Brownian Motion[a]

- Brownian motion is a stochastic process $\{ X(t), t \geq 0 \}$ with the following properties.

  1. $X(0) = 0$, unless stated otherwise.

  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

  $$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.[b]

  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where μ and $\sigma \neq 0$ are real numbers.

---

[a] Robert Brown (1773–1858).

[b] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

Brownian Motion (concluded)

- Such a process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2.
- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is also called the Wiener process.

[a]Norbert Wiener (1894–1964).

Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then $X(t) - X(s) \sim N(0, t-s)$.
- A (μ, σ) Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \tag{47}$$

- Note that $Y(t+s) - Y(t) \sim N(\mu s, \sigma^2 s)$.

Brownian Motion as Limit of Random Walk

Claim 1 *A (μ, σ) Brownian motion is the limiting case of random walk.*

- A particle moves Δx to the left with probability $1 - p$.
- It moves to the right with probability p after Δt time.
- Assume $n \equiv t/\Delta t$ is an integer.
- Its position at time t is

$$Y(t) \equiv \Delta x \left(X_1 + X_2 + \cdots + X_n \right).$$

Brownian Motion as Limit of Random Walk (continued)

- (continued)
  - Here
    $$X_i \equiv \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$
  - X_i are independent with
    $\text{Prob}[\, X_i = 1 \,] = p = 1 - \text{Prob}[\, X_i = -1 \,]$.
- Recall $E[\, X_i \,] = 2p - 1$ and $\text{Var}[\, X_i \,] = 1 - (2p-1)^2$.

Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$
$$\mathrm{Var}[\,Y(t)\,] = n(\Delta x)^2\left[\,1-(2p-1)^2\,\right].$$

- With $\Delta x \equiv \sigma\sqrt{\Delta t}$ and $p \equiv [\,1+(\mu/\sigma)\sqrt{\Delta t}\,]/2$,

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \mathrm{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1-(\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t, \end{aligned}$$

as $\Delta t \to 0$.

Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{ Y(t), t \geq 0 \}$ converges to a (μ, σ) Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Note that

$$\begin{aligned} & \mathrm{Var}[\, Y(t + \Delta t) - Y(t) \,] \\ = & \mathrm{Var}[\, \Delta x \, X_{n+1} \,] = (\Delta x)^2 \times \mathrm{Var}[\, X_{n+1} \,] \to \sigma^2 \Delta t. \end{aligned}$$

- Similarity to the the BOPM: The p is identical to the probability in Eq. (24) on p. 246 and $\Delta x = \ln u$.

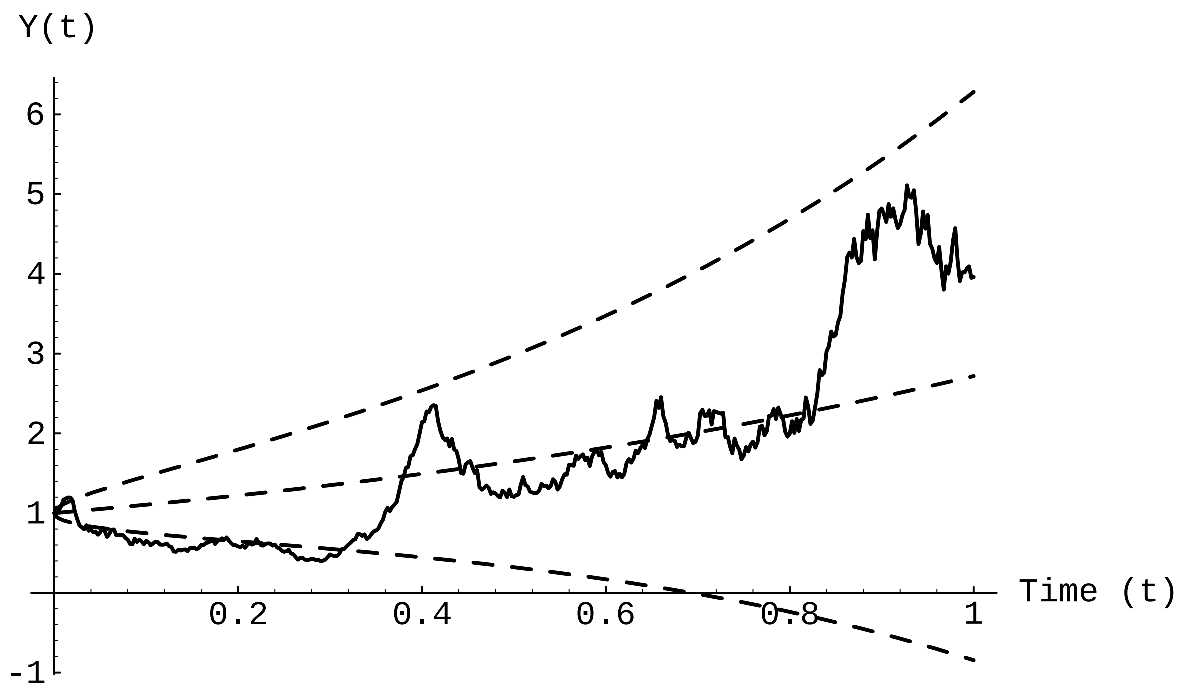
Geometric Brownian Motion

- Let $X \equiv \{ X(t), t \geq 0 \}$ be a Brownian motion process.
- The process
$$\{ Y(t) \equiv e^{X(t)}, t \geq 0 \},$$
is called geometric Brownian motion.
- Suppose further that X is a (μ, σ) Brownian motion.
- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function
$$E\left[e^{sX(t)} \right] = E\left[Y(t)^s \right] = e^{\mu ts + (\sigma^2 ts^2/2)}$$
from Eq. (17) on p 145.

Geometric Brownian Motion (continued)

- In particular,

$$
\begin{aligned}
E[\,Y(t)\,] &= e^{\mu t+(\sigma^2 t/2)},\\
\mathrm{Var}[\,Y(t)\,] &= E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2\\
&= e^{2\mu t+\sigma^2 t}\left(e^{\sigma^2 t} - 1\right).
\end{aligned}
$$


Y(t)
6
5
4
3
2
1
-1
0.2
0.4
0.6
0.8
1
Time (t)

Geometric Brownian Motion (continued)

- It is useful for situations in which percentage changes are independent and identically distributed.
- Let Y_n denote the stock price at time n and $Y_0 = 1$.
- Assume relative returns

$$X_i \equiv \frac{Y_i}{Y_{i-1}}$$

are independent and identically distributed.

Geometric Brownian Motion (concluded)

- Then
$$\ln Y_n = \sum_{i=1}^{n} \ln X_i$$
is a sum of independent, identically distributed random variables.
- Thus $\{\ln Y_n, n \geq 0\}$ is approximately Brownian motion.
  - And $\{Y_n, n \geq 0\}$ is approximately geometric Brownian motion.

Continuous-Time Financial Mathematics

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
Science and the Modern World

Stochastic Integrals

- Use $W \equiv \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of X from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \equiv \int_0^t X\, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of X with respect to W.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X\, dW$.

[a] Kiyoshi Ito (1915–2008).

Stochastic Integrals (concluded)

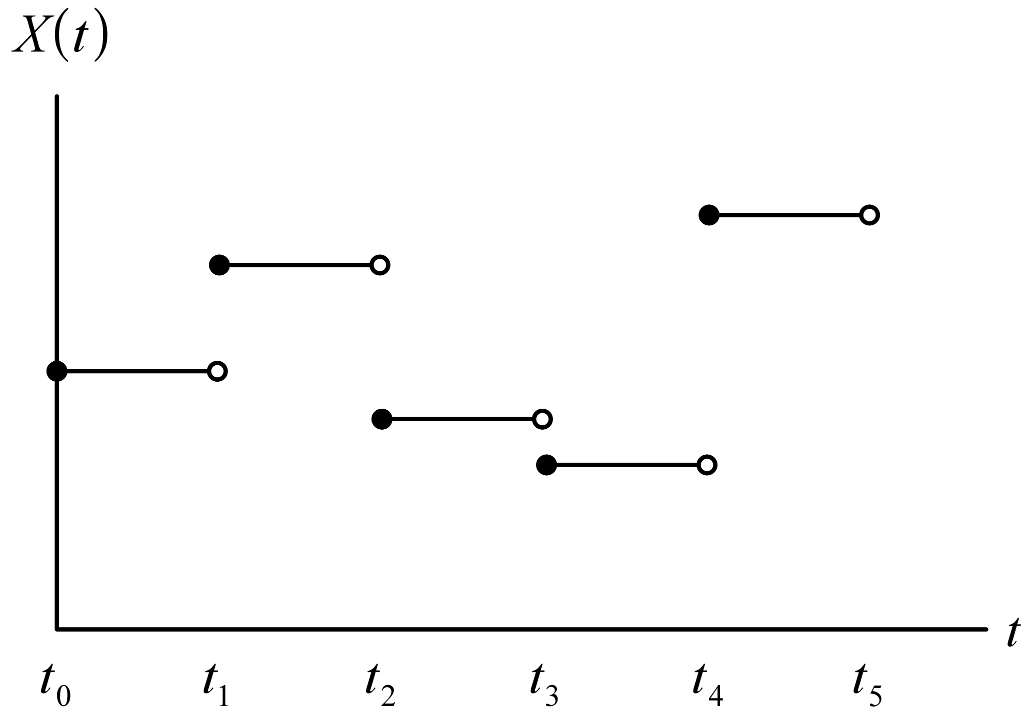
- Typical requirements for X in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\,ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\,ds < \infty$.
  - The information set at time t includes the history of X and W up to that point in time.
  - But it contains nothing about the evolution of X or W after t (nonanticipating, so to speak).
  - The future cannot influence the present.
- $\{X(s), 0 \leq s \leq t\}$ is independent of $\{W(t+u) - W(t), u > 0\}$.

Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist $0 = t_0 < t_1 < \cdots$ such that

$$X(t) = X(t_{k-1}) \quad \text{for } t \in [\, t_{k-1}, t_k), \, k = 1, 2, \ldots$$

for any realization (see figure on next page).



X(t)
t
t0
t1
t2
t3
t4
t5

Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \equiv \sum_{k=0}^{n-1} X(t_k)[\,W(t_{k+1}) - W(t_k)\,], \tag{48}$$

  where $t_n = t$.

  – The integrand X is evaluated at t_k, not t_{k+1}.

- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (continued)

- Let $X = \{ X(t), t \geq 0 \}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that X_n converges in probability to X.
- If X is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as $\delta_n \equiv \max_{1 \leq k \leq n}(t_k - t_{k-1})$ goes to zero.

Ito Integral (concluded)

- It is a fundamental fact that $\int X \, dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.
  - A corollary is the mean value formula
$$E\left[\int_a^b X \, dW\right] = 0.$$

Theorem 15 *The Ito integral $\int X \, dW$ is a martingale.*

Discrete Approximation

- Recall Eq. (48) on p. 478.
- The following simple stochastic process $\{\widehat{X}(t)\}$ can be used in place of X to approximate the stochastic integral $\int_0^t X\, dW$,

$$\widehat{X}(s) \equiv X(t_{k-1}) \text{ for } s \in [t_{k-1}, t_k),\ k = 1, 2, \ldots, n.$$

- Note the nonanticipating feature of $\widehat{X}$.
  - The information up to time s,

$$\{\widehat{X}(t), W(t), 0 \leq t \leq s\},$$

    cannot determine the future evolution of X or W.

Discrete Approximation (concluded)

- Suppose we defined the stochastic integral as

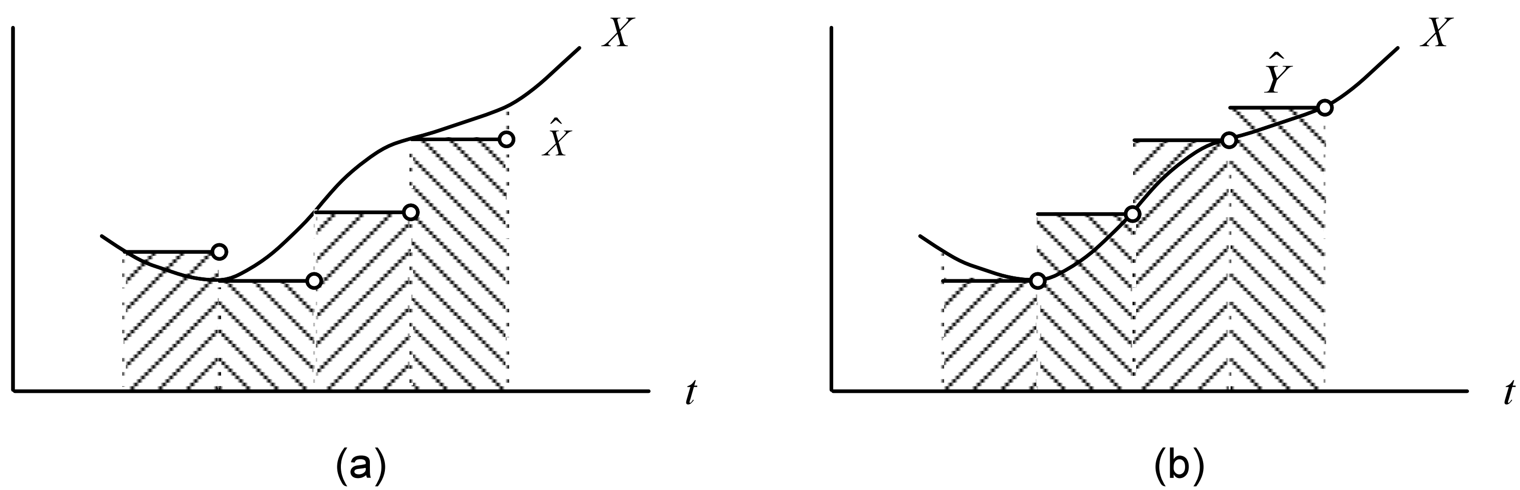
$$\sum_{k=0}^{n-1} X(t_{k+1})[\,W(t_{k+1}) - W(t_k)\,].$$

- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \equiv X(t_k) \quad \text{for} \quad s \in [\,t_{k-1}, t_k),\; k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of X.[a]

[a]See Exercise 14.1.2 of the textbook for an example where it matters.



X
$\hat{X}$
t
(a)
X
$\hat{Y}$
t
(b)

Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

  is called an Ito process.

  - X_0 is a scalar starting point.
  - $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.

- The terms $a(X_t, t)$ and $b(X_t, t)$ are the drift and the diffusion, respectively.

Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{49}$$

  – Or simply $dX_t = a_t\, dt + b_t\, dW_t$.

- This is Brownian motion with an instantaneous drift a_t and an instantaneous variance b_t^2.

- X is a martingale if the drift a_t is zero by Theorem 15 (p. 480).

[a]Paul Langevin (1904).

Ito Process (concluded)

- dW is normally distributed with mean zero and variance dt.
- An equivalent form to Eq. (49) is

$$dX_t = a_t \, dt + b_t \sqrt{dt} \; \xi, \tag{50}$$

where $\xi \sim N(0, 1)$.

Euler Approximation

- The following approximation follows from Eq. (50),

$$\begin{aligned}&\widehat{X}(t_{n+1})\\=&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\,\Delta W(t_n),\end{aligned}\tag{51}$$

  where $t_n \equiv n\Delta t$.

- It is called the Euler or Euler-Maruyama method.
- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.
- Recall that $\Delta W(t_n)$ should be interpreted as $W(t_{n+1}) - W(t_n)$ instead of $W(t_n) - W(t_{n-1})$.

More Discrete Approximations

- Under fairly loose regularity conditions, approximation (51) on p. 487 can be replaced by

$$
\begin{aligned}
&\widehat{X}(t_{n+1})\\
=&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,Y(t_n).
\end{aligned}
$$

  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\sqrt{\Delta t}\,\xi.\end{aligned}$$

  - $\text{Prob}[\,\xi=1\,]=\text{Prob}[\,\xi=-1\,]=1/2$.
  - Note that $E[\,\xi\,]=0$ and $\text{Var}[\,\xi\,]=1$.
- This clearly defines a binomial model.
- As Δt goes to zero, $\widehat{X}$ converges to X.

Trading and the Ito Integral

- Consider an Ito process $d\boldsymbol{S}_t = \mu_t \, dt + \sigma_t \, dW_t$.
  - $\boldsymbol{S}_t$ is the vector of security prices at time t.
- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time t.
  - Hence the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time t.
- $\boldsymbol{\phi}_t \, d\boldsymbol{S}_t \equiv \boldsymbol{\phi}_t(\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time t.

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \equiv \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

  measures the gains realized by the trading strategy over the period $[\,0, T\,]$.

Ito's Lemma

A smooth function of an Ito process is itself an Ito process.

Theorem 16 *Suppose* $f : R \to R$ *is twice continuously differentiable and* $dX = a_t\, dt + b_t\, dW$*. Then* $f(X)$ *is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \quad & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

for $t \geq 0$.

Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$df(X) = f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2}\, f''(X)\, b^2\, dt. \tag{52}$$

- Compared with calculus, the interesting part is the third term on the right-hand side.
- A convenient formulation of Ito's lemma is

$$df(X) = f'(X)\, dX + \frac{1}{2}\, f''(X)(dX)^2.$$

Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

$\times$	dW	dt
dW	dt	0
dt	0	0

  – The $(dW)^2 = dt$ entry is justified by a known result.

- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (continued)

Theorem 17 (Higher-Dimensional Ito's Lemma) *Let* $W_1, W_2, \ldots, W_n$ *be independent Wiener processes and* $X \equiv (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* X_i *is an Ito process with* $dX_i = a_i \, dt + \sum_{j=1}^{n} b_{ij} \, dW_j$*. Then* $df(X)$ *is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^{m} f_i(X) \, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X) \, dX_i \, dX_k,$$

where $f_i \equiv \partial f / \partial x_i$ *and* $f_{ik} \equiv \partial^2 f / \partial x_i \partial x_k$*.*

Ito's Lemma (continued)

- The multiplication table for Theorem 17 is

$\times$	dW_i	dt
dW_k	$\delta_{ik}\, dt$	0
dt	0	0

  in which

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Ito's Lemma (continued)

Theorem 18 (Alternative Ito's Lemma) *Let* $W_1, W_2, \ldots, W_m$ *be Wiener processes and* $X \equiv (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* X_i *is an Ito process with* $dX_i = a_i\, dt + b_i\, dW_i$*. Then* $df(X)$ *is the following Ito process,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k.$$

Ito's Lemma (concluded)

- The multiplication table for Theorem 18 is

$\times$	dW_i	dt
dW_k	$\rho_{ik}\, dt$	0
dt	0	0

- Here, ρ_{ik} denotes the correlation between dW_i and dW_k.

Geometric Brownian Motion

- Consider the geometric Brownian motion process $Y(t) \equiv e^{X(t)}$
  - $X(t)$ is a (μ, σ) Brownian motion.
  - Hence $dX = \mu\, dt + \sigma\, dW$ by Eq. (47) on p. 462.
- As $\partial Y/\partial X = Y$ and $\partial^2 Y/\partial X^2 = Y$, Ito's formula (52) on p. 493 implies

$$
\begin{aligned}
dY &= Y\, dX + (1/2)\, Y\, (dX)^2 \\
&= Y\, (\mu\, dt + \sigma\, dW) + (1/2)\, Y\, (\mu\, dt + \sigma\, dW)^2 \\
&= Y\, (\mu\, dt + \sigma\, dW) + (1/2)\, Y \sigma^2\, dt.
\end{aligned}
$$

Geometric Brownian Motion (concluded)

- Hence
$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right) dt + \sigma\, dW.$$
- The annualized instantaneous rate of return is $\mu + \sigma^2/2$ not μ.

Product of Geometric Brownian Motion Processes

- Let

$$\begin{aligned} dY/Y &= a\,dt + b\,dW_Y, \\ dZ/Z &= f\,dt + g\,dW_Z. \end{aligned}$$

- Consider the Ito process $U \equiv YZ$.
- Apply Ito's lemma (Theorem 18 on p. 497):

$$\begin{aligned} dU &= Z\,dY + Y\,dZ + dY\,dZ \\ &= ZY(a\,dt + b\,dW_Y) + YZ(f\,dt + g\,dW_Z) \\ &\quad + YZ(a\,dt + b\,dW_Y)(f\,dt + g\,dW_Z) \\ &= U(a + f + bg\rho)\,dt + Ub\,dW_Y + Ug\,dW_Z. \end{aligned}$$

Product of Geometric Brownian Motion Processes (continued)

- The product of two (or more) correlated geometric Brownian motion processes thus remains geometric Brownian motion.
- Note that

$$
\begin{aligned}
Y &= \exp\left[\left(a - b^2/2\right) dt + b\, dW_Y\right], \\
Z &= \exp\left[\left(f - g^2/2\right) dt + g\, dW_Z\right], \\
U &= \exp\left[\,\left(a + f - \left(b^2 + g^2\right)/2\right) dt + b\, dW_Y + g\, dW_Z\,\right].
\end{aligned}
$$

Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.
- This holds even if Y and Z are correlated.
- Finally, $\ln Y$ and $\ln Z$ have correlation ρ.

Quotients of Geometric Brownian Motion Processes

- Suppose Y and Z are drawn from p. 501.
- Let $U \equiv Y/Z$.
- We now show that[a]
$$\frac{dU}{U} = (a - f + g^2 - bg\rho)\, dt + b\, dW_Y - g\, dW_Z. \tag{53}$$
- Keep in mind that dW_Y and dW_Z have correlation ρ.

[a] Exercise 14.3.6 of the textbook is erroneous.

Quotients of Geometric Brownian Motion Processes (concluded)

- The multidimensional Ito's lemma (Theorem 18 on p. 497) can be employed to show that

$$
\begin{aligned}
& dU \\
= \;& (1/Z)\, dY - (Y/Z^2)\, dZ - (1/Z^2)\, dY\, dZ + (Y/Z^3)\, (dZ)^2 \\
= \;& (1/Z)(aY\, dt + bY\, dW_Y) - (Y/Z^2)(fZ\, dt + gZ\, dW_Z) \\
& -(1/Z^2)(bgYZ\rho\, dt) + (Y/Z^3)(g^2 Z^2\, dt) \\
= \;& U(a\, dt + b\, dW_Y) - U(f\, dt + g\, dW_Z) \\
& -U(bg\rho\, dt) + U(g^2\, dt) \\
= \;& U(a - f + g^2 - bg\rho)\, dt + Ub\, dW_Y - Ug\, dW_Z .
\end{aligned}
$$

Forward Price

- Suppose S follows
$$\frac{dS}{S} = \mu\, dt + \sigma\, dW.$$
- Consider $F(S, t) \equiv S e^{y(T-t)}$.
- Observe that
$$\begin{aligned} \frac{\partial F}{\partial S} &= e^{y(T-t)}, \\ \frac{\partial F}{\partial t} &= -y S e^{y(T-t)}. \end{aligned}$$

Forward Prices (concluded)

- Then

$$\begin{aligned} dF &= e^{y(T-t)}\, dS - ySe^{y(T-t)}\, dt \\ &= Se^{y(T-t)}\, (\mu\, dt + \sigma\, dW) - ySe^{y(T-t)}\, dt \\ &= F(\mu - y)\, dt + F\sigma\, dW \end{aligned}$$

  by Theorem 17 (p. 495).

- Thus F follows

$$\frac{dF}{F} = (\mu - y)\, dt + \sigma\, dW.$$

- This result has applications in forward and futures contracts.[a]

[a] This is also consistent with p. 453.

Ornstein-Uhlenbeck Process

- The Ornstein-Uhlenbeck process:

$$dX = -\kappa X\, dt + \sigma\, dW,$$

where $\kappa, \sigma \geq 0$.

- It is known that

$$\begin{aligned}
E[\, X(t)\,] &= e^{-\kappa(t-t_0)}\, E[\, x_0\,], \\
\mathrm{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-t_0)}\right) + e^{-2\kappa(t-t_0)}\, \mathrm{Var}[\, x_0\,], \\
\mathrm{Cov}[\, X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa(t-s)} \left[\, 1 - e^{-2\kappa(s-t_0)}\,\right] \\
&\quad + e^{-\kappa(t+s-2t_0)}\, \mathrm{Var}[\, x_0\,],
\end{aligned}$$

for $t_0 \leq s \leq t$ and $X(t_0) = x_0$.

Ornstein-Uhlenbeck Process (continued)

- $X(t)$ is normally distributed if x_0 is a constant or normally distributed.
- X is said to be a normal process.
- $E[x_0] = x_0$ and $\text{Var}[x_0] = 0$ if x_0 is a constant.
- The Ornstein-Uhlenbeck process has the following mean reversion property.
  - When $X > 0$, X is pulled toward zero.
  - When $X < 0$, it is pulled toward zero again.

Ornstein-Uhlenbeck Process (continued)

- Another version:

$$dX = \kappa(\mu - X)\, dt + \sigma\, dW,$$

  where $\sigma \geq 0$.

- Given $X(t_0) = x_0$, a constant, it is known that

$$\begin{aligned} E[\, X(t)\,] &= \mu + (x_0 - \mu)\, e^{-\kappa(t-t_0)}, \qquad (54) \\ \text{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa} \left[\, 1 - e^{-2\kappa(t-t_0)}\, \right], \end{aligned}$$

  for $t_0 \leq t$.

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly μ and $\sigma/\sqrt{2\kappa}$, respectively.
- For large t, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$.
- The process is mean-reverting.
  - X tends to move toward μ.
  - Useful for modeling term structure, stock price volatility, and stock price return.

Square-Root Process

- Suppose X is an Ornstein-Uhlenbeck process.
- Ito's lemma says $V \equiv X^2$ has the differential,

$$\begin{aligned} dV &= 2X\,dX + (dX)^2 \\ &= 2\sqrt{V}\,(-\kappa\sqrt{V}\,dt + \sigma\,dW) + \sigma^2\,dt \\ &= \left(-2\kappa V + \sigma^2\right) dt + 2\sigma\sqrt{V}\,dW, \end{aligned}$$

a square-root process.

Square-Root Process (continued)

- In general, the square-root process has the stochastic differential equation,

$$dX = \kappa(\mu - X)\, dt + \sigma\sqrt{X}\, dW,$$

  where $\kappa, \sigma \geq 0$ and the initial value of X is a nonnegative constant.

- Like the Ornstein-Uhlenbeck process, it possesses mean reversion: X tends to move toward μ, but the volatility is proportional to $\sqrt{X}$ instead of a constant.

Square-Root Process (continued)

- When X hits zero and $\mu \geq 0$, the probability is one that it will not move below zero.
  - Zero is a reflecting boundary.
- Hence, the square-root process is a good candidate for modeling interest rate movements.[a]
- The Ornstein-Uhlenbeck process, in contrast, allows negative interest rates.
- The two processes are related (see p. 512).

[a]Cox, Ingersoll, and Ross (1985).

Square-Root Process (concluded)

- The random variable $2cX(t)$ follows the noncentral chi-square distribution,[a]

$$\chi\left(\frac{4\kappa\mu}{\sigma^2}, 2cX(0)\, e^{-\kappa t}\right),$$

where $c \equiv (2\kappa/\sigma^2)(1-e^{-\kappa t})^{-1}$.

- Given $X(0) = x_0$, a constant,

$$\begin{aligned} E[\,X(t)\,] &= x_0 e^{-\kappa t} + \mu\left(1 - e^{-\kappa t}\right), \\ \mathrm{Var}[\,X(t)\,] &= x_0\,\frac{\sigma^2}{\kappa}\,\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \mu\,\frac{\sigma^2}{2\kappa}\,\left(1 - e^{-\kappa t}\right)^2, \end{aligned}$$

for $t \geq 0$.

[a] William Feller (1906–1970) in 1951.

Continuous-Time Derivatives Pricing

I have hardly met a mathematician
who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Fischer [Black] is the only real genius
I've ever met in finance. Other people,
like Robert Merton or Stephen Ross,
are just very smart and quick,
but they think like me.
Fischer came from someplace else entirely.
— John C. Cox, quoted in Mehrling (2005)

Toward the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation.
- The key step is recognizing that the same random process drives both securities.
- As their prices are perfectly correlated, we figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio's return to be the riskless rate.

Assumptions

- The stock price follows $dS = \mu S\,dt + \sigma S\,dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at r.
- There is unlimited riskless borrowing and lending.
- t is the current time, T is the expiration time, and $\tau \equiv T - t$.

Black-Scholes Differential Equation

- Let C be the price of a derivative on S.
- From Ito's lemma (p. 495),

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} \, dW.$$

  - The same W drives both C and S.
- Short one derivative and long $\partial C / \partial S$ shares of stock (call it Π).
- By construction,

$$\Pi = -C + S(\partial C / \partial S).$$

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is[a]

$$d\Pi = -dC + \frac{\partial C}{\partial S}\, dS.$$

- Substitute the formulas for dC and dS into the partial differential equation to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r\Pi\, dt$.

[a] Mathematically speaking, it is not quite right (Bergman, 1982).

Black-Scholes Differential Equation (concluded)

- So
$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2}\right) dt = r\left(C - S\,\frac{\partial C}{\partial S}\right) dt.$$
- Equate the terms to finally obtain
$$\frac{\partial C}{\partial t} + rS\,\frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2} = rC.$$
- When there is a dividend yield q,
$$\frac{\partial C}{\partial t} + (r-q)\,S\,\frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 C}{\partial S^2} = rC.$$

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC. \tag{55}$$

- Identity (55) leads to an alternative way of computing Θ numerically from Δ and Γ.
- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC.$$

  - A definite relation thus exists between Γ and Θ.

PDEs for Asian Options

- Add the new variable $A(t) \equiv \int_0^t S(u)\, du$.
- Then the value V of the Asian option satisfies this two-dimensional PDE:[a]

$$\frac{\partial V}{\partial t} + rS\, \frac{\partial V}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 V}{\partial S^2} + S\, \frac{\partial V}{\partial A} = rV.$$

- The terminal conditions are

$$\begin{aligned} V(T, S, A) &= \max\left(\frac{A}{T} - X, 0 \right) \quad \text{for call,} \\ V(T, S, A) &= \max\left(X - \frac{A}{T}, 0 \right) \quad \text{for put.} \end{aligned}$$

[a] Kemna and Vorst (1990).

PDEs for Asian Options (continued)

- The two-dimensional PDE produces algorithms similar to that on pp. 346ff.
- But one-dimensional PDEs are available for Asian options.[a]
- For example, Večeř (2001) derives the following PDE for Asian calls:

$$\frac{\partial u}{\partial t} + r\left(1 - \frac{t}{T} - z\right)\frac{\partial u}{\partial z} + \frac{\left(1 - \frac{t}{T} - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

with the terminal condition $u(T, z) = \max(z, 0)$.

[a] Rogers and Shi (1995); Večeř (2001); Dubois and Lelièvre (2005).

PDEs for Asian Options (concluded)

- For Asian puts:

$$\frac{\partial u}{\partial t} + r\left(\frac{t}{T} - 1 - z\right)\frac{\partial u}{\partial z} + \frac{\left(\frac{t}{T} - 1 - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

  with the same terminal condition.

- One-dimensional PDEs lead to highly efficient numerical methods.

Heston's Stochastic-Volatility Model[a]

- Heston assumes the stock price follows

$$\frac{dS}{S} = (\mu - q)\, dt + \sqrt{V}\, dW_1, \tag{56}$$

$$dV = \kappa(\theta - V)\, dt + \sigma\sqrt{V}\, dW_2. \tag{57}$$

  - V is the instantaneous variance, which follows a square-root process.
  - dW_1 and dW_2 have correlation ρ.
  - The riskless rate r is constant.

- It may be the most popular continuous-time stochastic-volatility model.

[a]Heston (1993).

Heston's Stochastic-Volatility Model (continued)

- Heston assumes the market price of risk is $b_2\sqrt{V}$.
- So $\mu = r + b_2 V$.
- Define

$$\begin{aligned} dW_1^* &= dW_1 + b_2\sqrt{V}\,dt, \\ dW_2^* &= dW_2 + \rho b_2\sqrt{V}\,dt, \\ \kappa^* &= \kappa + \rho b_2 \sigma, \\ \theta^* &= \frac{\theta\kappa}{\kappa + \rho b_2 \sigma}. \end{aligned}$$

- dW_1^* and dW_2^* have correlation ρ.
- Under the risk-neutral probability measure Q, both W_1^* and W_2^* are Wiener processes.

Heston's Stochastic-Volatility Model (continued)

- Heston's model becomes, under probability measure Q,

$$\begin{aligned}\frac{dS}{S} &= (r-q)\,dt + \sqrt{V}\,dW_1^*,\\ dV &= \kappa^*(\theta^* - V)\,dt + \sigma\sqrt{V}\,dW_2^*.\end{aligned}$$

Heston's Stochastic-Volatility Model (continued)

- Define

$$
\begin{aligned}
\phi(u,\tau) &= \exp\Big\{\, \imath u(\ln S + (r-q)\,\tau) \\
&\quad + \theta^{*}\kappa^{*}\sigma^{-2}\left[(\kappa^{*} - \rho\sigma u\imath - d)\,\tau - 2\ln\frac{1-ge^{-d\tau}}{1-g}\right] \\
&\quad + \frac{v\sigma^{-2}(\kappa^{*} - \rho\sigma u\imath - d)\left(1-e^{-d\tau}\right)}{1-ge^{-d\tau}}\Big\}, \\
d &= \sqrt{(\rho\sigma u\imath - \kappa^{*})^{2} - \sigma^{2}(-\imath u - u^{2})}\,, \\
g &= (\kappa^{*} - \rho\sigma u\imath - d)/(\kappa^{*} - \rho\sigma u\imath + d).
\end{aligned}
$$

Heston's Stochastic-Volatility Model (concluded)

The formulas are[a]

$$
\begin{aligned}
C &= S\left[\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(\frac{X^{-\imath u}\phi(u-\imath,\tau)}{\imath u S e^{r\tau}}\right)\,du\right] \\
&\quad -Xe^{-r\tau}\left[\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(\frac{X^{-\imath u}\phi(u,\tau)}{\imath u}\right)\,du\right], \\
P &= Xe^{-r\tau}\left[\frac{1}{2}-\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(\frac{X^{-\imath u}\phi(u,\tau)}{\imath u}\right)\,du\right], \\
&\quad -S\left[\frac{1}{2}-\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(\frac{X^{-\imath u}\phi(u-\imath,\tau)}{\imath u S e^{r\tau}}\right)\,du\right],
\end{aligned}
$$

where $\imath = \sqrt{-1}$ and $\mathrm{Re}(x)$ denotes the real part of the complex number x.

[a]Contributed by Mr. Chen, Chun-Ying (`D95723006`) on August 17, 2008 and Mr. Liou, Yan-Fu (`R92723060`) on August 26, 2008.

Stochastic-Volatility Models and Further Extensions[a]

- How to explain the October 1987 crash?
- Stochastic-volatility models require an implausibly high-volatility level prior to *and* after the crash.
- Merton (1976) proposed jump models.
- Discontinuous jump models *in the asset price* can alleviate the problem somewhat.

[a] Eraker (2004).

Stochastic-Volatility Models and Further Extensions (continued)

- But if the jump intensity is a constant, it cannot explain the tendency of large movements to cluster over time.
- This assumption also has no impacts on option prices.
- Jump-diffusion models combine both.
  - E.g., add a jump process to Eq. (56) on p. 527.

Stochastic-Volatility Models and Further Extensions (concluded)

- But they still do not adequately describe the systematic variations in option prices.[a]
- Jumps in volatility are alternatives.[b]
  - E.g., add correlated jump processes to Eqs. (56) and Eq. (57) on p. 527.
- Such models allow high level of volatility caused by a jump to volatility.[c]

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[a] Bates (2000) and Pan (2002).
[b] Duffie, Pan, and Singleton (2000).
[c] Eraker, Johnnes, and Polson (2000).