Example

- Consider the stochastic process
$$\{ Z_n \equiv \sum_{i=1}^{n} X_i, n \geq 1 \},$$
where X_i are independent random variables with zero mean.

- This process is a martingale because
$$\begin{aligned} & E[\, Z_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \;& E[\, Z_n + X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \;& E[\, Z_n \,|\, Z_1, Z_2, \ldots, Z_n \,] + E[\, X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \;& Z_n + E[\, X_{n+1} \,] = Z_n. \end{aligned}$$

Probability Measure

- A probability measure assigns probabilities to states of the world.
- A martingale is defined with respect to a probability measure, under which the expectation is taken.
- A martingale is also defined with respect to an information set.
  - In the characterizations (41)–(42) on p. 437, the information set contains the current and past values of X by default.
  - But it need not be so.

Probability Measure (continued)

- A stochastic process $\{ X(t), t \geq 0 \}$ is a martingale with respect to information sets $\{ I_t \}$ if, for all $t \geq 0$, $E[\, | X(t) | \,] < \infty$ and

$$E[\, X(u) \,|\, I_t \,] = X(t)$$

for all $u > t$.

- The discrete-time version: For all $n > 0$,

$$E[\, X_{n+1} \,|\, I_n \,] = X_n,$$

given the information sets $\{ I_n \}$.

Probability Measure (concluded)

- The above implies $E[\, X_{n+m} \,|\, I_n \,] = X_n$ for any $m > 0$ by Eq. (16) on p. 143.
  - A typical I_n is the price information up to time n.
  - Then the above identity says the FVs of X will not deviate systematically from today's value given the price history.

Example

- Consider the stochastic process $\{ Z_n - n\mu, n \geq 1 \}$.
  - $Z_n \equiv \sum_{i=1}^{n} X_i$.
  - $X_1, X_2, \ldots$ are independent random variables with mean μ.
- Now,

$$
\begin{aligned}
& E[\, Z_{n+1} - (n+1)\,\mu \,|\, X_1, X_2, \ldots, X_n \,] \\
= \;& E[\, Z_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\
= \;& E[\, Z_n + X_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\
= \;& Z_n + \mu - (n+1)\,\mu \\
= \;& Z_n - n\mu.
\end{aligned}
$$

Example (concluded)

- Define

$$I_n \equiv \{ X_1, X_2, \ldots, X_n \}.$$

- Then

$$\{ Z_n - n\mu, n \geq 1 \}$$

is a martingale with respect to $\{ I_n \}$.

Martingale Pricing

- Recall that the price of a European option is the expected discounted future payoff at expiration in a risk-neutral economy.
- This principle can be generalized using the concept of martingale.
- Recall the recursive valuation of European option via

$$C = [\, pC_u + (1 - p)\, C_d\,]/R.$$

  - p is the risk-neutral probability.
  - \$1 grows to \$$R$ in a period.

Martingale Pricing (continued)

- Let $C(i)$ denote the value of the option at time i.
- Consider the discount process

$$\left\{ \frac{C(i)}{R^i}, i = 0, 1, \ldots, n \right\}.$$

- Then,

$$E\left[\left. \frac{C(i+1)}{R^{i+1}} \right| C(i) = C \right] = \frac{pC_u + (1-p)\, C_d}{R^{i+1}} = \frac{C}{R^i}.$$

Martingale Pricing (continued)

- It is easy to show that

$$E\left[\frac{C(k)}{R^k} \,\middle|\, C(i) = C \right] = \frac{C}{R^i}, \quad i \leq k. \tag{44}$$

- This formulation assumes:[a]
  1. The model is Markovian: The distribution of the future is determined by the present (time i) and not the past.
  2. The payoff depends only on the terminal price of the underlying asset (Asian options do not qualify).

---

[a]Contributed by Mr. Wang, Liang-Kai (Ph.D. student, ECE, University of Wisconsin-Madison) and Mr. Hsiao, Huan-Wen (B90902081) on May 3, 2006.

Martingale Pricing (continued)

- In general, the discount process is a martingale in that

$$E_i^{\pi}\left[\frac{C(k)}{R^k}\right] = \frac{C(i)}{R^i}, \quad i \le k. \tag{45}$$

  - E_i^{π} is taken under the risk-neutral probability conditional on the price information up to time i.

- This risk-neutral probability is also called the EMM, or the equivalent martingale (probability) measure.

Martingale Pricing (continued)

- Equation (45) holds for all assets, not just options.
- When interest rates are stochastic, the equation becomes

$$\frac{C(i)}{M(i)} = E_i^{\pi}\left[\frac{C(k)}{M(k)}\right], \quad i \leq k. \tag{46}$$

  - $M(j)$ is the balance in the money market account at time j using the rollover strategy with an initial investment of \$1.
  - So it is called the bank account process.
- It says the discount process is a martingale under π.

Martingale Pricing (continued)

- If interest rates are stochastic, then $M(j)$ is a random variable.
  - $M(0) = 1$.
  - $M(j)$ is known at time $j - 1$.
- Identity (46) on p. 452 is the general formulation of risk-neutral valuation.

Martingale Pricing (concluded)

Theorem 14 *A discrete-time model is arbitrage-free if and only if there exists a probability measure such that the discount process is a martingale. This probability measure is called the risk-neutral probability measure.*

Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability.
  - The expected futures price in the next period is

$$p_{\mathrm{f}} Fu + (1 - p_{\mathrm{f}})\, Fd = F\left(\frac{1-d}{u-d}\, u + \frac{u-1}{u-d}\, d \right) = F$$

(p. 412).

- Can be generalized to

$$F_i = E_i^{\pi}[\, F_k \,], \quad i \le k,$$

where F_i is the futures price at time i.

- It holds under stochastic interest rates, too.

Martingale Pricing and Numeraire[a]

- The martingale pricing formula (46) on p. 452 uses the money market account as numeraire.[b]
  - It expresses the price of any asset *relative to* the money market account.
- The money market account is not the only choice for numeraire.
- Suppose asset S's value is positive at all times.

[a] John Law (1671–1729), "Money to be qualified for exchaning goods and for payments need not be certain in its value."

[b] Leon Walras (1834–1910).

Martingale Pricing and Numeraire (concluded)

- Choose S as numeraire.
- Martingale pricing says there exists a risk-neutral probability π under which the relative price of any asset C is a martingale:

$$\frac{C(i)}{S(i)} = E_i^{\pi}\left[\frac{C(k)}{S(k)}\right], \quad i \leq k.$$

  - $S(j)$ denotes the price of S at time j.
- So the discount process remains a martingale.

Example

- Take the binomial model with two assets.
- In a period, asset one's price can go from S to S_1 or S_2.
- In a period, asset two's price can go from P to P_1 or P_2.
- Assume

$$\frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2}$$

to rule out arbitrage opportunities.

Example (continued)

- For any derivative security, let C_1 be its price at time one if asset one's price moves to S_1.
- Let C_2 be its price at time one if asset one's price moves to S_2.
- Replicate the derivative by solving

$$\begin{aligned} \alpha S_1 + \beta P_1 &= C_1, \\ \alpha S_2 + \beta P_2 &= C_2, \end{aligned}$$

using α units of asset one and β units of asset two.

Example (continued)

- This yields

$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \quad \text{and} \quad \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}.$$

- The derivative costs

$$\begin{aligned} C &= \alpha S + \beta P \\ &= \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2. \end{aligned}$$

Example (concluded)

- It is easy to verify that

$$\frac{C}{P} = p\,\frac{C_1}{P_1} + (1-p)\,\frac{C_2}{P_2}.$$

  – Above,

$$p \equiv \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}.$$

- The derivative's price using asset two as numeraire (i.e., C/P) is a martingale under the risk-neutral probability p.
- The expected returns of the two assets are irrelevant.

Brownian Motion[a]

- Brownian motion is a stochastic process $\{ X(t), t \geq 0 \}$ with the following properties.

  1. $X(0) = 0$, unless stated otherwise.

  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

  $$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.[b]

  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where μ and $\sigma \neq 0$ are real numbers.

---

[a] Robert Brown (1773–1858).

[b] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

Brownian Motion (concluded)

- Such a process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2.
- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is also called the Wiener process.

[a] Norbert Wiener (1894–1964).

Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then $X(t) - X(s) \sim N(0, t-s)$.
- A (μ, σ) Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \tag{47}$$

- Note that $Y(t+s) - Y(t) \sim N(\mu s, \sigma^2 s)$.

Brownian Motion as Limit of Random Walk

Claim 1 *A* (μ, σ) *Brownian motion is the limiting case of random walk.*

- A particle moves Δx to the left with probability $1 - p$.
- It moves to the right with probability p after Δt time.
- Assume $n \equiv t / \Delta t$ is an integer.
- Its position at time t is

$$Y(t) \equiv \Delta x \left(X_1 + X_2 + \cdots + X_n \right).$$

Brownian Motion as Limit of Random Walk (continued)

- (continued)
  - Here
    $$X_i \equiv \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$
  - X_i are independent with
    $\text{Prob}[\,X_i = 1\,] = p = 1 - \text{Prob}[\,X_i = -1\,]$.
- Recall $E[\,X_i\,] = 2p - 1$ and $\text{Var}[\,X_i\,] = 1 - (2p-1)^2$.

Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$
$$\mathrm{Var}[\,Y(t)\,] = n(\Delta x)^2\left[\,1-(2p-1)^2\,\right].$$

- With $\Delta x \equiv \sigma\sqrt{\Delta t}$ and $p \equiv [\,1+(\mu/\sigma)\sqrt{\Delta t}\,]/2$,

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \mathrm{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1-(\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t, \end{aligned}$$

as $\Delta t \to 0$.

Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{ Y(t), t \geq 0 \}$ converges to a (μ, σ) Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Note that

$$\begin{aligned} & \text{Var}[\, Y(t + \Delta t) - Y(t)\,] \\ =& \text{Var}[\, \Delta x\, X_{n+1}\,] = (\Delta x)^2 \times \text{Var}[\, X_{n+1}\,] \to \sigma^2 \Delta t. \end{aligned}$$

- Similarity to the the BOPM: The p is identical to the probability in Eq. (24) on p. 248 and $\Delta x = \ln u$.

Geometric Brownian Motion

- Let $X \equiv \{ X(t), t \geq 0 \}$ be a Brownian motion process.
- The process

$$\{ Y(t) \equiv e^{X(t)}, t \geq 0 \},$$

is called geometric Brownian motion.

- Suppose further that X is a (μ, σ) Brownian motion.
- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function

$$E\left[e^{sX(t)} \right] = E\left[Y(t)^s \right] = e^{\mu t s + (\sigma^2 t s^2/2)}$$

from Eq. (17) on p 145.

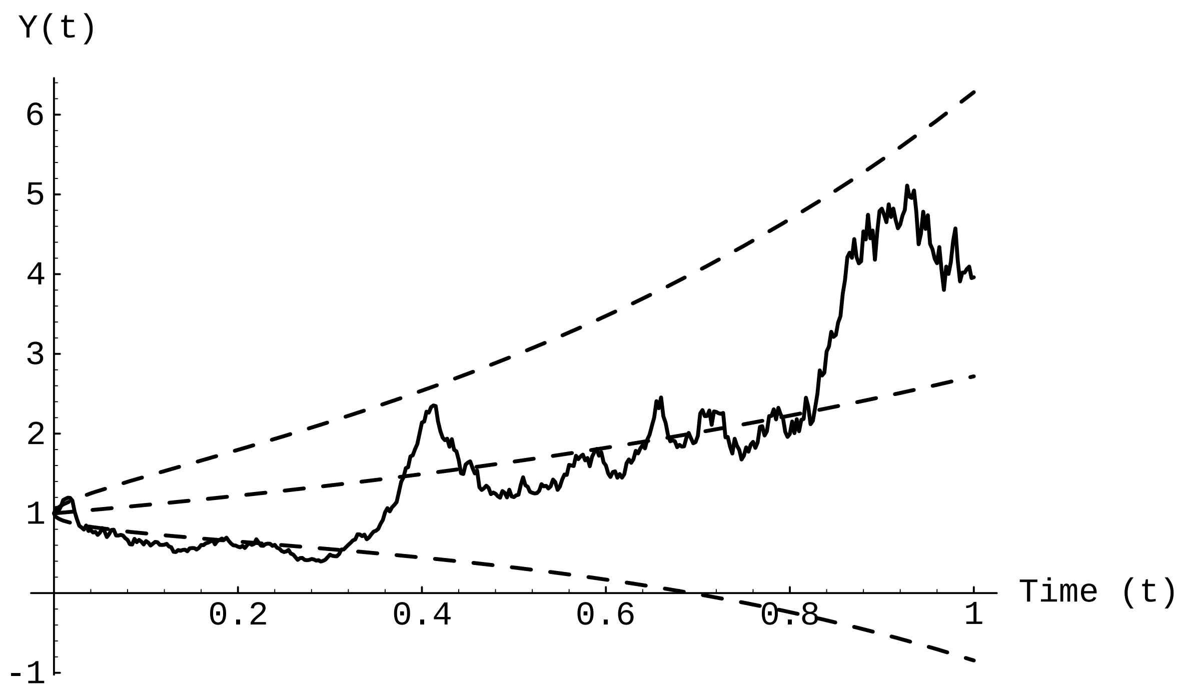
Geometric Brownian Motion (continued)

- In particular,

$$E[\,Y(t)\,] = e^{\mu t + (\sigma^2 t/2)},$$

$$\mathrm{Var}[\,Y(t)\,] = E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2$$

$$= e^{2\mu t + \sigma^2 t}\left(e^{\sigma^2 t} - 1\right).$$



Y(t)
6
5
4
3
2
1
-1
0.2
0.4
0.6
0.8
1
Time (t)

Geometric Brownian Motion (continued)

- It is useful for situations in which percentage changes are independent and identically distributed.
- Let Y_n denote the stock price at time n and $Y_0 = 1$.
- Assume relative returns

$$X_i \equiv \frac{Y_i}{Y_{i-1}}$$

are independent and identically distributed.

Geometric Brownian Motion (concluded)

- Then

$$\ln Y_n = \sum_{i=1}^{n} \ln X_i$$

  is a sum of independent, identically distributed random variables.

- Thus $\{ \ln Y_n, n \geq 0 \}$ is approximately Brownian motion.
  - And $\{ Y_n, n \geq 0 \}$ is approximately geometric Brownian motion.

Continuous-Time Financial Mathematics

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
Science and the Modern World

Stochastic Integrals

- Use $W \equiv \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of X from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \equiv \int_0^t X \, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of X with respect to W.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X \, dW$.

[a] Kiyoshi Ito (1915–2008).

Stochastic Integrals (concluded)

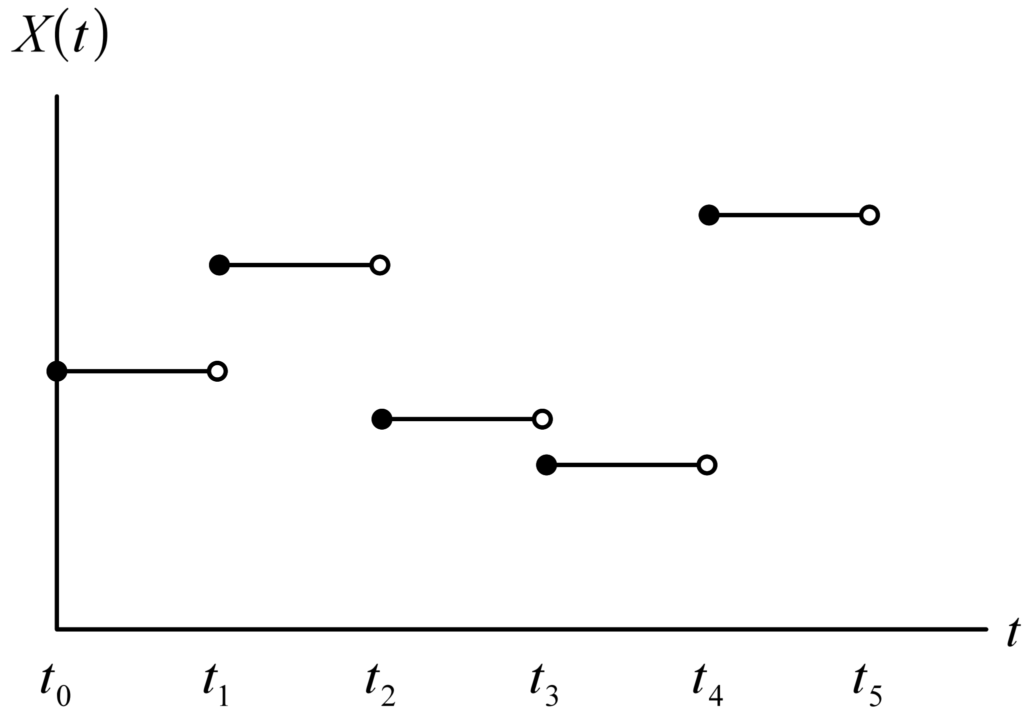
- Typical requirements for X in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\, ds < \infty$.
  - The information set at time t includes the history of X and W up to that point in time.
  - But it contains nothing about the evolution of X or W after t (nonanticipating, so to speak).
  - The future cannot influence the present.
- $\{X(s), 0 \leq s \leq t\}$ is independent of $\{W(t+u) - W(t), u > 0\}$.

Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist $0 = t_0 < t_1 < \cdots$ such that

$$X(t) = X(t_{k-1}) \text{ for } t \in [\, t_{k-1}, t_k), \; k = 1, 2, \ldots$$

for any realization (see figure on next page).



X(t)
t
t0
t1
t2
t3
t4
t5

Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \equiv \sum_{k=0}^{n-1} X(t_k)[\, W(t_{k+1}) - W(t_k)\,], \tag{48}$$

where $t_n = t$.

  - The integrand X is evaluated at t_k, not t_{k+1}.

- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (continued)

- Let $X = \{ X(t), t \geq 0 \}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that X_n converges in probability to X.
- If X is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as $\delta_n \equiv \max_{1 \leq k \leq n}(t_k - t_{k-1})$ goes to zero.

Ito Integral (concluded)

- It is a fundamental fact that $\int X\,dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.
  - A corollary is the mean value formula

$$E\left[\int_a^b X\,dW\right] = 0.$$

Theorem 15 *The Ito integral* $\int X\,dW$ *is a martingale.*

Discrete Approximation

- Recall Eq. (48) on p. 480.
- The following simple stochastic process $\{ \widehat{X}(t) \}$ can be used in place of X to approximate the stochastic integral $\int_0^t X \, dW$,

$$\widehat{X}(s) \equiv X(t_{k-1}) \text{ for } s \in [\, t_{k-1}, t_k), \, k = 1, 2, \ldots, n.$$

- Note the nonanticipating feature of $\widehat{X}$.
  - The information up to time s,

$$\{ \widehat{X}(t), W(t), 0 \leq t \leq s \},$$

    cannot determine the future evolution of X or W.

Discrete Approximation (concluded)

- Suppose we defined the stochastic integral as

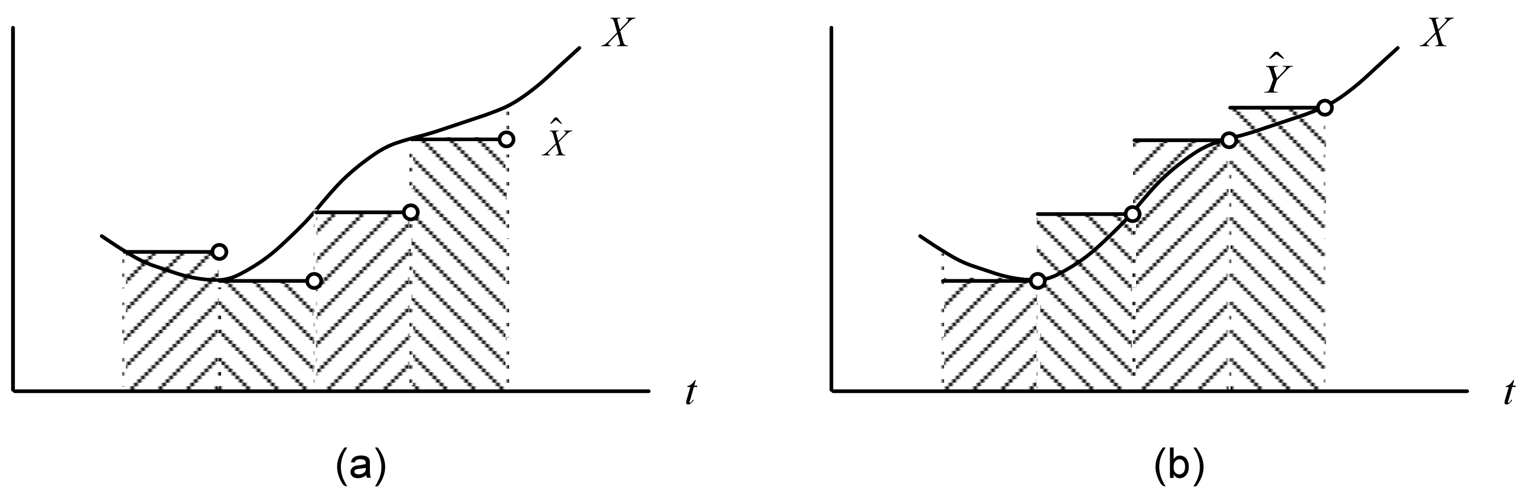
$$\sum_{k=0}^{n-1} X(t_{k+1})[\,W(t_{k+1}) - W(t_k)\,].$$

- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \equiv X(t_k) \quad \text{for} \quad s \in [\,t_{k-1}, t_k), \; k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of X.[a]

[a]See Exercise 14.1.2 of the textbook for an example where it matters.



X
$\hat{X}$
t
(a)
X
$\hat{Y}$
t
(b)

Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

is called an Ito process.

  - X_0 is a scalar starting point.
  - $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.

- The terms $a(X_t, t)$ and $b(X_t, t)$ are the drift and the diffusion, respectively.

Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{49}$$

  – Or simply $dX_t = a_t\, dt + b_t\, dW_t$.

- This is Brownian motion with an instantaneous drift a_t and an instantaneous variance b_t^2.

- X is a martingale if the drift a_t is zero by Theorem 15 (p. 482).

[a]Paul Langevin (1904).

Ito Process (concluded)

- dW is normally distributed with mean zero and variance dt.
- An equivalent form to Eq. (49) is

$$dX_t = a_t\, dt + b_t \sqrt{dt}\; \xi, \tag{50}$$

where $\xi \sim N(0,1)$.

Euler Approximation

- The following approximation follows from Eq. (50),

$$\begin{aligned}&\widehat{X}(t_{n+1})\\=&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\,\Delta W(t_n),\end{aligned} \tag{51}$$

  where $t_n \equiv n\Delta t$.

- It is called the Euler or Euler-Maruyama method.
- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.
- Recall that $\Delta W(t_n)$ should be interpreted as $W(t_{n+1}) - W(t_n)$ instead of $W(t_n) - W(t_{n-1})$.

More Discrete Approximations

- Under fairly loose regularity conditions, approximation (51) on p. 489 can be replaced by

$$
\begin{aligned}
&\widehat{X}(t_{n+1})\\
=&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,Y(t_n).
\end{aligned}
$$

  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

$$\begin{aligned}&\widehat{X}(t_{n+1})\\=&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\sqrt{\Delta t}\,\xi.\end{aligned}$$

  - $\text{Prob}[\,\xi=1\,]=\text{Prob}[\,\xi=-1\,]=1/2$.
  - Note that $E[\,\xi\,]=0$ and $\text{Var}[\,\xi\,]=1$.
- This clearly defines a binomial model.
- As Δt goes to zero, $\widehat{X}$ converges to X.

Trading and the Ito Integral

- Consider an Ito process $d\boldsymbol{S}_t = \mu_t \, dt + \sigma_t \, dW_t$.
  - $\boldsymbol{S}_t$ is the vector of security prices at time t.
- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time t.
  - Hence the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time t.
- $\boldsymbol{\phi}_t \, d\boldsymbol{S}_t \equiv \boldsymbol{\phi}_t(\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time t.

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \equiv \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

  measures the gains realized by the trading strategy over the period $[\,0, T\,]$.

Ito's Lemma

A smooth function of an Ito process is itself an Ito process.

Theorem 16 *Suppose* $f : R \to R$ *is twice continuously differentiable and* $dX = a_t\, dt + b_t\, dW$. *Then* $f(X)$ *is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \; & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

for $t \geq 0$.

Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$df(X) = f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2}\, f''(X)\, b^2\, dt. \tag{52}$$

- Compared with calculus, the interesting part is the third term on the right-hand side.
- A convenient formulation of Ito's lemma is

$$df(X) = f'(X)\, dX + \frac{1}{2}\, f''(X)(dX)^2.$$

Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

$\times$	dW	dt
dW	dt	0
dt	0	0

  - The $(dW)^2 = dt$ entry is justified by a known result.

- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (continued)

Theorem 17 (Higher-Dimensional Ito's Lemma) *Let $W_1, W_2, \ldots, W_n$ be independent Wiener processes and $X \equiv (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : R^m \to R$ is twice continuously differentiable and X_i is an Ito process with $dX_i = a_i\,dt + \sum_{j=1}^{n} b_{ij}\,dW_j$. Then $df(X)$ is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\,dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\,dX_i\,dX_k,$$

where $f_i \equiv \partial f/\partial x_i$ and $f_{ik} \equiv \partial^2 f/\partial x_i \partial x_k$.

Ito's Lemma (continued)

- The multiplication table for Theorem 17 is

$\times$	dW_i	dt
dW_k	$\delta_{ik}\, dt$	0
dt	0	0

in which

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Ito's Lemma (continued)

- In applying the higher-dimensional Ito's lemma, usually one of the variables, say X_1, is the time variable t and $dX_1 = dt$.
- In this case, $b_{1j} = 0$ for all j and $a_1 = 1$.
- Assume $dX_t = a_t\, dt + b_t\, dW_t$.
- Consider the process $f(X_t, t)$.

Ito's Lemma (continued)

- Then

$$
\begin{aligned}
df(X_t, t) &= \frac{\partial f}{\partial X_t}\, dX_t + \frac{\partial f}{\partial t}\, dt + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\,(dX_t)^2 \\
&= \frac{\partial f}{\partial X_t}\,(a_t\, dt + b_t\, dW_t) + \frac{\partial f}{\partial t}\, dt \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\,(a_t\, dt + b_t\, dW_t)^2 \\
&= \left(\frac{\partial f}{\partial X_t}\, a_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\, b_t^2\right) dt \\
&\quad + \frac{\partial f}{\partial X_t}\, b_t\, dW_t. \qquad (53)
\end{aligned}
$$