The Extended CIR Model

- In the extended CIR model the short rate follows

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\sqrt{r}\, dW.$$

- The functions $\theta(t)$, $a(t)$, and $\sigma(t)$ are implied from market observables.
- With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.

The Hull-White Model: Calibration[a]

- We describe a trinomial forward induction scheme to calibrate the Hull-White model given a and σ.
- As with the Ho-Lee model, the set of achievable short rates is evenly spaced.
- Let r_0 be the annualized, continuously compounded short rate at time zero.
- Every short rate on the tree takes on a value $r_0 + j\Delta r$ for some integer j.

[a] Hull and White (1993).

The Hull-White Model: Calibration (continued)

- Time increments on the tree are also equally spaced at Δt apart.
- Hence nodes are located at times $i\Delta t$ for $i = 0, 1, 2, \ldots$.
- We shall refer to the node on the tree with $t_i \equiv i\Delta t$ and $r_j \equiv r_0 + j\Delta r$ as the (i, j) node.
- The short rate at node (i, j), which equals r_j, is effective for the time period $[\, t_i, t_{i+1})$.

The Hull-White Model: Calibration (continued)

- Use

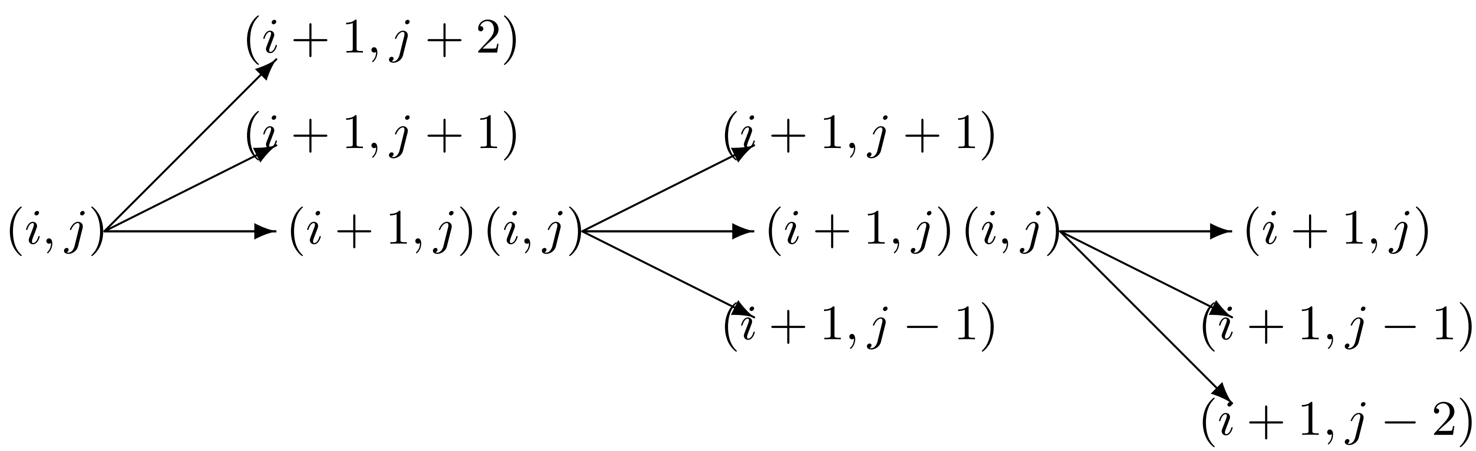
$$\mu_{i,j} \equiv \theta(t_i) - a r_j \tag{118}$$

to denote the drift rate, or the expected change, of the short rate as seen from node (i, j).

- The three distinct possibilities for node (i, j) with three branches incident from it are displayed on p. 996.

- The interest rate movement described by the middle branch may be an increase of Δr, no change, or a decrease of Δr.

The Hull-White Model: Calibration (continued)



The Hull-White Model: Calibration (continued)

- The upper and the lower branches bracket the middle branch.
- Define

$$
\begin{aligned}
p_1(i,j) &\equiv \text{the probability of following the upper branch from node } (i,j)\\
p_2(i,j) &\equiv \text{the probability of following the middle branch from node } (i,j)\\
p_3(i,j) &\equiv \text{the probability of following the lower branch from node } (i,j)
\end{aligned}
$$

- The root of the tree is set to the current short rate r_0.
- Inductively, the drift $\mu_{i,j}$ at node (i,j) is a function of $\theta(t_i)$.

The Hull-White Model: Calibration (continued)

- Once $\theta(t_i)$ is available, $\mu_{i,j}$ can be derived via Eq. (118) on p. 995.
- This in turn determines the branching scheme at every node (i, j) for each j, as we will see shortly.
- The value of $\theta(t_i)$ must thus be made consistent with the spot rate $r(0, t_{i+2})$.

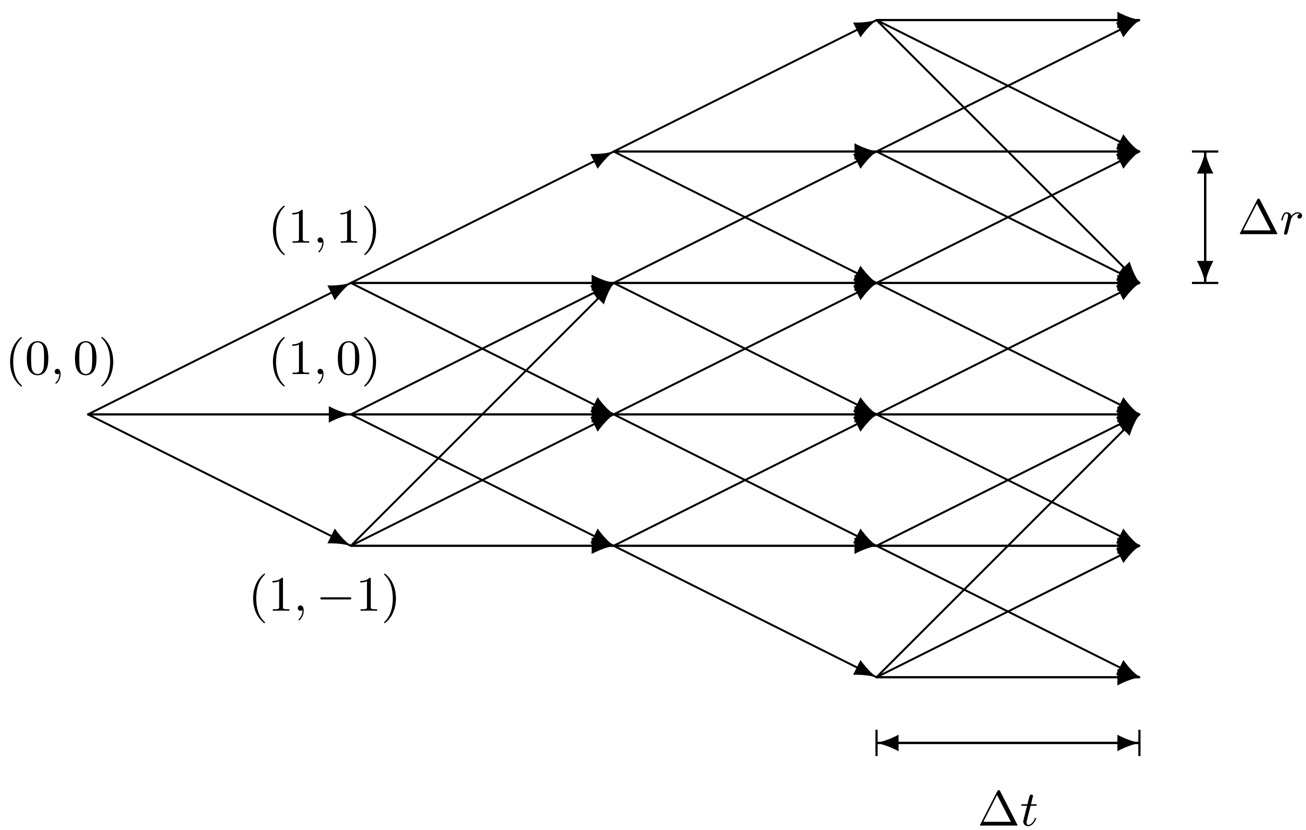
The Hull-White Model: Calibration (continued)

- The branches emanating from node (i, j) with their accompanying probabilities[a] must be chosen to be consistent with $\mu_{i,j}$ and σ.
- This is accomplished by letting the middle node be as close as possible to the current value of the short rate plus the drift.
- Let k be the number among $\{ j-1, j, j+1 \}$ that makes the short rate reached by the middle branch, r_k, closest to $r_j + \mu_{i,j} \Delta t$.

[a] $p_1(i, j)$, $p_2(i, j)$, and $p_3(i, j)$.

The Hull-White Model: Calibration (continued)

- Then the three nodes following node (i, j) are nodes $(i+1, k+1)$, $(i+1, k)$, and $(i+1, k-1)$.
- The resulting tree may have the geometry depicted on p. 1001.
- The resulting tree combines because of the constant jump sizes to reach k.


(0, 0)
(1, 1)
(1, 0)
(1, −1)
Δr
Δt

The Hull-White Model: Calibration (continued)

- The probabilities for moving along these branches are functions of $\mu_{i,j}$, σ, j, and k:

$$p_1(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} + \frac{\eta}{2\Delta r} \tag{119}$$

$$p_2(i,j) = 1 - \frac{\sigma^2 \Delta t + \eta^2}{(\Delta r)^2} \tag{119$'$}$$

$$p_3(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} - \frac{\eta}{2\Delta r} \tag{119$''$}$$

where $\eta \equiv \mu_{i,j} \Delta t + (j-k)\,\Delta r$.

The Hull-White Model: Calibration (continued)

- As trinomial tree algorithms are but explicit methods in disguise, certain relations must hold for Δr and Δt to guarantee stability.
- It can be shown that their values must satisfy

$$\frac{\sigma\sqrt{3\Delta t}}{2} \leq \Delta r \leq 2\sigma\sqrt{\Delta t}$$

for the probabilities to lie between zero and one.

  – For example, Δr can be set to $\sigma\sqrt{3\Delta t}$.[a]

[a]Hull and White (1988).

The Hull-White Model: Calibration (continued)

- Now it only remains to determine $\theta(t_i)$.
- At this point at time t_i, $r(0, t_1)$, $r(0, t_2)$, ..., $r(0, t_{i+1})$ have already been matched.
- Let $Q(i, j)$ denote the value of the state contingent claim that pays one dollar at node (i, j) and zero otherwise.
- By construction, the state prices $Q(i, j)$ for all j are known by now.
- We begin with state price $Q(0, 0) = 1$.

The Hull-White Model: Calibration (continued)

- Let $\hat{r}(i)$ refer to the short rate value at time t_i.
- The value at time zero of a zero-coupon bond maturing at time t_{i+2} is then

$$
\begin{aligned}
& e^{-r(0,t_{i+2})(i+2)\,\Delta t} \\
= & \sum_j Q(i,j)\, e^{-r_j\,\Delta t}\, E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right]. \quad (120)
\end{aligned}
$$

- The right-hand side represents the value of \$1 obtained by holding a zero-coupon bond until time t_{i+1} and then reinvesting the proceeds at that time at the prevailing short rate $\hat{r}(i+1)$, which is stochastic.

The Hull-White Model: Calibration (continued)

- The expectation (120) can be approximated by

$$
\begin{aligned}
& E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \\
\approx \quad & e^{-r_j\,\Delta t} \left(1 - \mu_{i,j} (\Delta t)^2 + \frac{\sigma^2 (\Delta t)^3}{2} \right). \qquad (121)
\end{aligned}
$$

- Substitute Eq. (121) into Eq. (120) and replace $\mu_{i,j}$ with $\theta(t_i) - a r_j$ to obtain

$$
\theta(t_i) \approx \frac{\sum_j Q(i,j)\, e^{-2r_j\,\Delta t} \left(1 + a r_j (\Delta t)^2 + \sigma^2 (\Delta t)^3 / 2 \right) - e^{-r(0,t_{i+2})(i+2)\,\Delta t}}{(\Delta t)^2 \sum_j Q(i,j)\, e^{-2r_j\,\Delta t}}.
$$

The Hull-White Model: Calibration (continued)

- For the Hull-White model, the expectation in Eq. (121) on p. 1006 is actually known analytically by Eq. (18) on p. 151:

$$E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] = e^{-r_j \Delta t + (-\theta(t_i) + a r_j + \sigma^2 \Delta t/2)(\Delta t)^2}.$$

- Therefore, alternatively,

$$\theta(t_i) = \frac{r(0, t_{i+2})(i+2)}{\Delta t} + \frac{\sigma^2 \Delta t}{2} + \frac{\ln \sum_j Q(i,j)\, e^{-2 r_j \Delta t + a r_j (\Delta t)^2}}{(\Delta t)^2}.$$

The Hull-White Model: Calibration (concluded)

- With $\theta(t_i)$ in hand, we can compute $\mu_{i,j}$, the probabilities, and finally the state prices at time t_{i+1}:

$$Q(i+1, j) = \sum_{(i,j^*) \text{ is connected to } (i+1,j) \text{ with probability } p_{j^*}} p_{j^*} e^{-r_{j^*} \Delta t} Q(i, j^*).$$

- There are at most 5 choices for j^*.
- The total running time is $O(n^2)$.
- The space requirement is $O(n)$ (why?).

Comments on the Hull-White Model

- One can try different values of a and σ for each option or have an a value common to all options but use a different σ value for each option.
- Either approach can match all the option prices exactly.
- If the demand is for a single set of parameters that replicate all option prices, the Hull-White model can be calibrated to all the observed option prices by choosing a and σ that minimize the mean-squared pricing error.[a]

[a] Hull and White (1995).

The Hull-White Model: Calibration with Irregular Trinomial Trees

- The previous calibration algorithm is quite general.
- For example, it can be modified to apply to cases where the diffusion term has the form σr^b.
- But it has at least two shortcomings.
- First, the resulting trinomial tree is irregular (p. 1001).
  - So higher complexity in programming.
- The second shortcoming is again a consequence of the tree's irregular shape.

The Hull-White Model: Calibration with Irregular Trinomial Trees (concluded)

- Recall that the algorithm figured out $\theta(t_i)$ that matches the spot rate $r(0, t_{i+2})$ in order to determine the branching schemes for the nodes at time t_i.
- But without those branches, the tree was not specified, and backward induction on the tree was not possible.
- To avoid this dilemma, the algorithm turned to the continuous-time model to evaluate Eq. (120) on p. 1005 that helps derive $\theta(t_i)$ later.
- The resulting $\theta(t_i)$ hence might not yield a tree that matches the spot rates exactly.

The Hull-White Model: Calibration with Regular Trinomial Trees[a]

- We will simplify the previous algorithm to exploit the fact that the Hull-White model has a constant diffusion term σ.
- The resulting trinomial tree will be regular.
- All the $\theta(t_i)$ terms can be chosen by backward induction to match the spot rates exactly.
- The tree is constructed in two phases.

[a]Hull and White (1994).

The Hull-White Model: Calibration with Regular Trinomial Trees (continued)

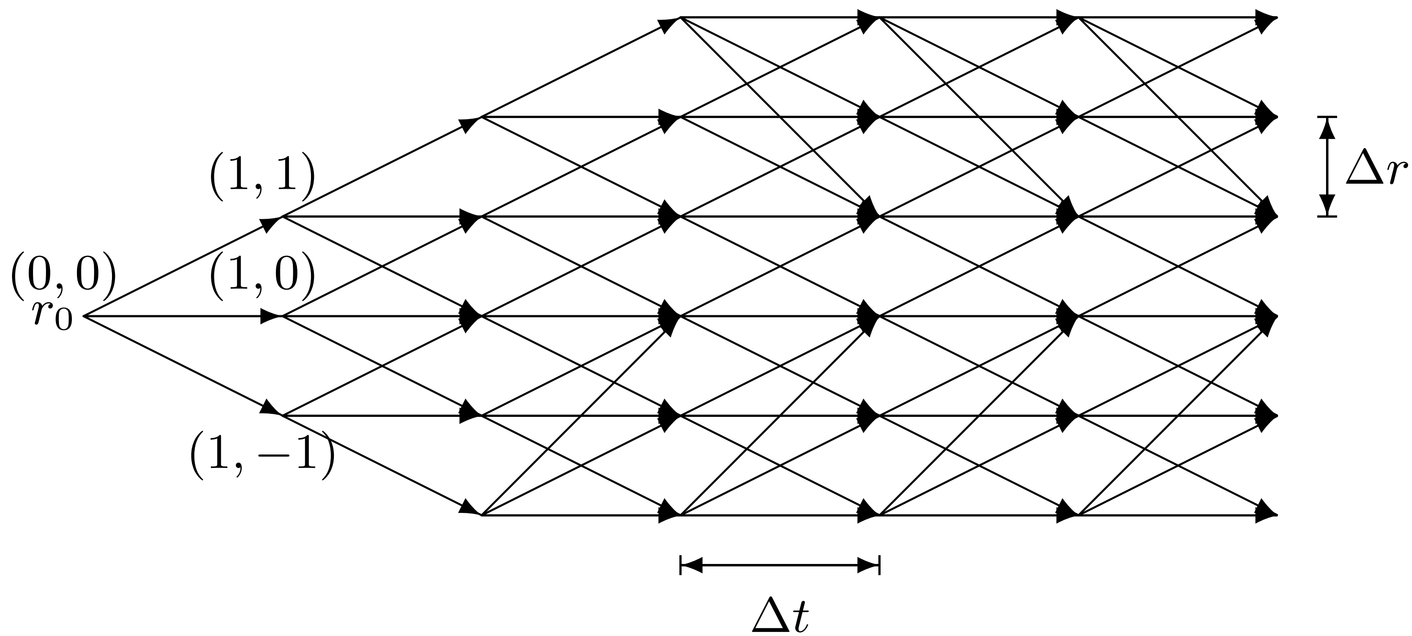
- In the first phase, a tree is built for the $\theta(t) = 0$ case, which is an Ornstein-Uhlenbeck process:

$$dr = -ar\, dt + \sigma\, dW, \quad r(0) = 0.$$

  - The tree is dagger-shaped (p. 1015).
  - The number of nodes above the r_0-line, $j_{\max}$, and that below the line, $j_{\min}$, will be picked so that the probabilities (119) on p. 1002 are positive for all nodes.
  - The tree's branches and probabilities are in place.

The Hull-White Model: Calibration with Regular Trinomial Trees (concluded)

- Phase two fits the term structure.
  - Backward induction is applied to calculate the β_i to add to the short rates on the tree at time t_i so that the spot rate $r(0, t_{i+1})$ is matched.



The short rate at node $(0, 0)$ equals $r_0 = 0$; here $j_{\text{max}} = 3$ and $j_{\text{min}} = 2$.

The Hull-White Model: Calibration

- Set $\Delta r = \sigma\sqrt{3\Delta t}$ and assume that $a > 0$.
- Node (i, j) is a top node if $j = j_{\max}$ and a bottom node if $j = -j_{\min}$.
- Because the root of the tree has a short rate of $r_0 = 0$, phase one adopts $r_j = j\Delta r$.
- Hence the probabilities in Eqs. (119) on p. 1002 use

$$\eta \equiv -aj\Delta r\Delta t + (j-k)\,\Delta r.$$

The Hull-White Model: Calibration (continued)

- The probabilities become

$$p_1(i,j) = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 - aj\Delta t + (j-k)}{2}, \quad (122)$$

$$p_2(i,j) = \frac{2}{3} - \left[a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 \right], \quad (123)$$

$$p_3(i,j) = \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 + aj\Delta t - (j-k)}{2}. \quad (124)$$

The Hull-White Model: Calibration (continued)

- The dagger shape dictates this:
  - Let $k = j - 1$ if node (i, j) is a top node.
  - Let $k = j + 1$ if node (i, j) is a bottom node.
  - Let $k = j$ for the rest of the nodes.
- Note that the probabilities are identical for nodes (i, j) with the same j.
- Furthermore, $p_1(i, j) = p_3(i, -j)$.

The Hull-White Model: Calibration (continued)

- The inequalities

$$\frac{3-\sqrt{6}}{3} < ja\Delta t < \sqrt{\frac{2}{3}} \tag{125}$$

ensure that all the branching probabilities are positive in the upper half of the tree, that is, $j > 0$ (verify this).

- Similarly, the inequalities

$$-\sqrt{\frac{2}{3}} < ja\Delta t < -\frac{3-\sqrt{6}}{3}$$

ensure that the probabilities are positive in the lower half of the tree, that is, $j < 0$.

The Hull-White Model: Calibration (continued)

- To further make the tree symmetric across the r_0-line, we let $j_{\min} = j_{\max}$.
- As $\frac{3-\sqrt{6}}{3} \approx 0.184$, a good choice is

$$j_{\max} = \lceil 0.184/(a\Delta t) \rceil.$$

- Phase two computes the β_is to fit the spot rates.
- We begin with state price $Q(0,0) = 1$.
- Inductively, suppose that spot rates $r(0,t_1)$, $r(0,t_2)$, $\ldots$, $r(0,t_i)$ have already been matched at time t_i.

The Hull-White Model: Calibration (continued)

- By construction, the state prices $Q(i, j)$ for all j are known by now.
- The value of a zero-coupon bond maturing at time t_{i+1} equals

$$e^{-r(0,t_{i+1})(i+1)\,\Delta t} = \sum_j Q(i, j)\, e^{-(\beta_i + r_j)\Delta t}$$

by risk-neutral valuation.

- Hence

$$\beta_i = \frac{r(0, t_{i+1})(i+1)\,\Delta t + \ln \sum_j Q(i, j)\, e^{-r_j\,\Delta t}}{\Delta t},$$

and the short rate at node (i, j) equals $\beta_i + r_j$.

The Hull-White Model: Calibration (concluded)

- The state prices at time t_{i+1},

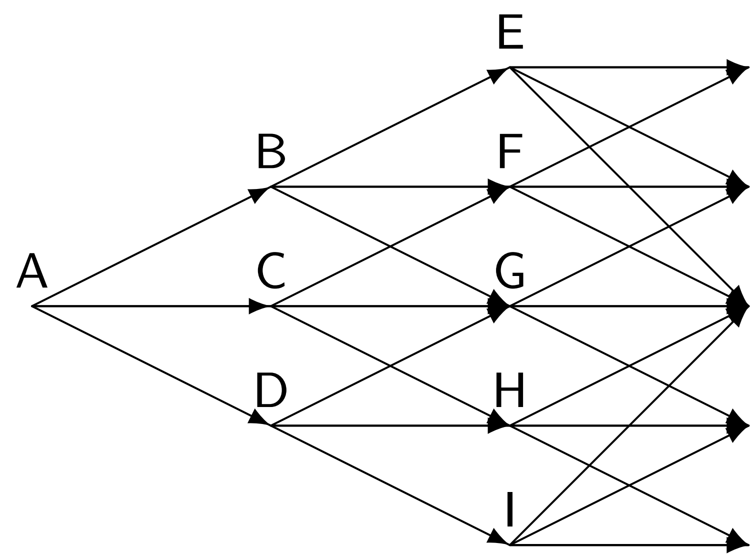
$$Q(i+1, j), \quad -j_{\max} \leq j \leq j_{\max},$$

can now be calculated as before.

- The total running time is $O(n j_{\max})$.
- The space requirement is $O(n)$.

A Numerical Example

- Assume $a = 0.1$, $\sigma = 0.01$, and $\Delta t = 1$ (year).
- Immediately, $\Delta r = 0.0173205$ and $j_{\max} = 2$.
- The plot on p. 1024 illustrates the 3-period trinomial tree after phase one.
- For example, the branching probabilities for node E are calculated by Eqs. (122)–(124) on p. 1017 with $j = 2$ and $k = 1$.



Node	A, C, G	B, F	E	D, H	I
r (%)	0.00000	1.73205	3.46410	−1.73205	−3.46410
p_1	0.16667	0.12167	0.88667	0.22167	0.08667
p_2	0.66667	0.65667	0.02667	0.65667	0.02667
p_3	0.16667	0.22167	0.08667	0.12167	0.88667

A Numerical Example (continued)

- Suppose that phase two is to fit the spot rate curve $0.08 - 0.05 \times e^{-0.18 \times t}$.
- The annualized continuously compounded spot rates are

$$r(0,1) = 3.82365\%, r(0,2) = 4.51162\%, r(0,3) = 5.08626\%.$$

- Start with state price $Q(0,0) = 1$ at node A.

A Numerical Example (continued)

- Now,

$$\beta_0 = r(0,1) + \ln Q(0,0)\, e^{-r_0} = r(0,1) = 3.82365\%.$$

- Hence the short rate at node A equals

$$\beta_0 + r_0 = 3.82365\%.$$

- The state prices at year one are calculated as

$$\begin{aligned} Q(1,1) &= p_1(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414, \\ Q(1,0) &= p_2(0,0)\, e^{-(\beta_0 + r_0)} = 0.641657, \\ Q(1,-1) &= p_3(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414. \end{aligned}$$

A Numerical Example (continued)

- The 2-year rate spot rate $r(0,2)$ is matched by picking

$$\beta_1 = r(0,2) \times 2 + \ln \left[Q(1,1)\, e^{-\Delta r} + Q(1,0) + Q(1,-1)\, e^{\Delta r} \right] = 5.20459\%.$$

- Hence the short rates at nodes B, C, and D equal

$$\beta_1 + r_j,$$

  where $j = 1, 0, -1$, respectively.

- They are found to be 6.93664%, 5.20459%, and 3.47254%.

A Numerical Example (continued)

- The state prices at year two are calculated as

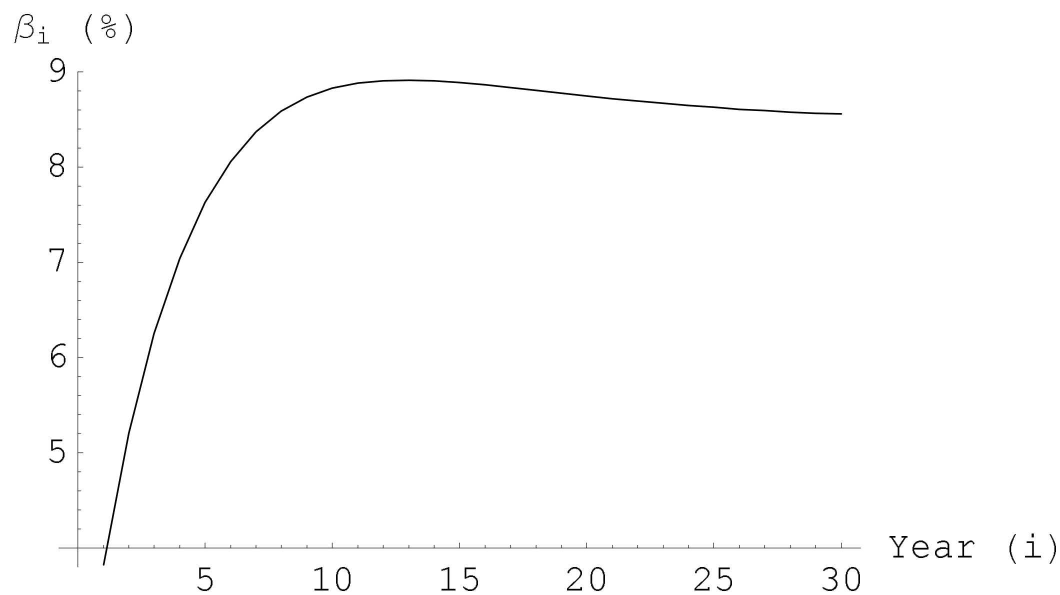
$$
\begin{aligned}
Q(2,2) &= p_1(1,1)\, e^{-(\beta_1 + r_1)} Q(1,1) = 0.018209, \\
Q(2,1) &= p_2(1,1)\, e^{-(\beta_1 + r_1)} Q(1,1) + p_1(1,0)\, e^{-(\beta_1 + r_0)} Q(1,0) \\
&= 0.199799, \\
Q(2,0) &= p_3(1,1)\, e^{-(\beta_1 + r_1)} Q(1,1) + p_2(1,0)\, e^{-(\beta_1 + r_0)} Q(1,0) \\
&\quad + p_1(1,-1)\, e^{-(\beta_1 + r_{-1})} Q(1,-1) = 0.473597, \\
Q(2,-1) &= p_3(1,0)\, e^{-(\beta_1 + r_0)} Q(1,0) + p_2(1,-1)\, e^{-(\beta_1 + r_{-1})} Q(1,-1) \\
&= 0.203263, \\
Q(2,-2) &= p_3(1,-1)\, e^{-(\beta_1 + r_{-1})} Q(1,-1) = 0.018851.
\end{aligned}
$$

A Numerical Example (concluded)

- The 3-year rate spot rate $r(0,3)$ is matched by picking

$$\beta_2 = r(0,3) \times 3 + \ln \Big[Q(2,2)\, e^{-2\times\Delta r} + Q(2,1)\, e^{-\Delta r} + Q(2,0) + Q(2,-1)\, e^{\Delta r} + Q(2,-2)\, e^{2\times\Delta r} \Big] = 6.25359\%.$$

- Hence the short rates at nodes E, F, G, H, and I equal $\beta_2 + r_j$, where $j = 2, 1, 0, -1, -2$, respectively.
- They are found to be 9.71769%, 7.98564%, 6.25359%, 4.52154%, and 2.78949%.
- The figure on p. 1030 plots β_i for $i = 0, 1, \ldots, 29$.


β_i (%)
9
8
7
6
5
Year (i)
5
10
15
20
25
30

Introduction to Mortgage-Backed Securities

Anyone stupid enough to promise to be responsible for a stranger's debts deserves to have his own property held to guarantee payment.

— Proverbs 27:13

Mortgages

- A mortgage is a loan secured by the collateral of real estate property.
- The lender — the mortgagee — can foreclose the loan by seizing the property if the borrower — the mortgagor — defaults, that is, fails to make the contractual payments.

Mortgage-Backed Securities

- A mortgage-backed security (MBS) is a bond backed by an undivided interest in a pool of mortgages.
- MBSs traditionally enjoy high returns, wide ranges of products, high credit quality, and liquidity.
- The mortgage market has witnessed tremendous innovations in product design.

Mortgage-Backed Securities (concluded)

- The complexity of the products and the prepayment option require advanced models and software techniques.
  - In fact, the mortgage market probably could not have operated efficiently without them.[a]
- They also consume lots of computing power.
- Our focus will be on residential mortgages.
- But the underlying principles are applicable to other types of assets.

---

[a] Merton (1994).

Types of MBSs

- An MBS is issued with pools of mortgage loans as the collateral.
- The cash flows of the mortgages making up the pool naturally reflect upon those of the MBS.
- There are three basic types of MBSs:
  1. Mortgage pass-through security (MPTS).
  2. Collateralized mortgage obligation (CMO).
  3. Stripped mortgage-backed security (SMBS).

Problems Investing in Mortgages

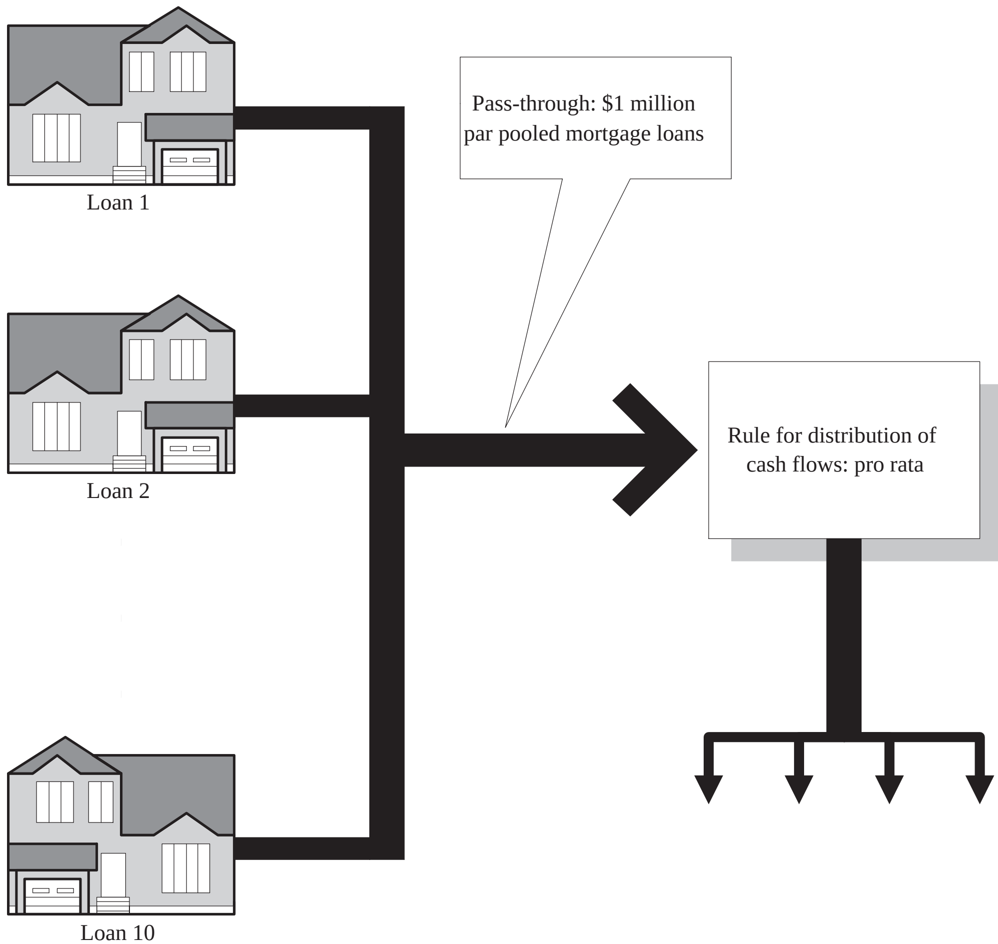
- The mortgage sector is one of the largest in the debt market (see text).
- Individual mortgages are unattractive for many investors.
- Often at hundreds of thousands of U.S. dollars or more, they demand too much investment.
- Most investors lack the resources and knowledge to assess the credit risk involved.

Problems Investing in Mortgages (concluded)

- Recall that a traditional mortgage is fixed rate, level payment, and fully amortized.
- So the percentage of principal and interest (P&I) varying from month to month, creating accounting headaches.
- Prepayment levels fluctuate with a host of factors, making the size and the timing of the cash flows unpredictable.

Mortgage Pass-Throughs

- The simplest kind of MBS.
- Payments from the underlying mortgages are passed from the mortgage holders through the servicing agency, after a fee is subtracted.
- They are distributed to the security holder on a pro rata basis.
  - The holder of a \$25,000 certificate from a \$1 million pool is entitled to $2\,1/2\%$ of the cash flow.
- Because of higher marketability, a pass-through is easier to sell than its individual loans.


Pass-through: $1 million par pooled mortgage loans
Loan 1
Loan 2
Loan 10
Rule for distribution of cash flows: pro rata

Collateralized Mortgage Obligations (CMOs)

- A pass-through exposes the investor to the total prepayment risk.
- Such risk is undesirable from an asset/liability perspective.
- To deal with prepayment uncertainty, CMOs were created.[a]
- Mortgage pass-throughs have a single maturity and are backed by individual mortgages.
- CMOs are *multiple*-maturity, *multi*class debt instruments collateralized by pass-throughs, stripped mortgage-backed securities, and whole loans.

[a]In June 1983 by Freddie Mac with the help of First Boston.

Collateralized Mortgage Obligations (CMOs) (concluded)

- The total prepayment risk is now divided among classes of bonds called classes or tranches.[a]
- The principal, scheduled and prepaid, is allocated on a prioritized basis so as to redistribute the prepayment risk among the tranches in an unequal way.

[a] *Tranche* is a French word for "slice."

Sequential Tranche Paydown

- In the sequential tranche paydown structure, Class A receives principal paydown and prepayments before Class B, which in turn does it before Class C, and so on.
- Each tranche thus has a different effective maturity.
- Each tranche may even have a different coupon rate.
- CMOs were the first successful attempt to alter mortgage cash flows in a security form that attracts a wide range of investors

An Example

- Consider a two-tranche sequential-pay CMO backed by \$1,000,000 of mortgages with a 12% coupon and six months to maturity.
- The cash flow pattern for each tranche with zero prepayment and zero servicing fee is shown on p. 1045.
- The calculation can be carried out first for the Total columns, which make up the amortization schedule, before the cash flow is allocated.
- Tranche A is retired after four months, and tranche B starts principal paydown at the end of month four.

CMO Cash Flows without Prepayments

Month	Interest			Principal			Remaining principal		
	A	B	Total	A	B	Total	A	B	Tota
							500,000	500,000	1,000,0
1	5,000	5,000	10,000	162,548	0	162,548	337,452	500,000	837,4
2	3,375	5,000	8,375	164,173	0	164,173	173,279	500,000	673,2
3	1,733	5,000	6,733	165,815	0	165,815	7,464	500,000	507,4
4	75	5,000	5,075	7,464	160,009	167,473	0	339,991	339,9
5	0	3,400	3,400	0	169,148	169,148	0	170,843	170,8
6	0	1,708	1,708	0	170,843	170,843	0	0	
Total	10,183	25,108	35,291	500,000	500,000	1,000,000			

The total monthly payment is \$172,548. Month-i numbers reflect the ith monthly payment.

Another Example

- When prepayments are present, the calculation is slightly more complex.
- Suppose the single monthly mortality (SMM) per month is 5%.
- This means the prepayment amount is 5% of the remaining principal.
- The remaining principal at month i *after* prepayment then equals the scheduled remaining principal as computed by Eq. (4) on p. 44 times $(0.95)^i$.
- This done for all the months, the total interest payment at any month is the remaining principal of the previous month times 1%.

Another Example (continued)

- The prepayment amount equals the remaining principal times 0.05/0.95.
  - The division by 0.95 yields the remaining principal *before* prepayment.
- Page 1049 tabulates the cash flows of the same two-tranche CMO under 5% SMM.
- For instance, the total principal payment at month one, $204,421, can be verified as follows.

Another Example (concluded)

- The scheduled remaining principal is \$837,452 from p. 1045.
- The remaining principal is hence $837452 \times 0.95 = 795579$, which makes the total principal payment $1000000 - 795579 = 204421$.
- As tranche A's remaining principal is \$500,000, all 204,421 dollars go to tranche A.
- Incidentally, the prepayment is $837452 \times 5\% = 41873$.
- Tranche A is retired after three months, and tranche B starts principal paydown at the end of month three.

CMO Cash Flows with Prepayments

Month	Interest			Principal			Remaining principal		
	A	B	Total	A	B	Total	A	B	Total
							500,000	500,000	1,000,00
1	5,000	5,000	10,000	204,421	0	204,421	295,579	500,000	795,57
2	2,956	5,000	7,956	187,946	0	187,946	107,633	500,000	607,63
3	1,076	5,000	6,076	107,633	64,915	172,548	0	435,085	435,08
4	0	4,351	4,351	0	158,163	158,163	0	276,922	276,92
5	0	2,769	2,769	0	144,730	144,730	0	132,192	132,19
6	0	1,322	1,322	0	132,192	132,192	0	0	
Total	9,032	23,442	32,474	500,000	500,000	1,000,000			

Month-i numbers reflect the ith monthly payment.

Stripped Mortgage-Backed Securities (SMBSs)

- They were created in February 1987 when Fannie Mae issued its Trust 1 stripped MBS.
- The principal and interest are divided between the PO strip and the IO strip.
- In the scenarios on p. 1044 and p. 1046:
  - The IO strip receives all the interest payments under the Interest/Total column.
  - The PO strip receives all the principal payments under the Principal/Total column.

Stripped Mortgage-Backed Securities (SMBSs) (concluded)

- These new instruments allow investors to better exploit anticipated changes in interest rates.
- The collateral for an SMBS is a pass-through,.
- CMOs and SMBSs are usually called derivative MBSs.

Prepayments

- The prepayment option sets MBSs apart from other fixed-income securities.
- The exercise of options on most securities is expected to be "rational."
- This kind of "rationality" is weakened when it comes to the homeowner's decision to prepay.
- Even when the prevailing mortgage rate exceeds the mortgage's loan rate, some loans are prepaid.

Prepayment Risk

- Prepayment risk is the uncertainty in the amount and timing of the principal prepayments in the pool of mortgages that collateralize the security.
- This risk can be divided into contraction risk and extension risk.
- Contraction risk is the risk of having to reinvest the prepayments at a rate lower than the coupon rate when interest rates decline.
- Extension risk is due to the slowdown of prepayments when interest rates climb, making the investor earn the security's lower coupon rate rather than the market's higher rate.

Prepayment Risk (concluded)

- Prepayments can be in whole or in part.
  - The former is called liquidation.
  - The latter is called curtailment.
- Prepayments, however, need not always result in losses.
- The holder of a pass-through security is exposed to the total prepayment risk associated with the underlying pool of mortgage loans.
- The CMO is designed to alter the distribution of that risk among the investors.

Other Risks

- Investors in mortgages are exposed to at least three other risks.
  - Interest rate risk is inherent in any fixed-income security.
  - Credit risk is the risk of loss from default.
    - For privately insured mortgage, the risk is related to the credit rating of the company that insures the mortgage.
  - Liquidity risk is the risk of loss if the investment must be sold quickly.

Prepayment: Causes

Prepayments have at least five components.

Home sale ("housing turnover"). The sale of a home generally leads to the prepayment of mortgage because of the full payment of the remaining principal.

Refinancing. Mortgagors can refinance their home mortgage at a lower mortgage rate. This is the most volatile component of prepayment and constitutes the bulk of it when prepayments are extremely high.

Prepayment: Causes (concluded)

Default. Caused by foreclosure and subsequent liquidation of a mortgage. Relatively minor in most cases.

Curtailment. As the extra payment above the scheduled payment, curtailment applies to the principal and shortens the maturity of fixed-rate loans. Its contribution to prepayments is minor.

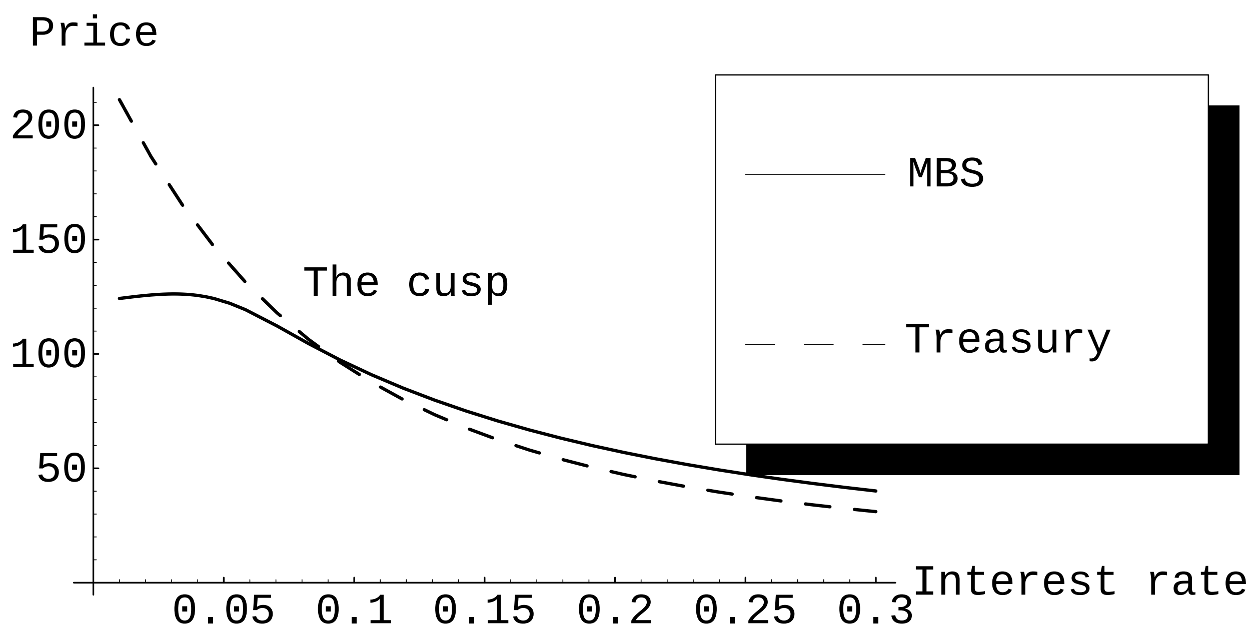
Full payoff (liquidation). There is evidence that many mortgagors pay off their mortgage completely when it is very seasoned and the remaining balance is small. Full payoff can also be due to natural disasters.

Prepayment: Characteristics

- Prepayments usually increase as the mortgage ages — first at an increasing rate and then at a decreasing rate.
- They are higher in the spring and summer and lower in the fall and winter.
- They vary by the geographic locations of the underlying properties.
- They increase when interest rates drop but with a time lag.

Prepayment: Characteristics (continued)

- If prepayments were higher for some time because of high refinancing rates, they tend to slow down.
  - Perhaps, homeowners who do not prepay when rates have been low for a prolonged time tend never to prepay.
- Plot on p. 1060 illustrates the typical price/yield curves of the Treasury and pass-through.



Price compression occurs as yields fall through a threshold. The cusp represents that point.

Prepayment: Characteristics (concluded)

- As yields fall and the pass-through's price moves above a certain price, it flattens and then follows a downward slope.
- This phenomenon is called the price compression of premium-priced MBSs.
- It demonstrates the negative convexity of such securities.

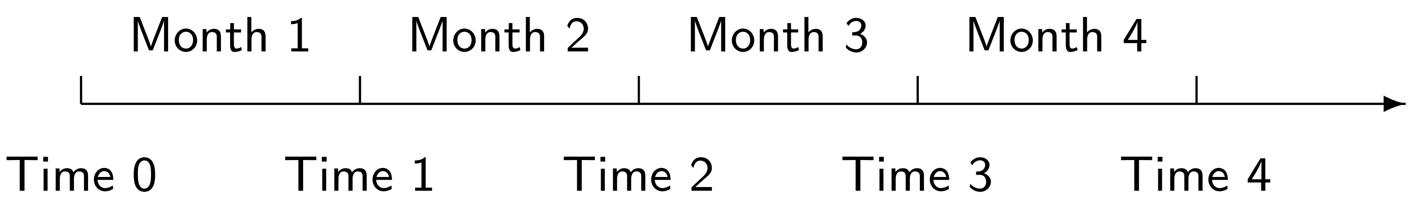
Analysis of Mortgage-Backed Securities

Oh, well, if you cannot measure,
measure anyhow.
— Frank H. Knight (1885–1972)

Uniqueness of MBS

- Compared with other fixed-income securities, the MBS is unique in two respects.
- Its cash flow consists of principal and interest (P&I).
- The cash flow may vary because of prepayments in the underlying mortgages.

Time Line

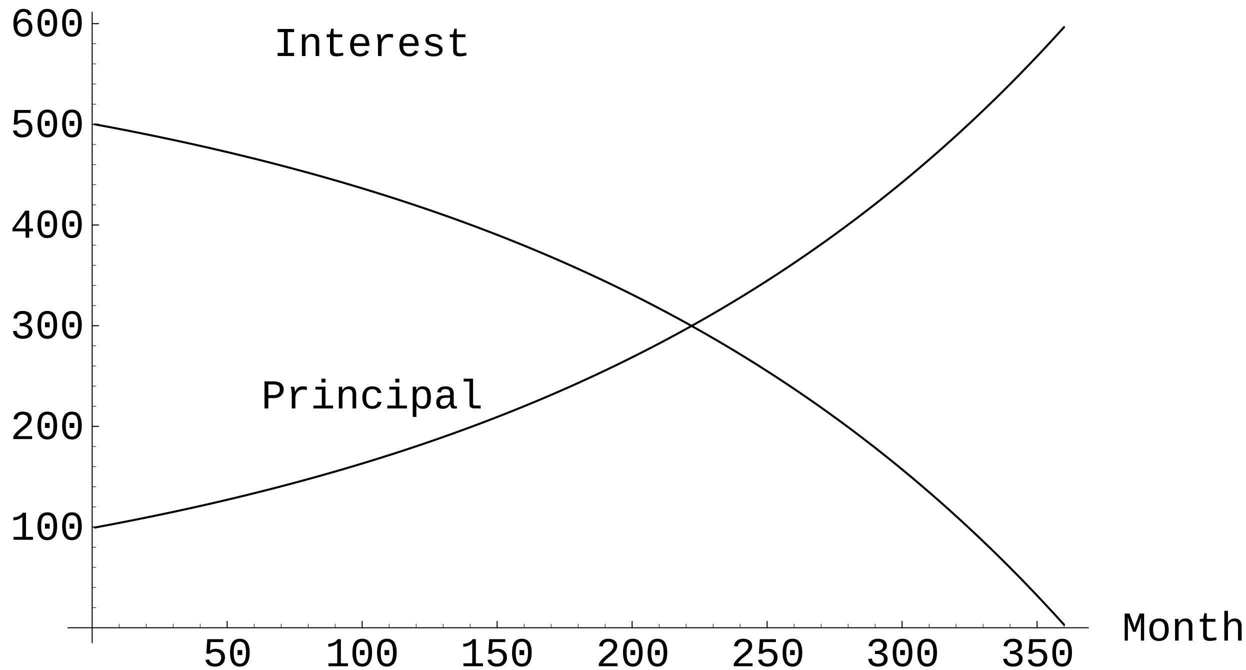


- Mortgage payments are paid in arrears.
- A payment for month i occurs at time i, that is, end of month i.
- The end of a month will be identified with the beginning of the coming month.

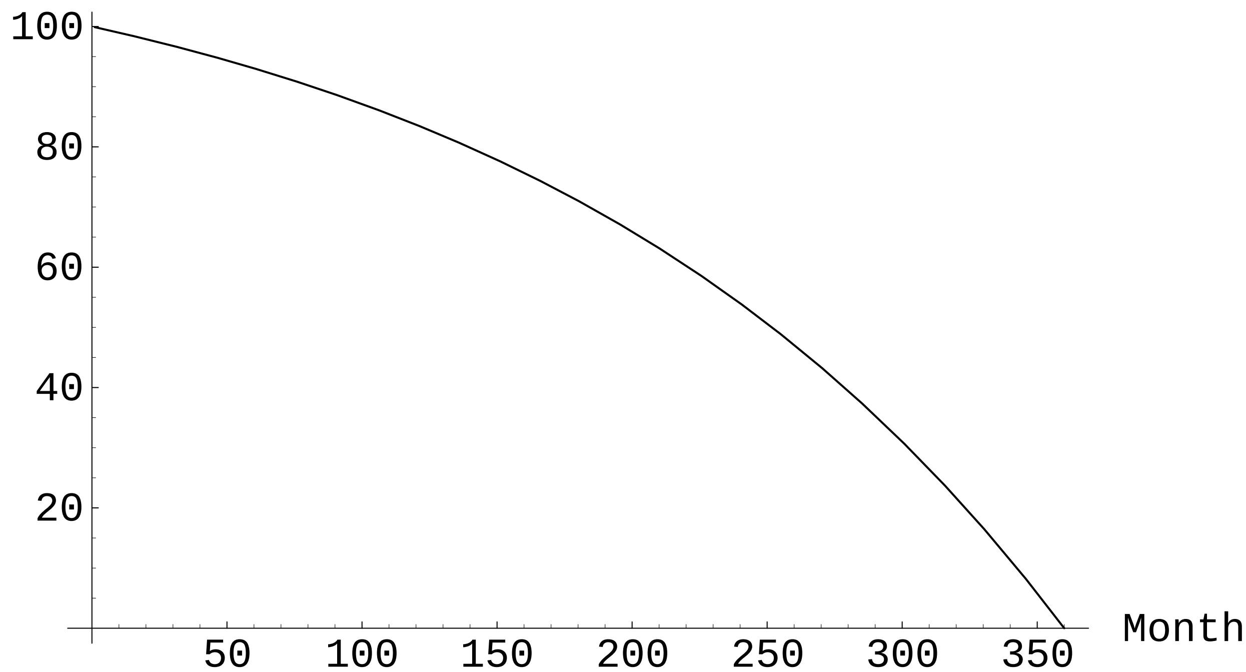
Cash Flow Analysis

- A traditional mortgage has a fixed term, a fixed interest rate, and a fixed monthly payment.
- Page 1067 illustrates the scheduled P&I for a 30-year, 6% mortgage with an initial balance of $100,000.
- Page 1068 depicts how the remaining principal balance decreases over time.

Scheduled Principal and Interest Payments



Scheduled Remaining Principal Balances



Cash Flow Analysis (continued)

- In the early years, the P&I consists mostly of interest.
- Then it gradually shifts toward principal payment with the passage of time.
- However, the total P&I payment remains the same each month, hence the term *level* pay.
- Identical characteristics hold for the pool's P&I payments in the absence of prepayments and servicing fees.

Cash Flow Analysis (continued)

- From Eq. (4) on p. 44 the remaining principal balance after the kth payment is

$$C \, \frac{1 - (1 + r/m)^{-n+k}}{r/m}. \tag{126}$$

  - C is the scheduled P&I payment of an n-month mortgage making m payments per year.
  - r is the annual mortgage rate.

- For mortgages, $m = 12$.

Cash Flow Analysis (continued)

- The remaining principal balance after k payments can be expressed as a portion of the original principal balance:

$$\begin{aligned} \text{Bal}_k &\equiv 1 - \frac{(1+r/m)^k - 1}{(1+r/m)^n - 1} \\ &= \frac{(1+r/m)^n - (1+r/m)^k}{(1+r/m)^n - 1}. \end{aligned} \tag{127}$$

- This equation can be verified by dividing Eq. (126) (p. 1070) by Bal_0.

Cash Flow Analysis (continued)

- The remaining principal balance after k payments is

$$\mathrm{RB}_k \equiv \mathcal{O} \times \mathrm{Bal}_k,$$

  where $\mathcal{O}$ will denote the original principal balance.

- The term factor denotes the portion of the remaining principal balance to its original principal balance expressed as a decimal.
- So Bal_k is the monthly factor when there are no prepayments.
- It is also known as the amortization factor.

Cash Flow Analysis (concluded)

- When the idea of factor is applied to a mortgage pool, it is called the paydown factor on the pool or simply the pool factor.

An Example

- The remaining balance of a 15-year mortgage with a 9% mortgage rate after 54 months is

$$\begin{aligned} & \mathcal{O} \times \frac{(1+(0.09/12))^{180} - (1+(0.09/12))^{54}}{(1+(0.09/12))^{180} - 1} \\ = \quad & \mathcal{O} \times 0.824866. \end{aligned}$$

- In other words, roughly 82.49% of the original loan amount remains after 54 months.

P&I Analysis

- By the amortization principle, the tth interest payment equals

$$I_t \equiv \mathrm{RB}_{t-1} \times \frac{r}{m} = \mathcal{O} \times \frac{r}{m} \times \frac{(1+r/m)^n - (1+r/m)^{t-1}}{(1+r/m)^n - 1}.$$

- The principal part of the tth monthly payment is

$$\begin{aligned} P_t &\equiv \mathrm{RB}_{t-1} - \mathrm{RB}_t \\ &= \mathcal{O} \times \frac{(r/m)(1+r/m)^{t-1}}{(1+r/m)^n - 1}. \end{aligned} \tag{128}$$

P&I Analysis (concluded)

- The scheduled P&I payment at month t, or $P_t + I_t$, is

$$\begin{aligned} & (\mathrm{RB}_{t-1} - \mathrm{RB}_t) + \mathrm{RB}_{t-1} \times \frac{r}{m} \\ = \quad & \mathcal{O} \times \left[\frac{(r/m)(1+r/m)^n}{(1+r/m)^n - 1} \right], \end{aligned} \tag{129}$$

  indeed a level pay independent of t.

- The term within the brackets, called the payment factor or annuity factor, is the monthly payment for each dollar of mortgage.

An Example

- The mortgage on pp. 39ff has a monthly payment of

$$250000 \times \frac{(0.08/12) \times (1 + (0.08/12))^{180}}{(1 + (0.08/12))^{180} - 1} = 2389.13$$

  by Eq. (129) on p. 1076.

- This number agrees with the number derived earlier.

Expressing Prepayment Speeds

- The cash flow of a mortgage derivative is determined from that of the mortgage pool.
- The single most important factor complicating this endeavor is the unpredictability of prepayments.
- Recall that prepayment represents the principal payment made in excess of the scheduled principal amortization.
- Compare the amortization factor Bal_t of the pool with the reported factor to determine if prepayments have occurred.

Expressing Prepayment Speeds (concluded)

- The amount by which the reported factor exceeds the amortization factor is the prepayment amount.

Single Monthly Mortality

- A SMM of ω means $\omega\%$ of the scheduled remaining balance at the end of the month will prepay.
- In other words, the SMM is the percentage of the remaining balance that prepays for the month.
- Suppose the remaining principal balance of an MBS at the beginning of a month is \$50,000, the SMM is 0.5%, and the scheduled principal payment is \$70.
- Then the prepayment for the month is

$$0.005 \times (50{,}000 - 70) \approx 250$$

dollars.

Single Monthly Mortality (concluded)

- If the same monthly prepayment speed s is maintained since the issuance of the pool, the remaining principal balance at month i will be $\mathrm{RB}_i \times (1 - s/100)^i$.
- It goes without saying that prepayment speeds must lie between 0% and 100%.

An Example

- Take the mortgage on p. 1074.
- Its amortization factor at the 54th month is 0.824866.
- If the actual factor is 0.8, then the SMM for the initial period of 54 months is

$$100 \times \left[1 - \left(\frac{0.8}{0.824866} \right)^{1/54} \right] = 0.0566677.$$

- In other words, roughly 0.057% of the remaining principal is prepaid per month.

Conditional Prepayment Rate

- The conditional prepayment rate (CPR) is the annualized equivalent of a SMM,

$$\text{CPR} = 100 \times \left[1 - \left(1 - \frac{\text{SMM}}{100} \right)^{12} \right].$$

- Conversely,

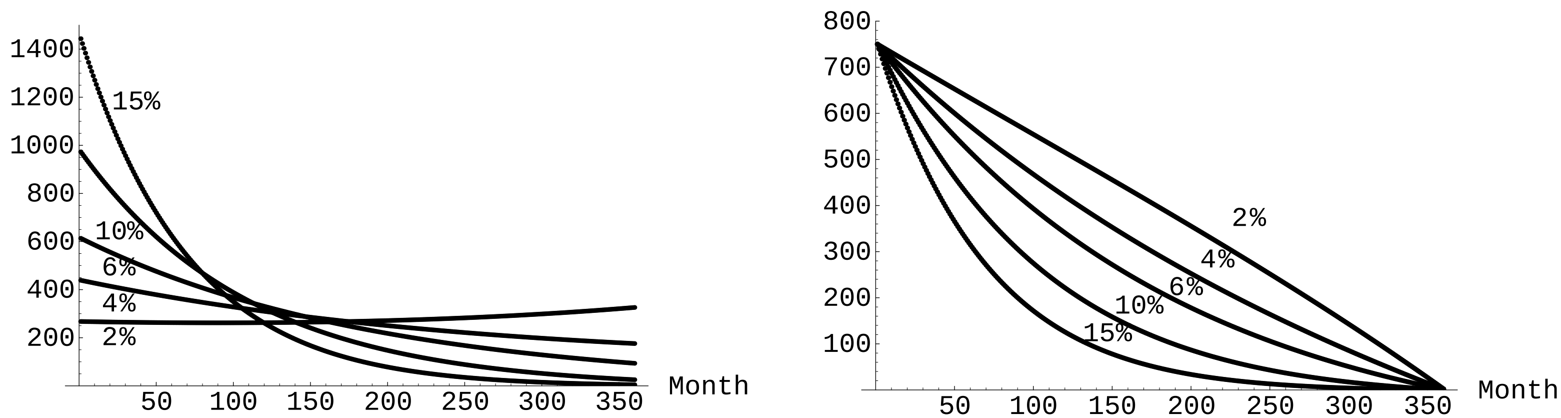
$$\text{SMM} = 100 \times \left[1 - \left(1 - \frac{\text{CPR}}{100} \right)^{1/12} \right].$$

Conditional Prepayment Rate (concluded)

- For example, the SMM of 0.0566677 on p. 1082 is equivalent to a CPR of

$$100 \times \left[1 - \left(1 - \left(\frac{0.0566677}{100} \right)^{12} \right) \right] = 0.677897.$$

- Roughly 0.68% of the remaining principal is prepaid annually.
- The figures on 1085 plot the principal and interest cash flows under various prepayment speeds.
- Observe that with accelerated prepayments, the principal cash flow is shifted forward in time.



Principal (left) and interest (right) cash flows at various CPRs. The 6% mortgage has 30 years to maturity and an original loan amount of $100,000.

PSA

- In 1985 the Public Securities Association (PSA) standardized a prepayment model.
- The PSA standard is expressed as a monthly series of CPRs.
  - It reflects the increase in CPR that occurs as the pool seasons.
- The PSA standard postulates the following prepayment speeds: The CPR is 0.2% for the first month, increases thereafter by 0.2% per month until it reaches 6% per year for the 30th month, and then stays at 6% for the remaining years.

PSA (continued)

- At the time the PSA proposed its standard, a seasoned 30-year GNMA's typical prepayment speed was approximately 6% CPR.
- The PSA benchmark is also referred to as 100 PSA.
- Other speeds are expressed as some percentage of PSA.
  - 50 PSA means one-half the PSA CPRs.
  - 150 PSA means one-and-a-half the PSA CPRs.

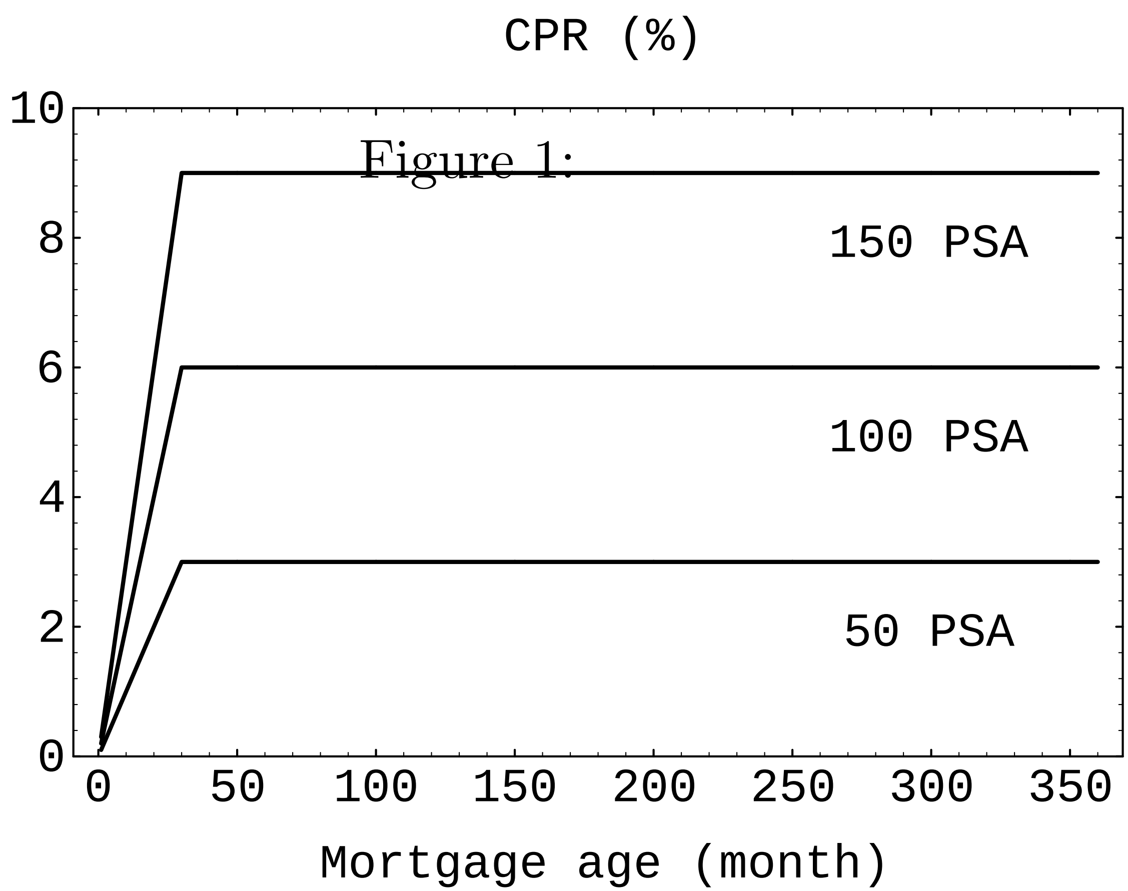

CPR (%)
10
8
6
4
2
0
0
50
100
150
200
250
300
350
150 PSA
100 PSA
50 PSA
Mortgage age (month)

Figure 1:

PSA (concluded)

- Mathematically,

$$\text{CPR} = \begin{cases} 6\% \times \frac{\text{PSA}}{100} & \text{if the pool age exceeds 30 months} \\ 0.2\% \times m \times \frac{\text{PSA}}{100} & \text{if the pool age } m \leq 30 \text{ months} \end{cases}$$

- Conversely,

$$\text{PSA} = \begin{cases} 100 \times \frac{\text{CPR}}{6} & \text{if the pool age exceeds 30 months} \\ 100 \times \frac{\text{CPR}}{0.2 \times m} & \text{if the pool age } m \leq 30 \text{ months} \end{cases}$$

Finis