The Cox-Ingersoll-Ross Model\textsuperscript{a}

- It is the following square-root short rate model:

\[ dr = \beta(\mu - r) \, dt + \sigma \sqrt{r} \, dW. \tag{110} \]

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.

- The parameter $\beta$ determines the speed of adjustment.

- The short rate can reach zero only if $2\beta \mu < \sigma^2$.

- See text for the bond pricing formula.

\textsuperscript{a}Cox, Ingersoll, and Ross (1985).
Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into $n$ periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will not combine.
Binomial CIR (continued)

• Instead, consider the transformed process

\[ x(r) \equiv 2\sqrt{r}/\sigma. \]

• It follows

\[ dx = m(x) \, dt + dW, \]

where

\[ m(x) \equiv 2\beta \mu / (\sigma^2 x) - (\beta x / 2) - 1/(2x). \]

• Since this new process has a constant volatility, its associated binomial tree combines.
Binomial CIR (continued)

- Construct the combining tree for \( r \) as follows.
- First, construct a tree for \( x \).
- Then transform each node of the tree into one for \( r \) via the inverse transformation \( r = f(x) \equiv x^2 \sigma^2 / 4 \) (p. 931).
Binomial CIR (concluded)

• The probability of an up move at each node $r$ is

$$p(r) \equiv \frac{\beta(\mu - r) \Delta t + r - r^-}{r^+ - r^-}. \quad (111)$$

- $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.

- $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

• Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.
Numerical Examples

• Consider the process,

\[ 0.2 (0.04 - r) \, dt + 0.1 \sqrt{r} \, dW, \]

for the time interval \([0, 1]\) given the initial rate \(r(0) = 0.04\).

• We shall use \(\Delta t = 0.2 \) (year) for the binomial approximation.

• See p. 934(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.
Numerical Examples (continued)

• Consider the node which is the result of an up move from the root.

• Since the root has $x = 2\sqrt{r(0)/\sigma} = 4$, this particular node’s $x$ value equals $4 + \sqrt{\Delta t} = 4.4472135955$.

• Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2 / 4 \approx 0.0494442719102$. 
Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).
A General Method for Constructing Binomial Models\textsuperscript{a}

- We are given a continuous-time process
  \[ dy = \alpha(y, t) \, dt + \sigma(y, t) \, dW. \]

- Make sure the binominal model’s drift and diffusion converge to the above process by setting the probability of an up move to
  \[
  \frac{\alpha(y, t) \, \Delta t + y - y_d}{y_u - y_d}.
  \]

- Here \( y_u \equiv y + \sigma(y, t) \sqrt{\Delta t} \) and \( y_d \equiv y - \sigma(y, t) \sqrt{\Delta t} \) represent the two rates that follow the current rate \( y \).

- The displacements are identical, at \( \sigma(y, t) \sqrt{\Delta t} \).

\textsuperscript{a}Nelson and Ramaswamy (1990).
A General Method (continued)

• But the binomial tree may not combine:

\[ \sigma(y, t)\sqrt{\Delta t} - \sigma(y_u, t)\sqrt{\Delta t} \neq -\sigma(y, t)\sqrt{\Delta t} + \sigma(y_d, t)\sqrt{\Delta t} \]

in general.

• When \( \sigma(y, t) \) is a constant independent of \( y \), equality holds and the tree combines.

• To achieve this, define the transformation

\[ x(y, t) \equiv \int_{y_u}^{y} \sigma(z, t)^{-1} \, dz. \]

• Then \( x \) follows \( dx = m(y, t) \, dt + dW \) for some \( m(y, t) \) (see text).
A General Method (continued)

• The key is that the diffusion term is now a constant, and the binomial tree for $x$ combines.

• The probability of an up move remains

$$\frac{\alpha(y(x, t), t) \Delta t + y(x, t) - y_d(x, t)}{y_u(x, t) - y_d(x, t)},$$

where $y(x, t)$ is the inverse transformation of $x(y, t)$ from $x$ back to $y$.

• Note that $y_u(x, t) \equiv y(x + \sqrt{\Delta t}, t + \Delta t)$ and $y_d(x, t) \equiv y(x - \sqrt{\Delta t}, t + \Delta t)$. 
A General Method (concluded)

- The transformation is
  \[
  \int_r^r (\sigma \sqrt{z})^{-1} \, dz = 2\sqrt{r}/\sigma
  \]
  for the CIR model.

- The transformation is
  \[
  \int_S^S (\sigma z)^{-1} \, dz = (1/\sigma) \ln S
  \]
  for the Black-Scholes model.

- The familiar binomial option pricing model in fact discretizes \( \ln S \) not \( S \).
Model Calibration

• In the time-series approach, the time series of short rates is used to estimate the parameters of the process.

• This approach may help in validating the proposed interest rate process.

• But it alone cannot be used to estimate the risk premium parameter $\lambda$.

• The model prices based on the estimated parameters may also deviate a lot from those in the market.
Model Calibration (concluded)

- The cross-sectional approach uses a cross section of bond prices observed at the same time.

- The parameters are to be such that the model prices closely match those in the market.

- After this procedure, the calibrated model can be used to price interest rate derivatives.

- Unlike the time-series approach, the cross-sectional approach is unable to separate out the interest rate risk premium from the model parameters.
On One-Factor Short Rate Models

• By using only the short rate, they ignore other rates on the yield curve.

• Such models also restrict the volatility to be a function of interest rate levels only.

• The prices of all bonds move in the same direction at the same time (their magnitudes may differ).

• The returns on all bonds thus become highly correlated.

• In reality, there seems to be a certain amount of independence between short- and long-term rates.
On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.
On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.

- But they are much harder to think about and work with.

- They also take much more computer time—the curse of dimensionality.

- These practical concerns limit the use of multifactor models to two-factor ones.
Options on Coupon Bonds\textsuperscript{a}

- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time $T$ on a bond with par value $1$.
- Let $X$ denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all $i$.
- The payoff for the option is
  \[
  \max \left( \sum_{i=1}^{n} c_i \left( \frac{P(r(T), T, t_i)}{1 - P(r(T), T, T)} \right) - X, 0 \right).
  \]

\textsuperscript{a}Jamshidian (1989).
Options on Coupon Bonds (continued)

• At time $T$, there is a unique value $r^*$ for $r(T)$ that renders the coupon bond’s price equal the strike price $X$.

• This $r^*$ can be obtained by solving
  
  \[ X = \sum_{i=1}^{n} c_i P(r, T, t_i) \]
  
  numerically for $r$.

• The solution is also unique for one-factor models whose bond price is a monotonically decreasing function of $r$.

• Let $X_i \equiv P(r^*, T, t_i)$, the value at time $T$ of a zero-coupon bond with par value $\$1$ and maturing at time $t_i$ if $r(T) = r^*$. 

Options on Coupon Bonds (concluded)

- Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$.

- As $X = \sum_i c_i X_i$, the option’s payoff equals

$$\max \left( \sum_{i=1}^{n} c_i P(r(T), T, t_i) - \sum_i c_i X_i, 0 \right)$$

$$= \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

- Thus the call is a package of $n$ options on the underlying zero-coupon bond.

- Why can’t we do the same thing for Asian options?\(^a\)

\(^a\)Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.
No-Arbitrage Term Structure Models
How much of the structure of our theories really tells us about things in nature, and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)
Motivations

• Recall the difficulties facing equilibrium models mentioned earlier.
  – They usually require the estimation of the market price of risk.
  – They cannot fit the market term structure.
  – But consistency with the market is often mandatory in practice.
No-Arbitrage Models\textsuperscript{a}

- No-arbitrage models utilize the full information of the term structure.

- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.

- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.

- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

\textsuperscript{a}Ho and Lee (1986).
No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.
The Ho-Lee Model

• The short rates at any given time are evenly spaced.

• Let $p$ denote the risk-neutral probability that the short rate makes an up move.

• We shall adopt continuous compounding.

---

$^a$Ho and Lee (1986).
The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t + 1), P(t, t + 2), \ldots$ at time $t$ identified with the root of the tree.

- Let the discount factors in the next period be
  
  $P_d(t + 1, t + 2), P_d(t + 1, t + 3), \ldots$ if short rate moves down
  $P_u(t + 1, t + 2), P_u(t + 1, t + 3), \ldots$ if short rate moves up

- By backward induction, it is not hard to see that for $n \geq 2$,
  
  $$P_u(t + 1, t + n) = P_d(t + 1, t + n) e^{-(v_2+\cdots+v_n)}$$

  (112)

  (see text).
The Ho-Lee Model (continued)

• It is also not hard to check that the $n$-period zero-coupon bond has yields

$$y_d(n) \equiv -\frac{\ln P_d(t + 1, t + n)}{n - 1}$$

$$y_u(n) \equiv -\frac{\ln P_u(t + 1, t + n)}{n - 1} = y_d(n) + \frac{v_2 + \cdots + v_n}{n - 1}$$

• The volatility of the yield to maturity for this bond is therefore

$$\kappa_n \equiv \sqrt{py_u(n)^2 + (1 - p) y_d(n)^2 - [py_u(n) + (1 - p) y_d(n)]^2}$$

$$= \sqrt{p(1 - p)} \ (y_u(n) - y_d(n))$$

$$= \sqrt{p(1 - p)} \ \frac{v_2 + \cdots + v_n}{n - 1}.$$
The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1 - p)} \, v_2.$$  \hfill (113)

- The variance of the short rate therefore equals $p(1 - p)(r_u - r_d)^2$, where $r_u$ and $r_d$ are the two successor rates.\(^a\)

\(^a\)Contrast this with the lognormal model.
The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of $\kappa_2, \kappa_3, \ldots$.
  - It is independent of the $r_i$.
- It is easy to compute the $v_i$s from the volatility structure, and vice versa.
- The $r_i$s can be computed by forward induction.
- The volatility structure is supplied by the market.
The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy

\[ P(t, t+n) = (pP_u(t+1, t+n) + (1-p) P_d(t+1, t+n)) P(t, t+1) \]

- Combine the above with Eq. (112) on p. 956 and assume \[ p = 1/2 \] to obtain\(^a\)

\[ P_d(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2 \times \exp[v_2 + \cdots + v_n]}{1 + \exp[v_2 + \cdots + v_n]}, \quad (114) \]

\[ P_u(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1 + \exp[v_2 + \cdots + v_n]}. \quad (114') \]

\(^a\)In the limit, only the volatility matters.
The Ho-Lee Model: Bond Price Process (concluded)

• The bond price tree combines.

• Suppose all $v_i$ equal some constant $v$ and $\delta \equiv e^v > 0$.

• Then

\[
P_d(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}},
\]

\[
P_u(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1+\delta^{n-1}}.
\]

• Short rate volatility $\sigma$ equals $v/2$ by Eq. (113) on p. 958.

• Price derivatives by taking expectations under the risk-neutral probability.
The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an \( n \)-period zero-coupon bond is

\[
    r(t, t + n) \equiv \ln \left( \frac{P(t + 1, t + n)}{P(t, t + n)} \right).
\]

- Its value is either \( \ln \frac{P_d(t+1,t+n)}{P(t,t+n)} \) or \( \ln \frac{P_u(t+1,t+n)}{P(t,t+n)} \).

- Thus the variance of return is

\[
    \text{Var}[r(t, t + n)] = p(1 - p)((n - 1) \nu)^2 = (n - 1)^2 \sigma^2.
\]
The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between \( r(t, t + n) \) and \( r(t, t + m) \) is \( (n - 1)(m - 1) \sigma^2 \) (see text).
- As a result, the correlation between any two one-period rates of return is unity.
- Strong correlation between rates is inherent in all one-factor Markovian models.
The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is

\[ dr = \theta(t) \, dt + \sigma \, dW. \]

- This is Vasicek’s model with the mean-reverting drift replaced by a deterministic, time-dependent drift.

- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying, i.e.,

\[ dr = \theta(t) \, dt + \sigma(t) \, dW. \]

- This corresponds to the discrete-time model in which \( v_i \) are not all identical.
The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model’s state variables (factors) not its parameters.
- Model parameters, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.
Problems with No-Arbitrage Models in General (concluded)

• This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.

• Consequently, a model’s intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.
The Black-Derman-Toy Model\textsuperscript{a}

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 807ff (repeated on next page).
- The volatility structure is given by the market.
- From it, the short rate volatilities (thus $v_i$) are determined together with $r_i$.

\textsuperscript{a}Black, Derman, and Toy (BDT) (1990).
The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes $v_i$ are given a priori.
  - A related model of Salomon Brothers takes $v_i$ to be constants.

- Lognormal models preclude negative short rates.
The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the $i$-period zero-coupon bond be denoted by $\kappa_i$.
- $P_u$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes an up move.
The BDT Model: Volatility Structure (concluded)

- $P_d$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes a down move.

- Corresponding to these two prices are the following yields to maturity,

$$y_u \equiv P_u^{-1/(i-1)} - 1,$$
$$y_d \equiv P_d^{-1/(i-1)} - 1.$$

- The yield volatility is defined as $\kappa_i \equiv (1/2) \ln(y_u/y_d)$. 

The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated $(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1})$.
  - They define the binomial tree up to period $i - 1$.
- We now proceed to calculate $r_i$ and $v_i$ to extend the tree to period $i$. 
The BDT Model: Calibration (continued)

- Assume the price of the \( i \)-period zero can move to \( P_u \) or \( P_d \) one period from now.

- Let \( y \) denote the current \( i \)-period spot rate, which is known.

- In a risk-neutral economy,

\[
\frac{P_u + P_d}{2(1 + r_1)} = \frac{1}{(1 + y)^i}.
\]  

\[ (115) \]

- Obviously, \( P_u \) and \( P_d \) are functions of the unknown \( r_i \) and \( v_i \).
The BDT Model: Calibration (continued)

• Viewed from now, the future \((i - 1)\)-period spot rate at time one is uncertain.

• Recall that \(y_u\) and \(y_d\) represent the spot rates at the up node and the down node, respectively (p. 972).

• With \(\kappa^2\) denoting their variance, we have

\[
\kappa_i = \frac{1}{2} \ln \left( \frac{P_u^{-1/(i-1)} - 1}{P_d^{-1/(i-1)} - 1} \right). \tag{116}
\]
The BDT Model: Calibration (continued)

• We will employ forward induction to derive a quadratic-time calibration algorithm.\(^{a}\)

• Recall that forward induction inductively figures out, by moving forward in time, how much $1 at a node contributes to the price (review p. 833(a)).

• This number is called the state price and is the price of the claim that pays $1 at that node and zero elsewhere.

\(^{a}\)Chen (R84526007) and Lyuu (1997); Lyuu (1999).
The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period $i$ be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i - 1$ be $P_1, P_2, \ldots, P_i$, corresponding to rates $r, rv, \ldots, rv^{i-1}$, respectively.
- One dollar at time $i$ has a present value of

$$f(r, v) \equiv \frac{P_1}{1 + r} + \frac{P_2}{1 + rv} + \frac{P_3}{1 + rv^2} + \cdots + \frac{P_i}{1 + rv^{i-1}}.$$
The BDT Model: Calibration (continued)

- The yield volatility is

\[ g(r, v) \equiv \frac{1}{2} \ln \left( \frac{\left( \frac{P_{u,1}}{1+r} + \frac{P_{u,2}}{1+rv} + \cdots + \frac{P_{u,i-1}}{1+rv^{i-1}} \right)^{-1/(i-1)} - 1}{\left( \frac{P_{d,1}}{1+r} + \frac{P_{d,2}}{1+rv} + \cdots + \frac{P_{d,i-1}}{1+rv^{i-2}} \right)^{-1/(i-1)} - 1} \right). \]

- Above, \( P_{u,1}, P_{u,2}, \ldots \) denote the state prices at time \( i-1 \) of the subtree rooted at the up node (like \( r_2v_2 \) on p. 969).

- And \( P_{d,1}, P_{d,2}, \ldots \) denote the state prices at time \( i-1 \) of the subtree rooted at the down node (like \( r_2 \) on p. 969).
The BDT Model: Calibration (concluded)

- Now solve

$$f(r, v) = \frac{1}{(1 + y)^i} \quad \text{and} \quad g(r, v) = \kappa_i$$

for $r = r_i$ and $v = v_i$.

- This $O(n^2)$-time algorithm appears in the text.
The BDT Model: Continuous-Time Limit

- The continuous-time limit of the BDT model is

\[ d \ln r = \left( \theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t) dW. \]

- The short rate volatility clearly should be a declining function of time for the model to display mean reversion.
  - That makes \( \sigma'(t) < 0 \).

- In particular, constant volatility will not attain mean reversion.
The Black-Karasinski Model\textsuperscript{a}

• The BK model stipulates that the short rate follows

\[ d \ln r = \kappa(t)(\theta(t) - \ln r) \, dt + \sigma(t) \, dW. \]

• This explicitly mean-reverting model depends on time through \( \kappa(\cdot), \theta(\cdot), \) and \( \sigma(\cdot). \)

• The BK model hence has one more degree of freedom than the BDT model.

• The speed of mean reversion \( \kappa(t) \) and the short rate volatility \( \sigma(t) \) are independent.

\textsuperscript{a}Black and Karasinski (1991).
The Black-Karasinski Model: Discrete Time

- The discrete-time version of the BK model has the same representation as the BDT model.
- To maintain a combining binomial tree, however, requires some manipulations.
- The next plot illustrates the ideas in which

\[
\begin{align*}
t_2 & \equiv t_1 + \Delta t_1, \\
t_3 & \equiv t_2 + \Delta t_2.
\end{align*}
\]
\[ \ln r_d(t_2) \]
\[ \ln r(t_1) \]
\[ \ln r_u(t_2) \]
\[ \ln r_{du}(t_3) = \ln r_{ud}(t_3) \]
The Black-Karasinski Model: Discrete Time (continued)

- Note that
  \[
  \ln r_d(t_2) = \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1)) \Delta t_1 - \sigma(t_1)\sqrt{\Delta t_1}, \\
  \ln r_u(t_2) = \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1)) \Delta t_1 + \sigma(t_1)\sqrt{\Delta t_1}.
  \]

- To ensure that an up move followed by a down move coincides with a down move followed by an up move, impose
  \[
  \ln r_d(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_d(t_2)) \Delta t_2 + \sigma(t_2)\sqrt{\Delta t_2}, \\
  = \ln r_u(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_u(t_2)) \Delta t_2 - \sigma(t_2)\sqrt{\Delta t_2}.
  \]
The Black-Karasinski Model: Discrete Time (concluded)

- They imply

\[ \kappa(t_2) = \frac{1 - (\sigma(t_2)/\sigma(t_1)) \sqrt{\Delta t_2/\Delta t_1}}{\Delta t_2}. \]  

(117)

- So from \( \Delta t_1 \), we can calculate the \( \Delta t_2 \) that satisfies the combining condition and then iterate.

\[ t_0 \to \Delta t_0 \to t_1 \to \Delta t_1 \to t_2 \to \Delta t_2 \to \cdots \to T \]
(roughly).

Problems with Lognormal Models in General

• Lognormal models such as BDT and BK share the problem that \( E^{\pi}[M(t)] = \infty \) for any finite \( t \) if they the continuously compounded rate.

• Hence periodic compounding should be used.

• Another issue is computational.

• Lognormal models usually do not give analytical solutions to even basic fixed-income securities.

• As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.
Problems with Lognormal Models in General (concluded)

- This problem can be somewhat mitigated by adopting different time steps: Use a fine time step up to the maturity of the short-dated derivative and a coarse time step beyond the maturity.\(^a\)

- A down side of this procedure is that it has to be carried out for each derivative.

- Finally, empirically, interest rates do not follow the lognormal distribution.

\(^a\)Hull and White (1993).
The Extended Vasicek Model\textsuperscript{a}

- Hull and White proposed models that extend the Vasicek model and the CIR model.
- They are called the extended Vasicek model and the extended CIR model.
- The extended Vasicek model adds time dependence to the original Vasicek model,

\[ dr = (\theta(t) - a(t)r) \, dt + \sigma(t) \, dW. \]

- Like the Ho-Lee model, this is a normal model, and the inclusion of \( \theta(t) \) allows for an exact fit to the current spot rate curve.

\textsuperscript{a}Hull and White (1990).
The Extended Vasicek Model (concluded)

- Function $\sigma(t)$ defines the short rate volatility, and $a(t)$ determines the shape of the volatility structure.

- Under this model, many European-style securities can be evaluated analytically, and efficient numerical procedures can be developed for American-style securities.
The Hull-White Model

• The Hull-White model is the following special case,

\[ dr = (\theta(t) - ar) \, dt + \sigma \, dW. \]

• When the current term structure is matched,\(^a\)

\[ \theta(t) = \frac{\partial f(0, t)}{\partial t} + af(0, t) + \frac{\sigma^2}{2a} \left(1 - e^{-2at}\right). \]

\(^a\)Hull and White (1993).