $O(1/\sqrt{n}\,)$.[b]

- The root-mean-square tracking error increases with σ at first and then decreases.

---

[a]Bertsimas, Kogan, and Lo (2000).
[b]See also Grannan and Swindle (1996).

Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, Δf, due to changes in the stock price, ΔS.

- When ΔS is not small, the second-order term, gamma $\Gamma \equiv \partial^2 f / \partial S^2$, helps (theoretically).

- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.

- To meet this extra condition, one more security needs to be brought in.

Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.

- A hedging call f_2 is brought in.

- To set up a delta-gamma hedge, we solve

$$
\begin{aligned}
-N \times f + n_1 \times S + n_2 \times f_2 - B &= 0 \quad \text{(self-financing)}, \\
-N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 &= 0 \quad \text{(delta neutrality)}, \\
-N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 &= 0 \quad \text{(gamma neutrality)},
\end{aligned}
$$

  for n_1, n_2, and B.

  – The gammas of the stock and bond are 0.

Other Hedges

- If volatility changes, delta-gamma hedge may not work well.

- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.

- To accomplish this, one more security has to be brought into the process.

- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.

Trees

I love a tree more than a man.
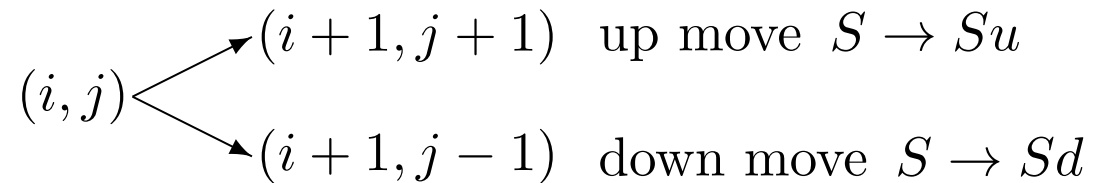— Ludwig van Beethoven (1770–1827)

And though the holes were rather small,
they had to count them all.
— The Beatles, *A Day in the Life* (1967)

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.

- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.

- We first used this method in the linear-time algorithm for standard European option pricing on p. 240.

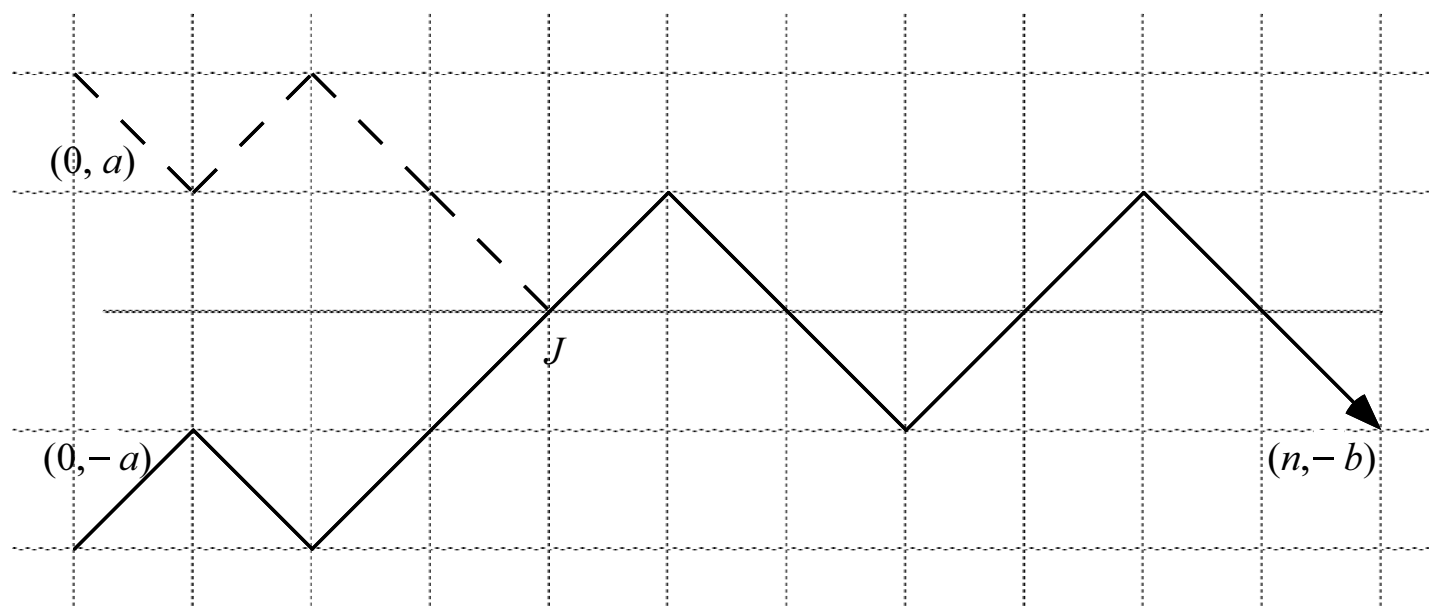- We will now apply it to price barrier options.

The Reflection Principle[a]

- Imagine a particle at position $(0, -a)$ on the integral lattice that is to reach $(n, -b)$.

- Without loss of generality, assume $a > 0$ and $b \geq 0$.

- This particle's movement:

$$(i, j) \begin{array}{c} \nearrow (i+1, j+1) \quad \text{up move } S \to Su \\ \\ \searrow (i+1, j-1) \quad \text{down move } S \to Sd \end{array}$$

- How many paths touch the x axis?

---

[a]André (1887).

$(0, a)$

$(0, -a)$

J

$(n, -b)$

The Reflection Principle (continued)

- For a path from $(0, -a)$ to $(n, -b)$ that touches the x axis, let J denote the first point this happens.

- Reflect the portion of the path from $(0, -a)$ to J.

- A path from $(0, a)$ to $(n, -b)$ is constructed.

- It also hits the x axis at J for the first time.

- The one-to-one mapping shows the number of paths from $(0, -a)$ to $(n, -b)$ that touch the x axis equals the number of paths from $(0, a)$ to $(n, -b)$.

The Reflection Principle (concluded)

- A path of this kind has $(n + b + a)/2$ down moves and $(n - b - a)/2$ up moves.

- Hence there are

$$
\binom{n}{\frac{n+a+b}{2}}
\tag{59}
$$

  such paths for even $n + a + b$.

  – Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.

- Assume $H < S$ without loss of generality.

- Define

$$
\begin{aligned}
a &\equiv \left\lceil \frac{\ln\left(X/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil, \\
h &\equiv \left\lfloor \frac{\ln\left(H/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor.
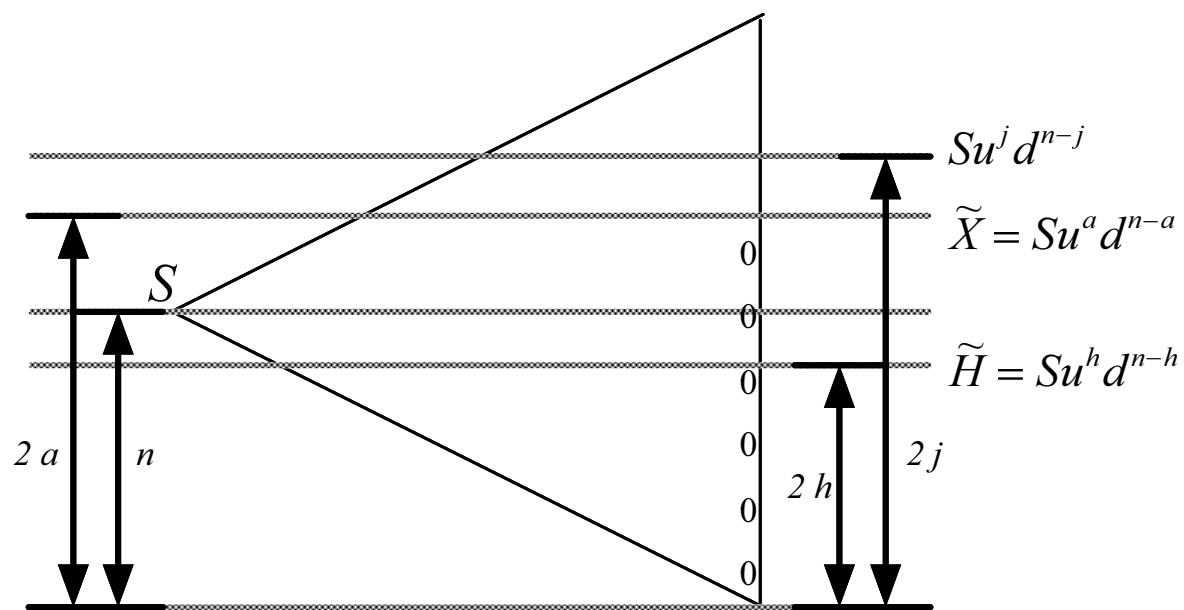\end{aligned}
$$

- h is such that $\tilde{H} \equiv Su^h d^{n-h}$ is the *terminal* price that is closest to, but does not exceed H.

- a is such that $\tilde{X} \equiv Su^a d^{n-a}$ is the terminal price that is closest to, but is not exceeded by X.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.

- A process with n moves hence ends up in the money if and only if the number of up moves is at least a.

- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price Sd^n on the binomial tree.

  –

$$Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-2k}. \qquad (60)$$

$Su^j d^{n-j}$

$\widetilde{X} = Su^a d^{n-a}$

$\widetilde{H} = Su^h d^{n-h}$

S

$2\,a$

n

$2\,h$

$2\,j$

0

0

0

0

0

0

Pricing Barrier Options (continued)

- The number of paths from S to the terminal price $Su^j d^{n-j}$ is $\binom{n}{j}$, each with probability $p^j(1-p)^{n-j}$.

- With reference to p. 568, the reflection principle can be applied with $\boldsymbol{a} = n - 2h$ and $\boldsymbol{b} = 2j - 2h$ in Eq. (59) on p. 565 by treating the $\tilde{H}$ line as the x axis.

- Therefore,

$$\binom{n}{\frac{n+(n-2h)+(2j-2h)}{2}} = \binom{n}{n - 2h + j}$$

paths hit $\tilde{H}$ in the process for $h \leq n/2$.

Pricing Barrier Options (concluded)

- The terminal price $S u^j d^{n-j}$ is reached by a path that hits the effective barrier with probability

$$\binom{n}{n-2h+j} p^j (1-p)^{n-j}.$$

- The option value equals

$$\frac{\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1-p)^{n-j} \left(S u^j d^{n-j} - X \right)}{R^n}. \qquad (61)$$

  - $R \equiv e^{r\tau/n}$ is the riskless return per period.

- It implies a linear-time algorithm.

Convergence of BOPM

- Equation (61) results in the sawtooth-like convergence shown on p. 329.

- The reasons are not hard to see.

- The true barrier most likely does not equal the effective barrier.

- The same holds for the strike price and the effective strike price.

- The issue of the strike price is less critical.

- But the issue of the barrier is not negligible.

Convergence of BOPM (continued)

- Convergence is actually good if we limit n to certain values—191, for example.

- These values make the true barrier coincide with or just above one of the stock price levels, that is,

$$H \approx S d^j = S e^{-j\sigma\sqrt{\tau/n}}$$

  for some integer j.

- The preferred n's are thus

$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

Convergence of BOPM (continued)

- There is only one minor technicality left.

- We picked the effective barrier to be one of the $n + 1$ possible terminal stock prices.

- However, the effective barrier above, Sd^j, corresponds to a terminal stock price only when $n - j$ is even.[a]

- To close this gap, we decrement n by one, if necessary, to make $n - j$ an even number.

---

[a]This is because $j = n - 2k$ for some k by Eq. (60) on p. 567. Of course we could have adopted the form Sd^j $(-n \leq j \leq n)$ for the effective barrier.
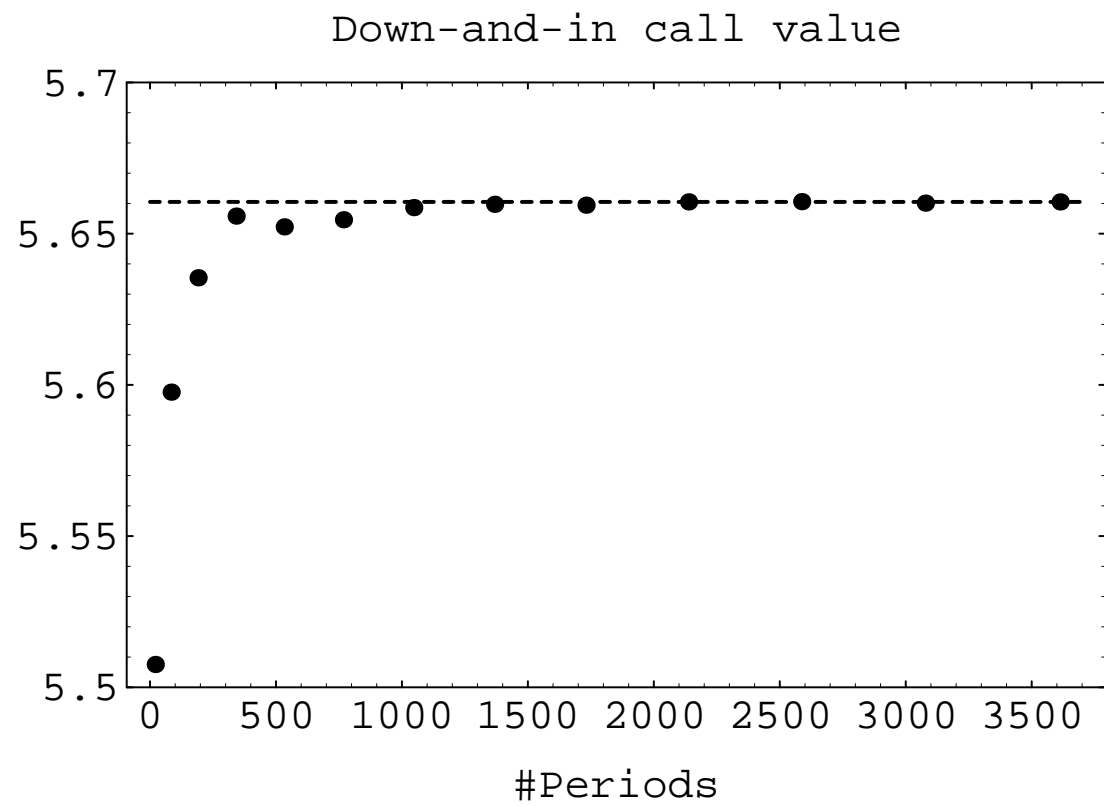
Convergence of BOPM (concluded)

- The preferred n's are now

$$
n = \begin{cases} \ell & \text{if } \ell - j \text{ is even} \\ \ell - 1 & \text{otherwise} \end{cases},
$$

  $j = 1, 2, 3, \ldots$, where

$$
\ell \equiv \left\lfloor \frac{\tau}{\left(\ln(S/H)/(j\sigma)\right)^2} \right\rfloor.
$$

- Evaluate pricing formula (61) on p. 570 only with the n's above.

Down-and-in call value

Practical Implications

- Now that barrier options can be efficiently priced, we can afford to pick very large n's (p. 577).

- This has profound consequences.

n	Combinatorial method	
	Value	Time (milliseconds)
21	5.507548	0.30
84	5.597597	0.90
191	5.635415	2.00
342	5.655812	3.60
533	5.652253	5.60
768	5.654609	8.00
1047	5.658622	11.10
1368	5.659711	15.00
1731	5.659416	19.40
2138	5.660511	24.70
2587	5.660592	30.20
3078	5.660099	36.70
3613	5.660498	43.70
4190	5.660388	44.10
4809	5.659955	51.60
5472	5.660122	68.70
6177	5.659981	76.70
6926	5.660263	86.90
7717	5.660272	97.20

Practical Implications (concluded)

- Pricing is prohibitively time consuming when $S \approx H$ because $n \sim 1/\ln^2(S/H)$.

  - This is called the barrier-too-close problem.

- This observation is indeed true of standard quadratic-time binomial tree algorithms.

- But it no longer applies to linear-time algorithms (see p. 579).

- In fact, this model is $O(1/n)$ convergent.[a]

---

[a]Lin (`R95221010`) (2008).

	Barrier at 95.0			Barrier at 99.5			Barrier at 99.9	
n	Value	Time	n	Value	Time	n	Value	Time
	.							
	.							
	.		795	7.47761	8	19979	8.11304	253
2743	2.56095	31.1	3184	7.47626	38	79920	8.11297	1013
3040	2.56065	35.5	7163	7.47682	88	179819	8.11300	2200
3351	2.56098	40.1	12736	7.47661	166	319680	8.11299	4100
3678	2.56055	43.8	19899	7.47676	253	499499	8.11299	6300
4021	2.56152	48.1	28656	7.47667	368	719280	8.11299	8500
True	2.5615			7.4767			8.1130	

(All times in milliseconds.)