Day Count Conventions: Actual/Actual

- The first “actual” refers to the actual number of days in a month.

- The second refers to the actual number of days in a coupon period.

- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.
Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ to date $D_2 \equiv (y_2, m_2, d_2)$ is
  \[360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1).\]
Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

\[ \omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}. \]  \hspace{1cm} (6)

- The price is now calculated by

\[ PV = \sum_{i=0}^{n-1} \left( \frac{C}{1 + \frac{r}{m}} \right)^{\omega + i} + \frac{F}{(1 + \frac{r}{m})^{\omega + n - 1}}. \]  \hspace{1cm} (7)
Accrued Interest

- The buyer pays the quoted price plus the accrued interest — the invoice price:

\[ C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega). \]

- The yield to maturity is the \( r \) satisfying Eq. (7) when \( P \) is the invoice price.

- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.
\[ C(1 - \omega) \]
Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.
Example ("30/360") (concluded)

- The accrued interest is \((10/2) \times \frac{180-60}{180} = 3.3333\) per $100 of par value.
- The yield to maturity is 3%.
- This can be verified by Eq. (7) on p. 68 with
  - \(\omega = 60/180\),
  - \(m = 2\),
  - \(C' = 5\),
  - \(PV= 111.2891 + 3.3333\),
  - \(r = 0.03\).
Price Behavior (2) Revisited

• Before: A bond selling at par if the yield to maturity equals the coupon rate.

• But it assumed that the settlement date is on a coupon payment date.

• Now suppose the settlement date for a bond selling at par (i.e., the quoted price is equal to the par value) falls between two coupon payment dates.

• Then its yield to maturity is less than the coupon rate.
  – The short reason: Exponential growth is replaced by linear growth, hence “overpaying” the coupon.
Bond Price Volatility
“Well, Beethoven, what is this?”
— Attribution to Prince Anton Esterházy
Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.
- Assume level-coupon bonds throughout.
Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?

- Define price volatility by

\[
\frac{\partial P}{\partial y} - \frac{P}{P}.
\]
Price Volatility of Bonds

- The price volatility of a coupon bond is

\[
\frac{(C/y) n - (C/y^2) ((1 + y)^{n+1} - (1 + y)) - nF}{(C/y) ((1 + y)^{n+1} - (1 + y)) + F(1 + y)}.
\]

- $F$ is the par value.
- $C$ is the coupon payment per period.

- For bonds without embedded options,

\[
- \frac{\partial P}{\partial y} > 0.
\]
Macaulay Duration

• The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.

• The weights are the cash flows’ PVs divided by the asset’s price.

• Formally,

\[
MD = \frac{1}{P} \sum_{i=1}^{n} \frac{iC_i}{(1 + y)^i}.
\]

• The Macaulay duration, in periods, is equal to

\[
MD = -(1 + y) \frac{\partial P}{\partial y} \frac{1}{P}.
\]  \(8\)
MD of Bonds

- The MD of a coupon bond is

\[
MD = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n} \right]. \quad (9)
\]

- It can be simplified to

\[
MD = \frac{c(1+y) \left[ (1+y)^n - 1 \right] + ny(y-c)}{cy \left[ (1+y)^n - 1 \right] + y^2},
\]

where \(c\) is the period coupon rate.

- The MD of a zero-coupon bond equals its term to maturity \(n\).

- The MD of a coupon bond is less than its maturity.
Remarks

• Equations (8) on p. 79 and (9) on p. 80 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
  – That is, if the cash flow is independent of yields.

• To see this point, suppose the market yield declines.

• The MD will be lengthened.

• But for securities whose maturity actually decreases as a result, the MD (as originally defined) may actually decrease.
How Not To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- The MD should be seen mainly as measuring price volatility.
- Many, if not most, duration-related terminology cannot be comprehended otherwise.
Conversion

• For the MD to be year-based, modify Eq. (9) on p. 80 to

\[
\frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} \left( \frac{C}{1 + \frac{y}{k}} \right)^i + \frac{n}{k} \frac{F}{(1 + \frac{y}{k})^n} \right],
\]

where \( y \) is the annual yield and \( k \) is the compounding frequency per annum.

• Equation (8) on p. 79 also becomes

\[
MD = - \left( 1 + \frac{y}{k} \right) \frac{\partial P}{\partial y} \frac{1}{P}.
\]

• By definition, MD (in years) = \( \frac{MD \ (in \ periods)}{k} \).
Modified Duration

- Modified duration is defined as

\[ \text{modified duration} \equiv - \frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}. \quad (10) \]

- By Taylor expansion,

  percent price change \( \approx -\text{modified duration} \times \text{yield change}. \)
Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.

- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

  \[-11.54 \times 0.001 = -0.01154 = -1.154\% .\]
Modified Duration of a Portfolio

- The modified duration of a portfolio equals

\[ \sum_{i} \omega_i D_i. \]

- \( D_i \) is the modified duration of the \( i \)th asset.
- \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

\[
\frac{P_- - P_+}{P_0(y_+ - y_-)}.
\]

- \(P_-\) is the price if the yield is decreased by \(\Delta y\).
- \(P_+\) is the price if the yield is increased by \(\Delta y\).
- \(P_0\) is the initial price, \(y\) is the initial yield.
- \(\Delta y\) is small.

- See plot on p. 88.
Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use
  \[ \frac{P_0 - P_+}{P_0 \Delta y} \]
  - More economical but less accurate.
The Practices

- Duration is usually expressed in percentage terms—call it \( D\% \)—for quick mental calculation.

- The percentage price change expressed in percentage terms is approximated by

\[-D\% \times \Delta r\]

when the yield increases instantaneously by \( \Delta r\% \).

- Price will drop by 20% if \( D\% = 10 \) and \( \Delta r = 2 \) because \( 10 \times 2 = 20 \).

- In fact, \( D\% \) equals modified duration as originally defined (prove it!).
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as

  \[
  \text{modified duration} \times \text{price (\% of par)} = -\frac{\partial P}{\partial y}.
  \]

- The approximate dollar price change per $100 of par value is

  \[
  \text{price change} \approx -\text{dollar duration} \times \text{yield change}.
  \]
Convexity

- Convexity is defined as

\[ \text{convexity (in periods)} = \frac{\partial^2 P}{\partial y^2} \frac{1}{P}. \]

- The convexity of a coupon bond is positive (prove it!).

- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

- Hence, between two bonds with the same duration, the one with a higher convexity is more valuable.
Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

\[
\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}
\]

when there are \( k \) periods per annum.
Use of Convexity

- The approximation $\Delta P/P \approx -\text{duration} \times \text{yield change}$ works for small yield changes.

- To improve upon it for larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2$$

$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$

- Recall the figure on p. 93.
The Practices

- Convexity is usually expressed in percentage terms—call it \( C\% \)—for quick mental calculation.

- The percentage price change expressed in percentage terms is approximated by 
  \(-D\% \times \Delta r + C\% \times (\Delta r)^2 / 2\)
  when the yield increases instantaneously by \( \Delta r\% \).

  - Price will drop by 17% if \( D\% = 10, C\% = 1.5, \) and \( \Delta r = 2 \) because

    \[-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.\]

- In fact, \( C\% \) equals convexity divided by 100 (prove it!).
Effective Convexity

- The effective convexity is defined as
  \[ \frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2}, \]
  - \( P_- \) is the price if the yield is decreased by \( \Delta y \).
  - \( P_+ \) is the price if the yield is increased by \( \Delta y \).
  - \( P_0 \) is the initial price, \( y \) is the initial yield.
  - \( \Delta y \) is small.

- Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

- Numerically, choosing the right \( \Delta y \) is a delicate matter.
Approximate \( d^2 f(x)^2/dx^2 \) at \( x = 1 \), Where \( f(x) = x^2 \)

The difference of \((1 + \Delta x)^2 + (1 - \Delta x)^2 - 2)/(\Delta x)^2\) and 2:
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest. Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds forms the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.
Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots yields to maturity against maturity.
- A par yield curve is constructed from bonds trading near par.
Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.
- The PV of one dollar $i$ periods from now is $\left[1 + S(i)\right]^{-i}$.
- The one-period spot rate is called the short rate.
- Spot rate curve: Plot of spot rates against maturity.
Problems with the PV Formula

- In the bond price formula,

\[ \sum_{i=1}^{n} \frac{C}{(1 + y)^i} + \frac{F}{(1 + y)^n}, \]

every cash flow is discounted at the same yield \( y \).

- Consider two riskless bonds with different yields to maturity because of their different cash flow streams:

\[ \sum_{i=1}^{n_1} \frac{C}{(1 + y_1)^i} + \frac{F}{(1 + y_1)^{n_1}}, \]

\[ \sum_{i=1}^{n_2} \frac{C}{(1 + y_2)^i} + \frac{F}{(1 + y_2)^{n_2}}. \]
Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.
- But shouldn’t they be discounted at the same rate?
Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the $i$th bond paying $C_i$ dollars at time $i$.

- So a level-coupon bond has the price

$$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}. \quad (11)$$

- This pricing method incorporates information from the term structure.

- Discount each cash flow at the corresponding spot rate.
Discount Factors

- In general, any riskless security having a cash flow \( C_1, C_2, \ldots, C_n \) should have a market price of
  \[
P = \sum_{i=1}^{n} C_i d(i).
  \]
  - Above, \( d(i) \equiv [1 + S(i)]^{-i} \), \( i = 1, 2, \ldots, n \), are called discount factors.
  - \( d(i) \) is the PV of one dollar \( i \) periods from now.

- The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

• Start with the short rate $S(1)$.
  – Note that short-term Treasuries are zero-coupon bonds.

• Compute $S(2)$ from the two-period coupon bond price $P$ by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.$$
Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price $P$ of the $n$-period coupon bond and $S(1), S(2), \ldots, S(n-1)$.
- Then $S(n)$ can be computed from Eq. (11) on p. 108, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.$$ 

- The running time is $O(n)$ (see text).
- The procedure is called bootstrapping.
Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.
  - Any economic justifications?
Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.
- Were this bond riskless, it would fetch
  \[
P^* = \sum_{t=1}^{n} \frac{C_t}{\left[1 + S(t)\right]^t}.
  \]
- Since riskiness must be compensated, $P < P^*$.
- Yield spread is the difference between the IRR of the risky bond and that of a riskless bond with comparable maturity.
Static Spread

- The static spread is the amount $s$ by which the spot rate curve has to shift in parallel to price the risky bond:

$$P = \sum_{t=1}^{n} \frac{C_t}{[1 + s + S(t)]^t}.$$ 

- Unlike the yield spread, the static spread incorporates information from the term structure.
Of Spot Rate Curve and Yield Curve

- $y_k$: yield to maturity for the $k$-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).

- If the yield curve is flat, the spot rate curve coincides with the yield curve.
Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).

- But this is only a trend not a mathematical truth.\(^a\)

\(^a\)See a counterexample in the text.
Forward Rates

- The yield curve contains information regarding future interest rates currently “expected” by the market.
- Invest $1 for $j$ periods to end up with $[1 + S(j)]^j$ dollars at time $j$.
  - The maturity strategy.
- Invest $1$ in bonds for $i$ periods and at time $i$ invest the proceeds in bonds for another $j - i$ periods where $j > i$.
- Will have $[1 + S(i)]^i[1 + S(i, j)]^{j-i}$ dollars at time $j$.
  - $S(i, j)$: $(j - i)$-period spot rate $i$ periods from now.
  - The rollover strategy.
Forward Rates (concluded)

- When $S(i, j)$ equals

$$f(i, j) \equiv \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1, \quad (12)$$

we will end up with $[1 + S(j)]^j$ dollars again.

- By definition, $f(0, j) = S(j)$.

- $f(i, j)$ is called the (implied) forward rates.
  - More precisely, the $(j - i)$-period forward rate $i$ periods from now.
Time Line

\[ f(0, 1) \quad f(1, 2) \quad f(2, 3) \quad f(3, 4) \]

Time 0

\[ S(1) \quad S(2) \quad S(3) \quad S(4) \]
Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i,j)$ and future spot rate $S(i,j)$.
  - This is the subject of the term structure theories.

- We merely looked for the future spot rate that, if realized, will equate two investment strategies.

- $f(i, i + 1)$ are instantaneous forward rates or one-period forward rates.
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,
  \[ f(i, j) > S(j) > \cdots > S(i). \]

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,
  \[ f(i, j) < S(j) < \cdots < S(i). \]