Problems with the Monte Carlo Method

- The error bound is only probabilistic.
- The probabilistic error bound of $\sqrt{N}$ does not benefit from regularity of the integrand function.
- The requirement that the points be independent random samples are wasteful because of clustering.
- In reality, pseudorandom numbers generated by completely deterministic means are used.
- Monte Carlo simulation exhibits a great sensitivity on the seed of the pseudorandom-number generator.

Matrix Computation

To set up a philosophy against physics is rash;
philosophers who have done so
have always ended in disaster.
— Bertrand Russell

Definitions and Basic Results

- Let $A \equiv [\, a_{ij} \,]_{1 \leq i \leq m, 1 \leq j \leq n}$, or simply $A \in \boldsymbol{R}^{m \times n}$, denote an $m \times n$ matrix.
- It can also be represented as $[\, a_1, a_2, \ldots, a_n \,]$ where $a_i \in \boldsymbol{R}^m$ are vectors.
  - Vectors are column vectors unless stated otherwise.
- A is a square matrix when $m = n$.
- The rank of a matrix is the largest number of linearly independent columns.

Definitions and Basic Results (continued)

- A square matrix A is said to be symmetric if $A^{\mathrm{T}} = A$.
- A real $n \times n$ matrix

$$A \equiv [\, a_{ij} \,]_{i,j}$$

  is diagonally dominant if $|\, a_{ii} \,| > \sum_{j \neq i} |\, a_{ij} \,|$ for $1 \leq i \leq n$.
  - Such matrices are nonsingular.
- The identity matrix is the square matrix

$$I \equiv \mathrm{diag}[\, 1, 1, \ldots, 1 \,].$$

Definitions and Basic Results (concluded)

- A matrix has full column rank if its columns are linearly independent.
- A real symmetric matrix A is positive definite if

$$x^{\mathrm{T}} A x = \sum_{i,j} a_{ij} x_i x_j > 0$$

  for any nonzero vector x.
- A matrix A is positive definite if and only if there exists a matrix W such that $A = W^{\mathrm{T}} W$ and W has full column rank.

Cholesky Decomposition

- Positive definite matrices can be factored as

$$A = LL^{\mathrm{T}},$$

called the Cholesky decomposition.

  - Above, L is a lower triangular matrix.

Generation of Multivariate Distribution

- Let $\boldsymbol{x} \equiv [\, x_1, x_2, \ldots, x_n \,]^{\mathrm{T}}$ be a vector random variable with a positive definite covariance matrix C.
- As usual, assume $E[\, \boldsymbol{x} \,] = \mathbf{0}$.
- This distribution can be generated by $P\boldsymbol{y}$.
  - $C = PP^{\mathrm{T}}$ is the Cholesky decomposition of C.[a]
  - $\boldsymbol{y} \equiv [\, y_1, y_2, \ldots, y_n \,]^{\mathrm{T}}$ is a vector random variable with a covariance matrix equal to the identity matrix.

---

[a]What if C is not positive definite? See Lai (`R93942114`) and Lyuu (2007).

Generation of Multivariate Normal Distribution

- Suppose we want to generate the multivariate normal distribution with a covariance matrix $C = PP^{\mathrm{T}}$.
- We start with independent standard normal distributions $y_1, y_2, \ldots, y_n$.
- Then $P[\, y_1, y_2, \ldots, y_n \,]^{\mathrm{T}}$ has the desired distribution.

Multivariate Derivatives Pricing

- Generating the multivariate normal distribution is essential for the Monte Carlo pricing of multivariate derivatives (p. 623).
- For example, the rainbow option on k assets has payoff

$$\max(\max(S_1, S_2, \ldots, S_k) - X, 0)$$

at maturity.

- The closed-form formula is a multi-dimensional integral.[a]

[a] Johnson (1987); Chen (D95723006) and Lyuu (2009).

Multivariate Derivatives Pricing (concluded)

- Suppose $dS_j/S_j = r\,dt + \sigma_j\,dW_j$, $1 \leq j \leq k$, where C is the correlation matrix for $dW_1, dW_2, \ldots, dW_k$.
- Let $C = PP^{\mathrm{T}}$.
- Let ξ consist of k independent random variables from $N(0,1)$.
- Let $\xi' = P\xi$.
- Similar to Eq. (76) on p. 662,

$$S_{i+1} = S_i e^{(r-\sigma_j^2/2)\,\Delta t + \sigma_j \sqrt{\Delta t}\;\xi'_j}, \quad 1 \leq j \leq k.$$

Least-Squares Problems

- The least-squares (LS) problem is concerned with

$$\min_{x \in R^n} \| Ax - b \|,$$

where $A \in \boldsymbol{R}^{m \times n}$, $b \in \boldsymbol{R}^m$, $m \geq n$.

- The LS problem is called regression analysis in statistics and is equivalent to minimizing the mean-square error.
- Often written as

$$Ax = b.$$

Polynomial Regression

- In polynomial regression, $x_0 + x_1 x + \cdots + x_n x^n$ is used to fit the data $\{ (a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m) \}$.
- This leads to the LS problem,

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^n \\ 1 & a_2 & a_2^2 & \cdots & a_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \cdots & a_m^n \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

- Consult the text for solutions.

American Option Pricing by Simulation

- The continuation value of an American option is the conditional expectation of the payoff from keeping the option alive now.
- The option holder must compare the immediate exercise value and the continuation value.
- In standard Monte Carlo simulation, each path is treated independently of other paths.
- But the decision to exercise the option cannot be reached by looking at only one path alone.

The Least-Squares Monte Carlo Approach

- The continuation value can be estimated from the cross-sectional information in the simulation by using least squares.[a]
- The result is a function (of the state) for estimating the continuation values.
- Use the function to estimate the continuation value for each path to determine its cash flow.
- This is called the least-squares Monte Carlo (LSM) approach and is provably convergent.[b]

---

[a] Longstaff and Schwartz (2001).

[b] Clément, Lamberton, and Protter (2002).

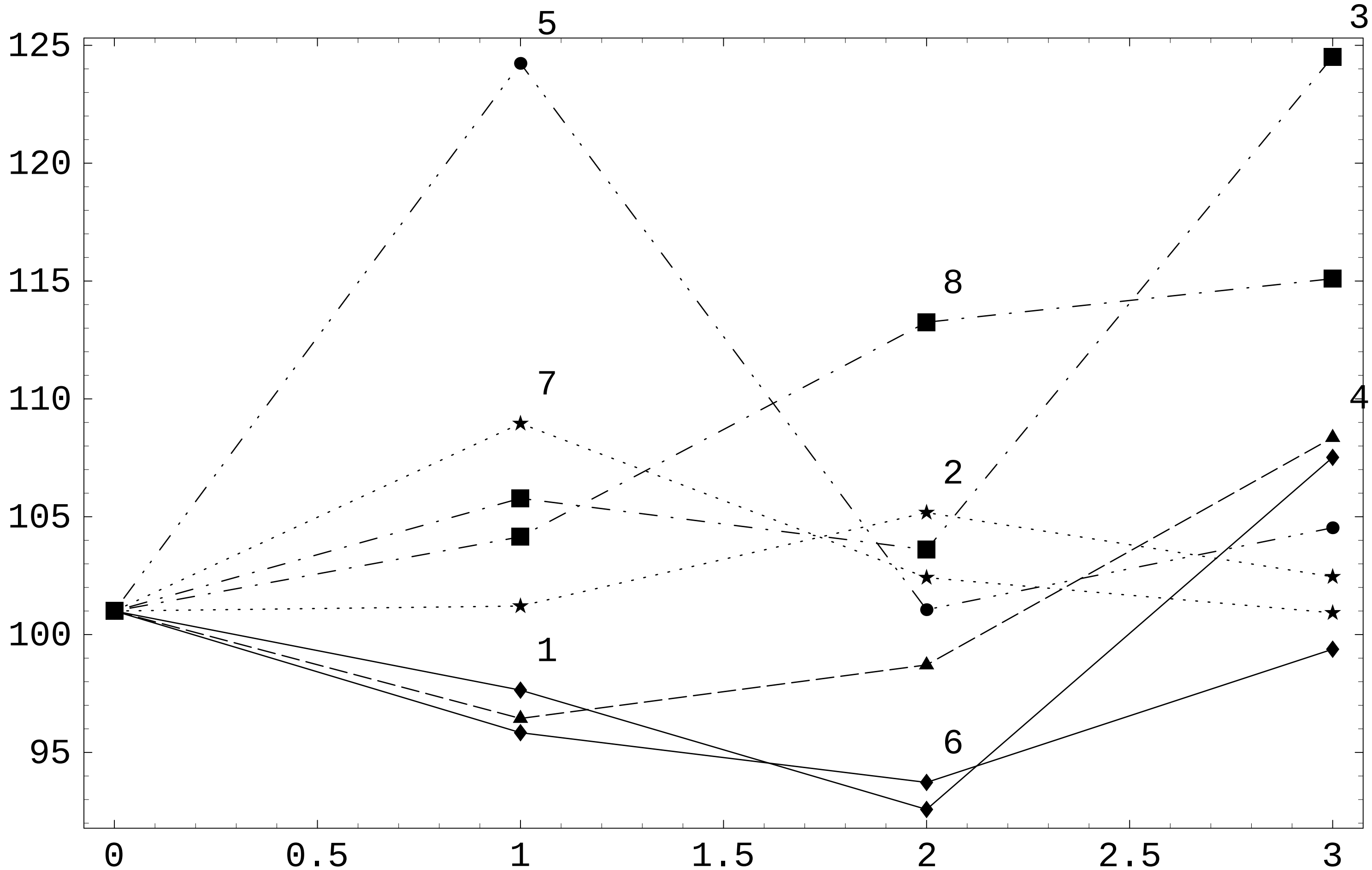
A Numerical Example

- Consider a 3-year American put on a non-dividend-paying stock.
- The put is exercisable at years 0, 1, 2, and 3.
- The strike price $X = 105$.
- The annualized riskless rate is $r = 5\%$.
- The spot stock price is 101.
  - The annual discount factor hence equals 0.951229.
- We use only 8 price paths to illustrate the algorithm.

A Numerical Example (continued)

Stock price paths

Path	Year 0	Year 1	Year 2	Year 3
1	**101**	**97.6424**	**92.5815**	107.5178
2	**101**	**101.2103**	105.1763	**102.4524**
3	**101**	105.7802	**103.6010**	124.5115
4	**101**	**96.4411**	**98.7120**	108.3600
5	**101**	124.2345	**101.0564**	**104.5315**
6	**101**	**95.8375**	**93.7270**	**99.3788**
7	**101**	108.9554	**102.4177**	**100.9225**
8	**101**	**104.1475**	113.2516	115.0994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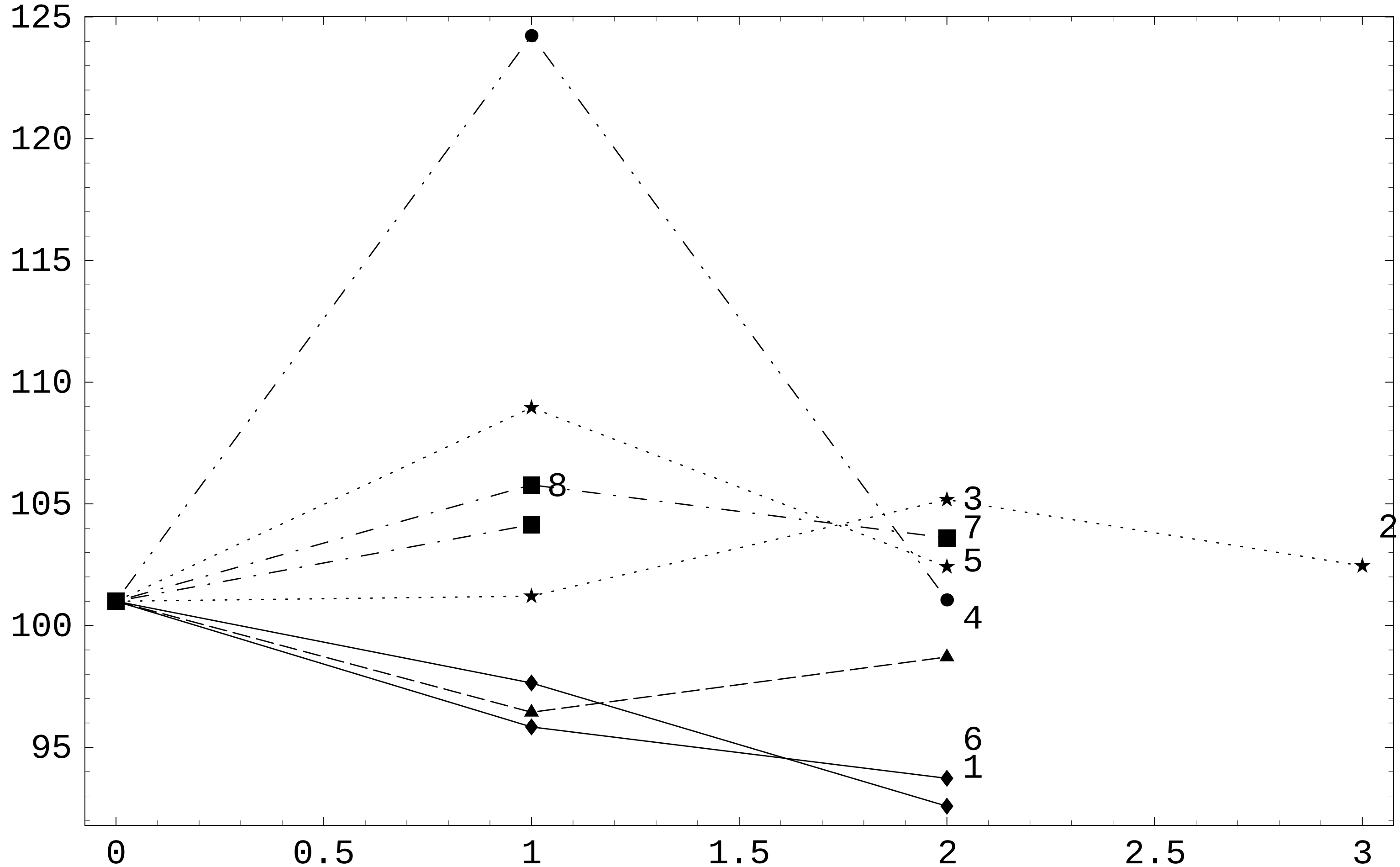
6

A Numerical Example (continued)

- We use the basis functions $1, x, x^2$.
  - Other basis functions are possible.[a]
- The plot next page shows the final estimated optimal exercise strategy given by LSM.
- We now proceed to tackle our problem.
- Our concrete problem is to calculate the cash flow along each path, using information from all paths.

[a]Laguerre polynomials, Hermite polynomials, Legendre polynomials, Chebyshev polynomials, Gedenbauer polynomials, and Jacobi polynomials.



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A Numerical Example (continued)

Cash flows at year 3

Path	Year 0	Year 1	Year 2	Year 3
1	—	—	—	0
2	—	—	—	2.5476
3	—	—	—	0
4	—	—	—	0
5	—	—	—	0.4685
6	—	—	—	5.6212
7	—	—	—	4.0775
8	—	—	—	0

A Numerical Example (continued)

- The cash flows at year 3 are the exercise value if the put is in the money.
- Only 4 paths are in the money: 2, 5, 6, 7.
- Some of the cash flows may not occur if the put is exercised earlier, which we will find out step by step.
- Incidentally, the European counterpart has a value of

$$0.951229^3 \times \frac{2.5476 + 0.4685 + 5.6212 + 4.0775}{8} = 1.3680.$$

A Numerical Example (continued)

- We move on to year 2.
- For each state that is in the money at year 2, we must decide whether to exercise it.
- There are 6 paths for which the put is in the money: 1, 3, 4, 5, 6, 7.
- Only in-the-money paths will be used in the regression because they are where early exercise is relevant.
  - If there were none, we would move on to year 1.

A Numerical Example (continued)

- Let x denote the stock prices at year 2 for those 6 paths.
- Let y denote the corresponding discounted future cash flows (at year 3) if the put is not exercised at year 2.

A Numerical Example (continued)

Regression at year 2

Path	x	y
1	92.5815	0×0.951229
2	—	—
3	103.6010	0×0.951229
4	98.7120	0×0.951229
5	101.0564	0.4685×0.951229
6	93.7270	5.6212×0.951229
7	102.4177	4.0775×0.951229
8	—	—

A Numerical Example (continued)

- We regress y on 1, x, and x^2.
- The result is

$$f(x) = 22.08 - 0.313114 \times x + 0.00106918 \times x^2.$$

- f estimates the continuation value conditional on the stock price at year 2.
- We next compare the immediate exercise value and the continuation value.

A Numerical Example (continued)

Optimal early exercise decision at year 2

Path	Exercise	Continuation
1	12.4185	$f(92.5815) = 2.2558$
2	—	—
3	1.3990	$f(103.6010) = 1.1168$
4	6.2880	$f(98.7120) = 1.5901$
5	3.9436	$f(101.0564) = 1.3568$
6	11.2730	$f(93.7270) = 2.1253$
7	2.5823	$f(102.4177) = 0.3326$
8	—	—

A Numerical Example (continued)

- Amazingly, the put should be exercised in all 6 paths: 1, 3, 4, 5, 6, 7.
- Now, any positive cash flow at year 3 should be set to zero for these paths as the put is exercised before year 3.
  - They are paths 5, 6, 7.
- Hence the cash flows on p. 717 become the next ones.

A Numerical Example (continued)

Cash flows at years 2 & 3

Path	Year 0	Year 1	Year 2	Year 3
1	—	—	12.4185	0
2	—	—	0	2.5476
3	—	—	1.3990	0
4	—	—	6.2880	0
5	—	—	3.9436	0
6	—	—	11.2730	0
7	—	—	2.5823	0
8	—	—	0	0

A Numerical Example (continued)

- We move on to year 1.
- For each state that is in the money at year 1, we must decide whether to exercise it.
- There are 5 paths for which the put is in the money: 1, 2, 4, 6, 8.
- Only in-the-money paths will be used in the regression because they are where early exercise is relevant.
  - If there were none, we would move on to year 0.

A Numerical Example (continued)

- Let x denote the stock prices at year 1 for those 5 paths.
- Let y denote the corresponding discounted future cash flows if the put is not exercised at year 1.
- From p. 725, we have the following table.

A Numerical Example (continued)

Regression at year 1

Path	x	y
1	97.6424	12.4185×0.951229
2	101.2103	2.5476×0.951229^2
3	—	—
4	96.4411	6.2880×0.951229
5	—	—
6	95.8375	11.2730×0.951229
7	—	—
8	104.1475	0

A Numerical Example (continued)

- We regress y on 1, x, and x^2.
- The result is

$$f(x) = -420.964 + 9.78113 \times x - 0.0551567 \times x^2.$$

- f estimates the continuation value conditional on the stock price at year 1.
- We next compare the immediate exercise value and the continuation value.

A Numerical Example (continued)

Optimal early exercise decision at year 1

Path	Exercise	Continuation
1	7.3576	$f(97.6424) = 8.2230$
2	3.7897	$f(101.2103) = 3.9882$
3	—	—
4	8.5589	$f(96.4411) = 9.3329$
5	—	—
6	9.1625	$f(95.8375) = 9.83042$
7	—	—
8	0.8525	$f(104.1475) = -0.551885$

A Numerical Example (continued)

- The put should be exercised for 1 path only: 8.
- Now, any positive future cash flow should be set to zero for this path as the put is exercised before years 2 and 3.
  - But there is none.
- Hence the cash flows on p. 725 become the next ones.
- They also confirm the plot on p. 716.

A Numerical Example (continued)

Cash flows at years 1, 2, & 3

Path	Year 0	Year 1	Year 2	Year 3
1	—	0	12.4185	0
2	—	0	0	2.5476
3	—	0	1.3990	0
4	—	0	6.2880	0
5	—	0	3.9436	0
6	—	0	11.2730	0
7	—	0	2.5823	0
8	—	0.8525	0	0

A Numerical Example (continued)

- We move on to year 0.
- The continuation value is, from p 732,

$$\begin{aligned}
& (12.4185 \times 0.951229^2 + 2.5476 \times 0.951229^3 \\
& +1.3990 \times 0.951229^2 + 6.2880 \times 0.951229^2 \\
& +3.9436 \times 0.951229^2 + 11.2730 \times 0.951229^2 \\
& +2.5823 \times 0.951229^2 + 0.8525 \times 0.951229)/8 \\
= \;& 4.66263.
\end{aligned}$$

A Numerical Example (concluded)

- As this is larger than the immediate exercise value of $105 - 101 = 4$, the put should not be exercised at year 0.
- Hence the put's value is estimated to be 4.66263.
- Compare this to the European put's value of 1.3680 (p. 718).
- Why is the LSM estimate a lower bound?[a]

---

[a]Contributed by Mr. Yang, Jui-Chung (`D97723002`) on April 29, 2009.

Time Series Analysis

The historian is a prophet in reverse.
— Friedrich von Schlegel (1772–1829)

Conditional Variance Models for Price Volatility

- Although a stationary model (see text for definition) has constant variance, its *conditional* variance may vary.
- Take for example an AR(1) process $X_t = aX_{t-1} + \epsilon_t$ with $|\, a \,| < 1$.
  - Here, ϵ_t is a stationary, uncorrelated process with zero mean and constant variance σ^2.
- The conditional variance,

$$\mathrm{Var}[\, X_t \mid X_{t-1}, X_{t-2}, \ldots \,],$$

equals σ^2, which is smaller than the *unconditional* variance $\mathrm{Var}[\, X_t \,] = \sigma^2/(1 - a^2)$.

Conditional Variance Models for Price Volatility (concluded)

- In the lognormal model, the conditional variance evolves independently of past returns.
- Suppose we assume that conditional variances are deterministic functions of past returns:

  $$V_t = f(X_{t-1}, X_{t-2}, \dots)$$

  for some function f.
- Then V_t can be computed given the information set of past returns:

  $$I_{t-1} \equiv \{ X_{t-1}, X_{t-2}, \dots \}.$$

ARCH Models[a]

- An influential model in this direction is the autoregressive conditional heteroskedastic (ARCH) model.
- Assume that $\{ U_t \}$ is a Gaussian stationary, uncorrelated process.

---

[a]Engle (1982), co-winner of the 2003 Nobel Prize in Economic Sciences.

ARCH Models (continued)

- The ARCH(p) process is defined by

$$X_t - \mu = \left(a_0 + \sum_{i=1}^{p} a_i (X_{t-i} - \mu)^2 \right)^{1/2} U_t,$$

where $a_1, \ldots, a_p \geq 0$ and $a_0 > 0$.

  – Thus $X_t \mid I_{t-1} \sim N(\mu, V_t^2)$.

- The variance V_t^2 satisfies

$$V_t^2 = a_0 + \sum_{i=1}^{p} a_i (X_{t-i} - \mu)^2.$$

- The volatility at time t as estimated at time $t-1$ depends on the p most recent observations on squared returns.

ARCH Models (concluded)

- The ARCH(1) process

$$X_t - \mu = (a_0 + a_1(X_{t-1} - \mu)^2)^{1/2} U_t$$

is the simplest.

- For it,

$$\mathrm{Var}[\, X_t \mid X_{t-1} = x_{t-1} \,] = a_0 + a_1(x_{t-1} - \mu)^2.$$

- The process $\{ X_t \}$ is stationary with finite variance if and only if $a_1 < 1$, in which case $\mathrm{Var}[\, X_t \,] = a_0/(1 - a_1)$.

GARCH Models[a]

- A very popular extension of the ARCH model is the generalized autoregressive conditional heteroskedastic (GARCH) process.
- The simplest GARCH$(1,1)$ process adds $a_2 V_{t-1}^2$ to the ARCH(1) process, resulting in

$$V_t^2 = a_0 + a_1 (X_{t-1} - \mu)^2 + a_2 V_{t-1}^2.$$

- The volatility at time t as estimated at time $t-1$ depends on the squared return and the estimated volatility at time $t-1$.

[a] Bollerslev (1986); Taylor (1986).

GARCH Models (concluded)

- The estimate of volatility averages past squared returns by giving heavier weights to recent squared returns (see text).
- It is usually assumed that $a_1 + a_2 < 1$ and $a_0 > 0$, in which case the unconditional, long-run variance is given by $a_0/(1 - a_1 - a_2)$.
- A popular special case of $\text{GARCH}(1, 1)$ is the exponentially weighted moving average process, which sets a_0 to zero and a_2 to $1 - a_1$.
- This model is used in J.P. Morgan's RiskMetrics™.

GARCH Option Pricing[a]

- Options can be priced when the underlying asset's return follows a GARCH process.
- Let S_t denote the asset price at date t.
- Let h_t^2 be the conditional variance of the return over the period $[t, t+1]$ given the information at date t.
  - "One day" is merely a convenient term for any elapsed time Δt.

[a] A Bloomberg quant said, on Feb 29, 2008, that GARCH option pricing is seldom used in trading.

GARCH Option Pricing (continued)

- Adopt the following risk-neutral process for the price dynamics:[a]

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{h_t^2}{2} + h_t \epsilon_{t+1}, \tag{79}$$

where

$$\begin{aligned} h_{t+1}^2 &= \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 (\epsilon_{t+1} - c)^2, \quad (80) \\ \epsilon_{t+1} &\sim N(0,1) \text{ given information at date } t, \\ r &= \text{daily riskless return}, \\ c &\geq 0. \end{aligned}$$

[a]Duan (1995).

GARCH Option Pricing (continued)

- The five unknown parameters of the model are c, h_0, β_0, β_1, and β_2.
- It is postulated that $\beta_0, \beta_1, \beta_2 \geq 0$ to make the conditional variance positive.
- The above process, called the nonlinear asymmetric GARCH model, generalizes the GARCH$(1, 1)$ model (see text).

GARCH Option Pricing (continued)

- It captures the volatility clustering in asset returns first noted by Mandelbrot (1963).[a]
  - When $c = 0$, a large ϵ_{t+1} results in a large h_{t+1}, which in turns tends to yield a large h_{t+2}, and so on.
- It also captures the negative correlation between the asset return and changes in its (conditional) volatility.[b]
  - For $c > 0$, a positive ϵ_{t+1} (good news) tends to decrease h_{t+1}, whereas a negative ϵ_{t+1} (bad news) tends to do the opposite.

[a] "... large changes tend to be followed by large changes—of either sign—and small changes tend to be followed by small changes ..."

[b] Noted by Black (1976): Volatility tends to rise in response to "bad news" and fall in response to "good news."

GARCH Option Pricing (concluded)

- With $y_t \equiv \ln S_t$ denoting the logarithmic price, the model becomes
$$y_{t+1} = y_t + r - \frac{h_t^2}{2} + h_t \epsilon_{t+1}. \tag{81}$$
- The pair (y_t, h_t^2) completely describes the current state.
- The conditional mean and variance of y_{t+1} are clearly
$$E[\, y_{t+1} \,|\, y_t, h_t^2 \,] = y_t + r - \frac{h_t^2}{2}, \tag{82}$$
$$\mathrm{Var}[\, y_{t+1} \,|\, y_t, h_t^2 \,] = h_t^2. \tag{83}$$

The Ritchken-Trevor (RT) Algorithm[a]

- The GARCH model is a continuous-state model.
- To approximate it, we turn to trees with *discrete* states.
- Path dependence in GARCH makes the tree for asset prices explode exponentially (why?).
- We need to mitigate this combinatorial explosion.

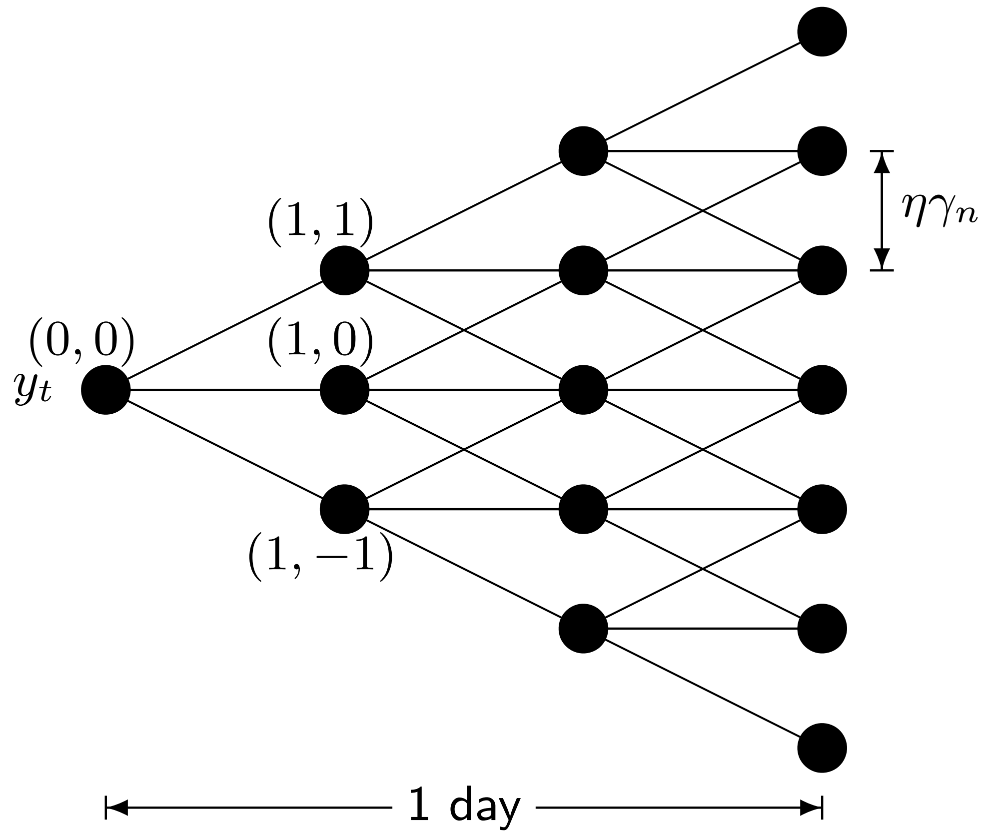
[a]Ritchken and Trevor (1999).

The Ritchken-Trevor Algorithm (continued)

- Partition a day into n periods.
- Three states follow each state (y_t, h_t^2) after a period.
- As the trinomial model combines, $2n+1$ states at date $t+1$ follow each state at date t (recall p. 579).
- These $2n+1$ values must approximate the distribution of (y_{t+1}, h_{t+1}^2).
- So the conditional moments (82)–(83) at date $t+1$ on p. 748 must be matched by the trinomial model to guarantee convergence to the continuous-state model.

The Ritchken-Trevor Algorithm (continued)

- It remains to pick the jump size and the three branching probabilities.
- The role of σ in the Black-Scholes option pricing model is played by h_t in the GARCH model.
- As a jump size proportional to $\sigma/\sqrt{n}$ is picked in the BOPM, a comparable magnitude will be chosen here.
- Define $\gamma \equiv h_0$, though other multiples of h_0 are possible, and
$$\gamma_n \equiv \frac{\gamma}{\sqrt{n}}.$$
- The jump size will be some integer multiple η of γ_n.
- We call η the jump parameter (p. 752).



The seven values on the right approximate the distribution of logarithmic price y_{t+1}.

The Ritchken-Trevor Algorithm (continued)

- The middle branch does not change the underlying asset's price.
- The probabilities for the up, middle, and down branches are

$$p_u = \frac{h_t^2}{2\eta^2\gamma^2} + \frac{r - (h_t^2/2)}{2\eta\gamma\sqrt{n}}, \tag{84}$$

$$p_m = 1 - \frac{h_t^2}{\eta^2\gamma^2}, \tag{85}$$

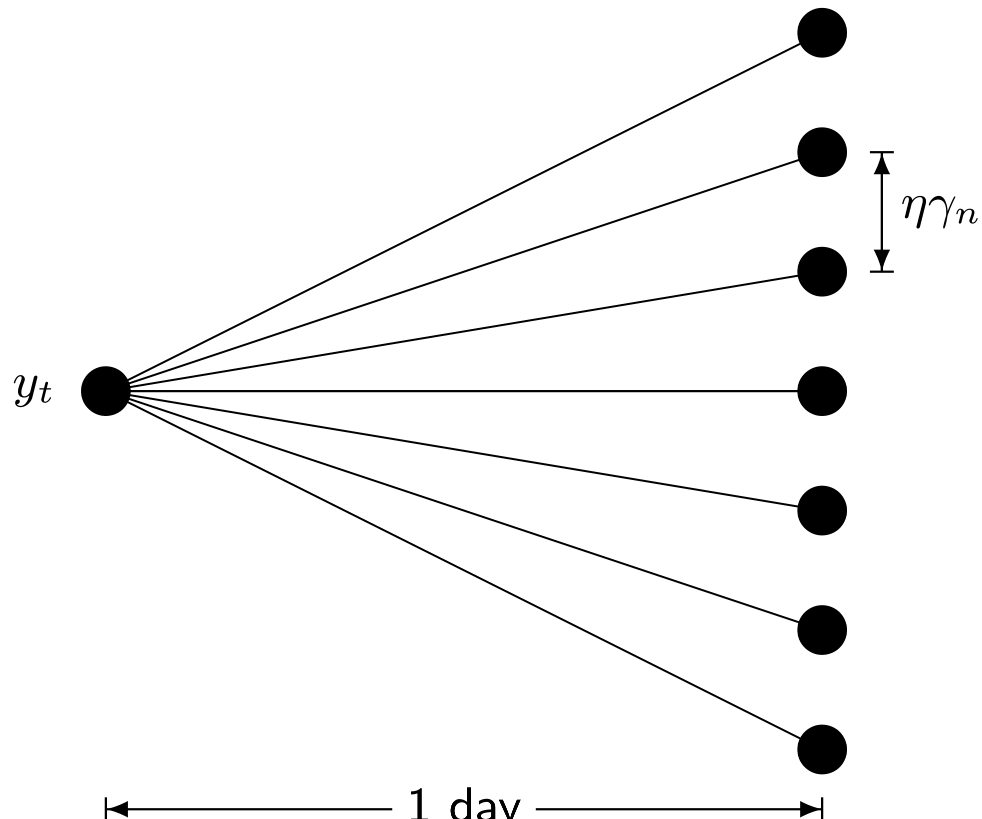
$$p_d = \frac{h_t^2}{2\eta^2\gamma^2} - \frac{r - (h_t^2/2)}{2\eta\gamma\sqrt{n}}. \tag{86}$$

The Ritchken-Trevor Algorithm (continued)

- It can be shown that:
  - The trinomial model takes on $2n+1$ values at date $t+1$ for y_{t+1}.
  - These values have a matching mean for y_{t+1}.
  - These values have an asymptotically matching variance for y_{t+1}.
- The central limit theorem thus guarantees the desired convergence as n increases.

The Ritchken-Trevor Algorithm (continued)

- We can dispense with the intermediate nodes *between* dates to create a $(2n+1)$-nomial tree (p. 756).
- The resulting model is multinomial with $2n+1$ branches from any state (y_t, h_t^2).
- There are two reasons behind this manipulation.
  - Interdate nodes are created merely to approximate the continuous-state model after one day.
  - Keeping the interdate nodes results in a tree that can be as much as n times larger.



This heptanomial tree is the outcome of the trinomial tree on p. 752 after its intermediate nodes are removed.

The Ritchken-Trevor Algorithm (continued)

- A node with logarithmic price $y_t + \ell\eta\gamma_n$ at date $t+1$ follows the current node at date t with price y_t, where $-n \leq \ell \leq n$.
- To reach that price in n periods, the number of up moves must exceed that of down moves by exactly ℓ.
- The probability that this happens is

$$P(\ell) \equiv \sum_{j_u, j_m, j_d} \frac{n!}{j_u!\, j_m!\, j_d!}\, p_u^{j_u} p_m^{j_m} p_d^{j_d},$$

with $j_u, j_m, j_d \geq 0$, $n = j_u + j_m + j_d$, and $\ell = j_u - j_d$.

The Ritchken-Trevor Algorithm (continued)

- A particularly simple way to calculate the $P(\ell)$s starts by noting that

$$(p_u x + p_m + p_d x^{-1})^n = \sum_{\ell=-n}^{n} P(\ell)\, x^{\ell}. \tag{87}$$

  - Convince yourself that this trick does the "accounting" correctly.

- So we expand $(p_u x + p_m + p_d x^{-1})^n$ and retrieve the probabilities by reading off the coefficients.

- It can be computed in $O(n^2)$ time.

The Ritchken-Trevor Algorithm (continued)

- The updating rule (80) on p. 745 must be modified to account for the adoption of the discrete-state model.
- The logarithmic price $y_t + \ell\eta\gamma_n$ at date $t+1$ following state (y_t, h_t^2) at date t has a variance equal to

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 (\epsilon'_{t+1} - c)^2, \tag{88}$$

  – Above,

$$\epsilon'_{t+1} = \frac{\ell\eta\gamma_n - (r - h_t^2/2)}{h_t}, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm n,$$

  is a discrete random variable with $2n+1$ values.

The Ritchken-Trevor Algorithm (continued)

- Different conditional variances h_t^2 may require different η so that the probabilities calculated by Eqs. (84)–(86) on p. 753 lie between 0 and 1.
- This implies varying jump sizes.
- The necessary requirement $p_m \geq 0$ implies $\eta \geq h_t/\gamma$.
- Hence we try

$$\eta = \lceil h_t/\gamma \rceil, \lceil h_t/\gamma \rceil + 1, \lceil h_t/\gamma \rceil + 2, \ldots$$

until valid probabilities are obtained or until their nonexistence is confirmed.

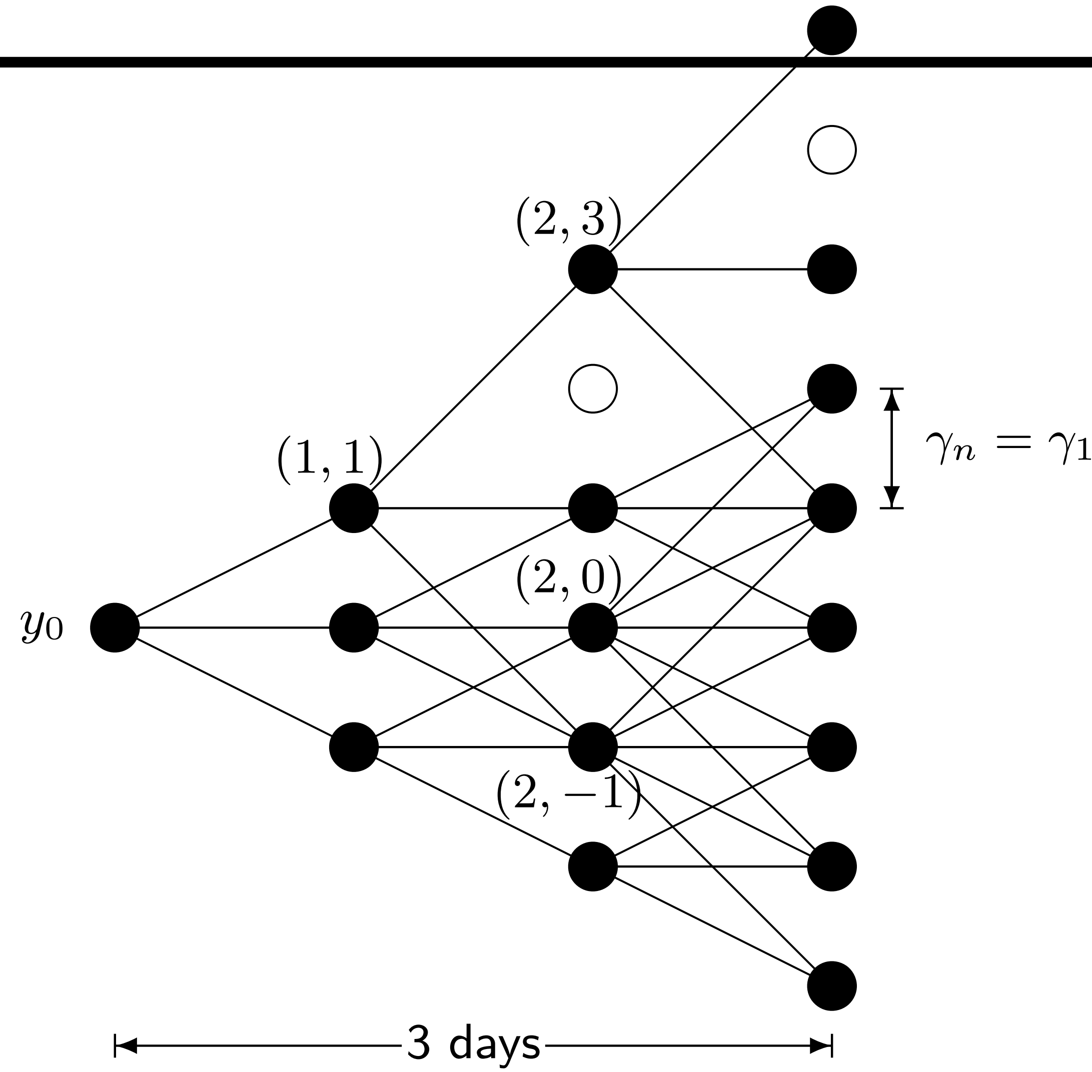
The Ritchken-Trevor Algorithm (continued)

- The sufficient and necessary condition for valid probabilities to exist is[a]

$$\frac{|\, r - (h_t^2/2) \,|}{2\eta\gamma\sqrt{n}} \leq \frac{h_t^2}{2\eta^2\gamma^2} \leq \min\left(1 - \frac{|\, r - (h_t^2/2) \,|}{2\eta\gamma\sqrt{n}}, \frac{1}{2}\right).$$

- Obviously, the magnitude of η tends to grow with h_t.
- The plot on p. 762 uses $n = 1$ to illustrate our points for a 3-day model.
- For example, node $(1, 1)$ of date 1 and node $(2, 3)$ of date 2 pick $\eta = 2$.

[a]Lyuu and Wu (R90723065) (2003).



$(2,3)$
$(1,1)$
$\gamma_n = \gamma_1$
y_0
$(2,0)$
$(2,-1)$
3 days

The Ritchken-Trevor Algorithm (continued)

- The topology of the tree is not a standard combining multinomial tree.
- For example, a few nodes on p. 762 such as nodes $(2, 0)$ and $(2, -1)$ have multiple jump sizes.
- The reason is the path dependence of the model.
  - Two paths can reach node $(2, 0)$ from the root node, each with a different variance for the node.
  - One of the variances results in $\eta = 1$, whereas the other results in $\eta = 2$.

The Ritchken-Trevor Algorithm (concluded)

- The number of possible values of h_t^2 at a node can be exponential.
  - Each path brings with it a different variance h_t^2.
- To address this problem, we record only the maximum and minimum h_t^2 at each node.[a]
- Therefore, each node on the tree contains only two states $(y_t, h_{\max}^2)$ and $(y_t, h_{\min}^2)$.
- Each of $(y_t, h_{\max}^2)$ and $(y_t, h_{\min}^2)$ carries its own η and set of $2n+1$ branching probabilities.

---

[a]Cakici and Topyan (2000). But see p. 797 for a potential problem.

Negative Aspects of the Ritchken-Trevor Algorithm[a]

- A small n may yield inaccurate option prices.
- But the tree will grow exponentially if n is large enough.
  - Specifically, $n > (1 - \beta_1)/\beta_2$ when $r = c = 0$.
- A large n has another serious problem: The tree cannot grow beyond a certain date.
- Thus the choice of n may be limited in practice.
- The RT algorithm can be modified to be free of shortened maturity and exponential complexity.[b]

---

[a] Lyuu and Wu (`R90723065`) (2003, 2005).

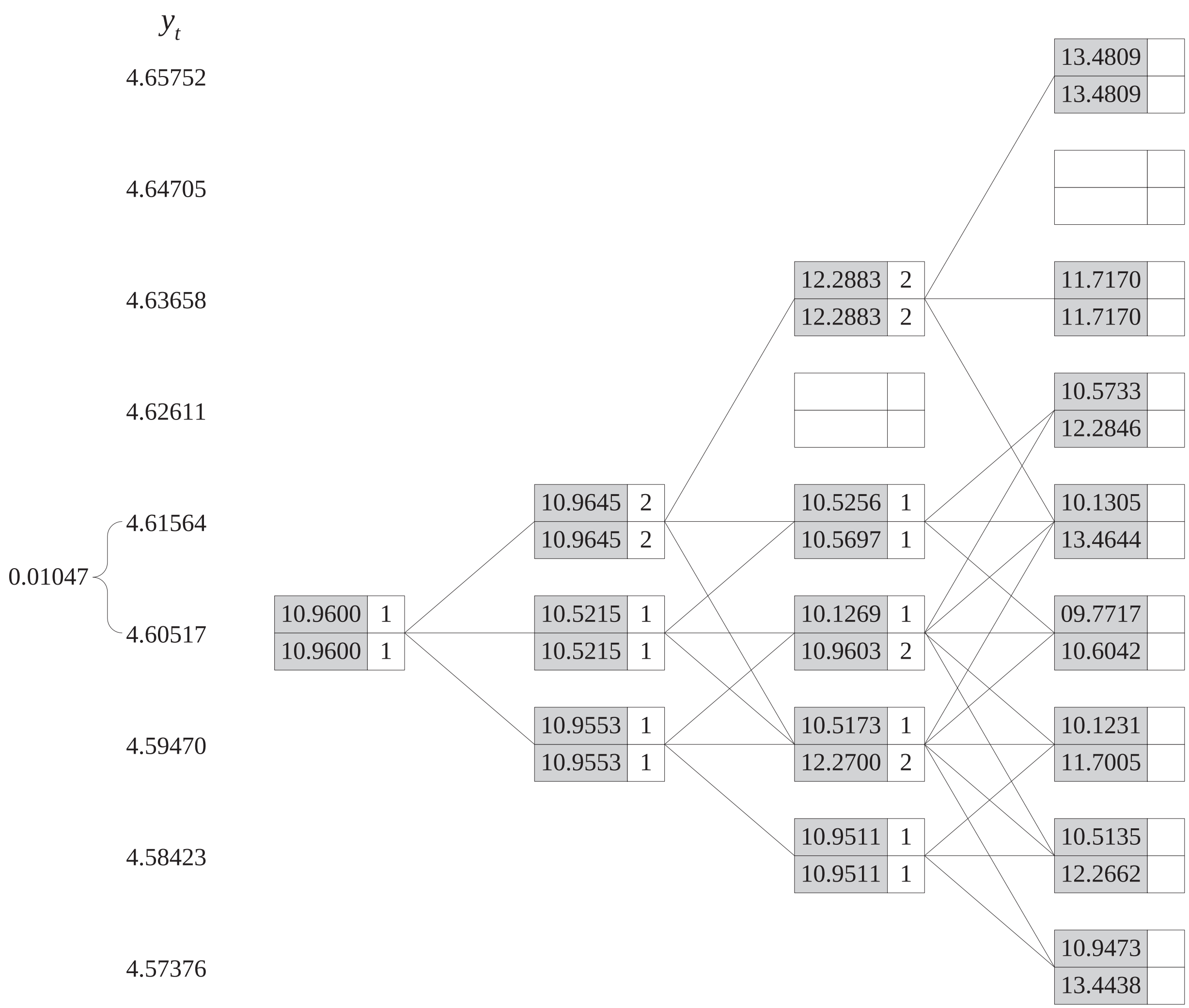
[b] It is only $O(n^2)$ if $n \leq (\sqrt{(1-\beta_1)/\beta_2} - c)^2$!

Numerical Examples

- Assume $S_0 = 100$, $y_0 = \ln S_0 = 4.60517$, $r = 0$, $h_0^2 = 0.0001096$, $\gamma = h_0 = 0.010469$, $n = 1$, $\gamma_n = \gamma/\sqrt{n} = 0.010469$, $\beta_0 = 0.000006575$, $\beta_1 = 0.9$, $\beta_2 = 0.04$, and $c = 0$.
- A daily variance of 0.0001096 corresponds to an annual volatility of $\sqrt{365 \times 0.0001096} \approx 20\%$.
- Let $h^2(i, j)$ denote the variance at node (i, j).
- Initially, $h^2(0, 0) = h_0^2 = 0.0001096$.

Numerical Examples (continued)

- Let $h^2_{\max}(i, j)$ denote the maximum variance at node (i, j).
- Let $h^2_{\min}(i, j)$ denote the minimum variance at node (i, j).
- Initially, $h^2_{\max}(0, 0) = h^2_{\min}(0, 0) = h^2_0$.
- The resulting three-day tree is depicted on p. 768.


y_t
4.65752
4.64705
4.63658
4.62611
4.61564
0.01047
4.60517
4.59470
4.58423
4.57376
10.9600 1
10.9600 1
10.9645 2
10.9645 2
10.5215 1
10.5215 1
10.9553 1
10.9553 1
12.2883 2
12.2883 2
10.5256 1
10.5697 1
10.1269 1
10.9603 2
10.5173 1
12.2700 2
10.9511 1
10.9511 1
13.4809
13.4809
11.7170
11.7170
10.5733
12.2846
10.1305
13.4644
09.7717
10.6042
10.1231
11.7005
10.5135
12.2662
10.9473
13.4438

A top (bottom) number inside a gray box refers to the minimum (maximum, resp.) variance $h^2_{\rm min}$ ($h^2_{\rm max}$, resp.) for the node. Variances are multiplied by 100,000 for readability. A top (bottom) number inside a white box refers to η corresponding to $h^2_{\rm min}$ ($h^2_{\rm max}$, resp.).

Numerical Examples (continued)

- Let us see how the numbers are calculated.
- Start with the root node, node $(0, 0)$.
- Try $\eta = 1$ in Eqs. (84)–(86) on p. 753 first to obtain

$$\begin{aligned} p_u &= 0.4974, \\ p_m &= 0, \\ p_d &= 0.5026. \end{aligned}$$

- As they are valid probabilities, the three branches from the root node use single jumps.

Numerical Examples (continued)

- Move on to node $(1,1)$.
- It has one predecessor node—node $(0,0)$—and it takes an up move to reach the current node.
- So apply updating rule (88) on p. 759 with $\ell = 1$ and $h_t^2 = h^2(0,0)$.
- The result is $h^2(1,1) = 0.000109645$.

Numerical Examples (continued)

- Because $\lceil h(1,1)/\gamma \rceil = 2$, we try $\eta = 2$ in Eqs. (84)–(86) on p. 753 first to obtain

$$\begin{aligned} p_u &= 0.1237, \\ p_m &= 0.7499, \\ p_d &= 0.1264. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(1,1)$ use double jumps.

Numerical Examples (continued)

- Carry out similar calculations for node $(1, 0)$ with $\ell = 0$ in updating rule (88) on p. 759.
- Carry out similar calculations for node $(1, -1)$ with $\ell = -1$ in updating rule (88).
- Single jump $\eta = 1$ works for both nodes.
- The resulting variances are

$$
\begin{aligned}
h^2(1, 0) &= 0.000105215, \\
h^2(1, -1) &= 0.000109553.
\end{aligned}
$$

Numerical Examples (continued)

- Node $(2, 0)$ has 2 predecessor nodes, $(1, 0)$ and $(1, -1)$.
- Both have to be considered in deriving the variances.
- Let us start with node $(1, 0)$.
- Because it takes a middle move to reach the current node, we apply updating rule (88) on p. 759 with $\ell = 0$ and $h_t^2 = h^2(1, 0)$.
- The result is $h_{t+1}^2 = 0.000101269$.

Numerical Examples (continued)

- Now move on to the other predecessor node $(1, -1)$.
- Because it takes an up move to reach the current node, apply updating rule (88) on p. 759 with $\ell = 1$ and $h_t^2 = h^2(1, -1)$.
- The result is $h_{t+1}^2 = 0.000109603$.
- We hence record

$$\begin{aligned} h^2_{\min}(2, 0) &= 0.000101269, \\ h^2_{\max}(2, 0) &= 0.000109603. \end{aligned}$$

Numerical Examples (continued)

- Consider state $h^2_{\max}(2,0)$ first.
- Because $\lceil h_{\max}(2,0)/\gamma \rceil = 2$, we first try $\eta = 2$ in Eqs. (84)–(86) on p. 753 to obtain

$$\begin{aligned} p_u &= 0.1237, \\ p_m &= 0.7500, \\ p_d &= 0.1263. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,0)$ with the maximum variance use double jumps.

Numerical Examples (continued)

- Now consider state $h^2_{\min}(2, 0)$.
- Because $\lceil h_{\min}(2, 0)/\gamma \rceil = 1$, we first try $\eta = 1$ in Eqs. (84)–(86) on p. 753 to obtain

$$
\begin{aligned}
p_u &= 0.4596, \\
p_m &= 0.0760, \\
p_d &= 0.4644.
\end{aligned}
$$

- As they are valid probabilities, the three branches from node $(2, 0)$ with the minimum variance use single jumps.

Numerical Examples (continued)

- Node $(2, -1)$ has 3 predecessor nodes.
- Start with node $(1, 1)$.
- Because it takes a down move to reach the current node, we apply updating rule (88) on p. 759 with $\ell = -1$ and $h_t^2 = h^2(1, 1)$.[a]
- The result is $h_{t+1}^2 = 0.0001227$.

[a]Note that it is not $\ell = -2$.

Numerical Examples (continued)

- Now move on to predecessor node $(1, 0)$.
- Because it also takes a down move to reach the current node, we apply updating rule (88) on p. 759 with $\ell = -1$ and $h_t^2 = h^2(1, 0)$.
- The result is $h_{t+1}^2 = 0.000105609$.

Numerical Examples (continued)

- Finally, consider predecessor node $(1, -1)$.
- Because it takes a middle move to reach the current node, we apply updating rule (88) on p. 759 with $\ell = 0$ and $h_t^2 = h^2(1, -1)$.
- The result is $h_{t+1}^2 = 0.000105173$.
- We hence record

$$\begin{aligned} h_{\min}^2(2, -1) &= 0.000105173, \\ h_{\max}^2(2, -1) &= 0.0001227. \end{aligned}$$

Numerical Examples (continued)

- Consider state $h_{\max}^2(2, -1)$.
- Because $\lceil h_{\max}(2, -1)/\gamma \rceil = 2$, we first try $\eta = 2$ in Eqs. (84)–(86) on p. 753 to obtain

$$
\begin{aligned}
p_u &= 0.1385, \\
p_m &= 0.7201, \\
p_d &= 0.1414.
\end{aligned}
$$

- As they are valid probabilities, the three branches from node $(2, -1)$ with the maximum variance use double jumps.

Numerical Examples (continued)

- Next, consider state $h^2_{\min}(2,-1)$.
- Because $\lceil h_{\min}(2,-1)/\gamma \rceil = 1$, we first try $\eta = 1$ in Eqs. (84)–(86) on p. 753 to obtain

$$\begin{aligned} p_u &= 0.4773, \\ p_m &= 0.0404, \\ p_d &= 0.4823. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,-1)$ with the minimum variance use single jumps.

Numerical Examples (concluded)

- Other nodes at dates 2 and 3 can be handled similarly.
- In general, if a node has k predecessor nodes, then $2k$ variances will be calculated using the updating rule.
  - This is because each predecessor node keeps two variance numbers.
- But only the maximum and minimum variances will be kept.

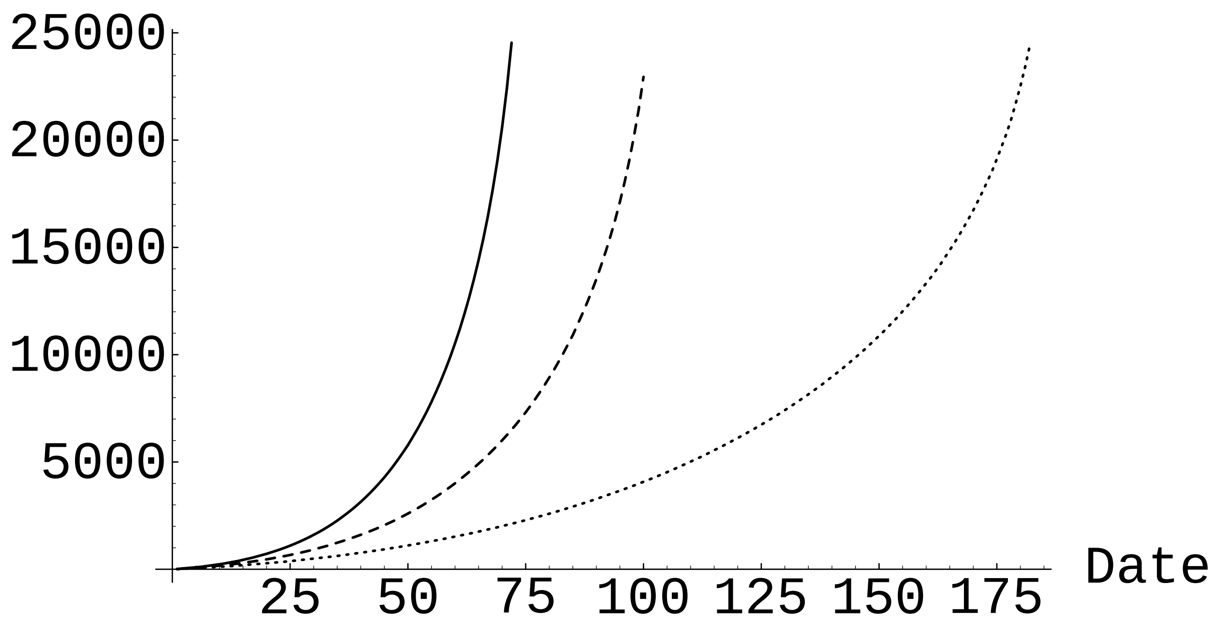
Negative Aspects of the RT Algorithm Revisited[a]

- Recall the problems mentioned on p. 765.
- In our case, combinatorial explosion occurs when

$$n > \frac{1-\beta_1}{\beta_2} = \frac{1-0.9}{0.04} = 2.5.$$

- Suppose we are willing to accept the exponential running time and pick $n = 100$ to seek accuracy.
- But the problem of shortened maturity forces the tree to stop at date 9!

[a] Lyuu and Wu (R90723065) (2003).



Dotted line: $n = 3$; dashed line: $n = 4$; solid line: $n = 5$.