Prepayment Vector

- The PSA tries to capture how prepayments vary with age.
- But it should be viewed as a market convention rather than a model.
- A vector of PSAs generated by a prepayment model should be used to describe the monthly prepayment speed through time.
- The monthly cash flows can be derived thereof.

Prepayment Vector (continued)

- Similarly, the CPR should be seen purely as a measure of speed rather than a model.
- If one treats a single CPR number as the true prepayment speed, that number will be called the constant prepayment rate.
- This simple model crashes with the empirical fact that pools with new production loans typically prepay at a slower rate than seasoned pools.
- A vector of CPRs should be preferred.

Prepayment Vector (concluded)

- A CPR/SMM vector is easier to work with than a PSA vector because of the lack of dependence on the pool age.
- But they are all equivalent as a CPR vector can always be converted into an equivalent PSA vector and vice versa.

Cash Flow Yield

- To price an MBS, one starts with its cash flow: The periodic P&I under a static prepayment assumption as given by a prepayment vector.
- The invoice price is now

$$\sum_{i=1}^{n} C_i/(1+r)^{\omega-1+i}.$$

  - C_i is the cash flow at time i.
  - n is the weighted average maturity (WAM).
  - r is the discount rate.
  - ω is the fraction of period from settlement until the first P&I payment date.

Cash Flow Yield (concluded)

- The WAM is the weighted average remaining term of the mortgages in the pool, where the weight for each mortgage is the remaining balance.
- The r that equates the above with the market price is called the (static) cash flow yield.
- The implied PSA is the single PSA speed producing the same cash flow yield.[a]

[a]Fabozzi (1991).

MBS Quotes

- MBSs are quoted in the same manner as U.S. Treasury notes and bonds.
- For example, a price of 94-05 means $94 \, 5/32\%$ of par value.
- Sixty-fourth of a percent is expressed by appending "+" to the price.
- Hence, the price 94-05+ represents $94 \, 11/64\%$ of par value.

Cash Flow Generation

- Each cash flow is composed of the principal payment, the interest payment, and the principal prepayment.
- Let B_k denote the actual remaining principal balance at month k.
- The pool's actual remaining principal balance at time $i-1$ is B_{i-1}.

Cash Flow Generation (continued)

- The principal and interest payments at time i are

$$\overline{P_i} \equiv B_{i-1}\left(\frac{\text{Bal}_{i-1}-\text{Bal}_i}{\text{Bal}_{i-1}}\right) \tag{120}$$

$$= B_{i-1}\,\frac{r/m}{(1+r/m)^{n-i+1}-1} \tag{121}$$

$$\overline{I_i} \equiv B_{i-1}\,\frac{r-\alpha}{m} \tag{122}$$

  - α is the servicing spread (or servicing fee rate), which consists of the servicing fee for the servicer as well as the guarantee fee.

Cash Flow Generation (continued)

- The prepayment at time i is

$$\mathrm{PP}_i = B_{i-1}\,\frac{\mathrm{Bal}_i}{\mathrm{Bal}_{i-1}} \times \mathrm{SMM}_i.$$

  - SMM_i is the prepayment speed for month i.

- If the total principal payment from the pool is $\overline{P_i} + \mathrm{PP}_i$, the remaining principal balance is

$$\begin{aligned} B_i &= B_{i-1} - \overline{P_i} - \mathrm{PP}_i \\ &= B_{i-1}\left[1 - \left(\frac{\mathrm{Bal}_{i-1} - \mathrm{Bal}_i}{\mathrm{Bal}_{i-1}}\right) - \frac{\mathrm{Bal}_i}{\mathrm{Bal}_{i-1}} \times \mathrm{SMM}_i\right] \\ &= \frac{B_{i-1} \times \mathrm{Bal}_i \times (1 - \mathrm{SMM}_i)}{\mathrm{Bal}_{i-1}}. \end{aligned} \tag{123}$$

Cash Flow Generation (continued)

- Equation (123) can be applied iteratively to yield

$$B_i = \mathrm{RB}_i \times \prod_{j=1}^{i} (1 - \mathrm{SMM}_j). \tag{124}$$

- Define

$$b_i \equiv \prod_{j=1}^{i} (1 - \mathrm{SMM}_j).$$

Cash Flow Generation (continued)

- Then the scheduled P&I is

$$\overline{P_i} = b_{i-1} P_i \quad \text{and} \quad \overline{I_i} = b_{i-1} I'_i. \tag{125}$$

  - $I'_i \equiv \text{RB}_{i-1} \times (r - \alpha)/m$ is the scheduled interest payment.

- The scheduled cash flow and the b_i determined by the prepayment vector are all that are needed to calculate the projected actual cash flows.

Cash Flow Generation (concluded)

- If the servicing fees do not exist (that is, $\alpha = 0$), the projected monthly payment *before* prepayment at month i becomes

$$\overline{P_i} + \overline{I_i} = b_{i-1}(P_i + I_i) = b_{i-1}C. \tag{126}$$

  - C is the scheduled monthly payment on the original principal.

- See Figure 29.10 in the text for a linear-time algorithm for generating the mortgage pool's cash flow.

Cash Flows of Sequential-Pay CMOs

- Take a 3-tranche sequential-pay CMO backed by \$3,000,000 of mortgages with a 12% coupon and 6 months to maturity.
- The 3 tranches are called A, B, and Z.
- All three tranches carry the same coupon rate of 12%.

Cash Flows of Sequential-Pay CMOs (continued)

- The Z tranche consists of Z bonds.
  - A Z bond receives no payments until all previous tranches are retired.
  - Although a Z bond carries an explicit coupon rate, the owed interest is accrued and added to the principal balance of that tranche.
  - The Z bond thus protects earlier tranches from extension risk
- When a Z bond starts receiving cash payments, it becomes a pass-through instrument.

Cash Flows of Sequential-Pay CMOs (continued)

- The Z tranche's coupon cash flows are initially used to pay down the tranches preceding it.
- Its existence (as in the ABZ structure here) accelerates the principal repayments of the sequential-pay bonds.
- Assume the ensuing monthly interest rates are 1%, 0.9%, 1.1%, 1.2%, 1.1%, 1.0%.
- Assume that the SMMs are 5%, 6%, 5%, 4%, 5%, 6%.
- We want to calculate the cash flow and the then fair price of each tranche.

Cash Flows of Sequential-Pay CMOs (continued)

- Compute the pool's cash flow by invoking the algorithm in Figure 29.10 in the text.
  - $n = 6$, $r = 0.01$, and
    $\text{SMM} = [\,0.05, 0.06, 0.05, 0.04, 0.05, 0.06\,]$.
- Individual tranches' cash flows and remaining principals thereof can be derived by allocating the pool's principal and interest cash flows based on the CMO structure.
- See the next table for the breakdown.

Month		1	2	3	4	5	6
Interest rate		1.0%	0.9%	1.1%	1.2%	1.1%	1.0%
SMM		5.0%	6.0%	5.0%	4.0%	5.0%	6.0%
Remaining principal (B_i)							
	3,000,000	2,386,737	1,803,711	1,291,516	830,675	396,533	0
A	1,000,000	376,737	0	0	0	0	0
B	1,000,000	1,000,000	783,611	261,215	0	0	0
Z	1,000,000	1,010,000	1,020,100	1,030,301	830,675	396,533	0
Interest ($\overline{I_i}$)		30,000	23,867	18,037	12,915	8,307	3,965
A		20,000	3,767	0	0	0	0
B		10,000	20,100	18,037	2,612	0	0
Z		0	0	0	10,303	8,307	3,965
Principal		613,263	583,026	512,195	460,841	434,142	396,534
A		613,263	376,737	0	0	0	0
B		0	206,289	512,195	261,215	0	0
Z		0	0	0	199,626	434,142	396,534

Cash Flows of Sequential-Pay CMOs (concluded)

- Note that the Z tranche's principal is growing at 1% per month until all previous tranches are retired.
- Before that time, the interest due the Z tranche is used to retire A's and B's principals.
- For example, the \$10,000 interest due tranche Z at month one is directed to tranche A instead.
  - It reduces A's remaining principal from \$386,737 by \$10,000 to \$376,737.
  - But it increases Z's from \$1,000,000 to \$1,010,000.
- At month four, the interest amount that goes into tranche Z, \$10,303, is exactly what is required of Z's remaining principal of \$1,030,301.

Pricing Sequential-Pay CMOs

- We now price the tranches:

$$
\begin{aligned}
\text{tranche A} &= \frac{20000 + 613263}{1.01} + \frac{3767 + 376737}{1.01 \times 1.009} = 1000369, \\
\text{tranche B} &= \frac{10000 + 0}{1.01} + \frac{20100 + 206289}{1.01 \times 1.009} + \frac{18037 + 512195}{1.01 \times 1.009 \times 1.011} \\
&\quad + \frac{2612 + 261215}{1.01 \times 1.009 \times 1.011 \times 1.012} \\
&= 999719, \\
\text{tranche Z} &= \frac{10303 + 199626}{1.01 \times 1.009 \times 1.011 \times 1.012} \\
&\quad + \frac{8307 + 434142}{1.01 \times 1.009 \times 1.011 \times 1.012 \times 1.011} \\
&\quad + \frac{3965 + 396534}{1.01 \times 1.009 \times 1.011 \times 1.012 \times 1.011 \times 1.01} \\
&= 997238.
\end{aligned}
$$

Pricing Sequential-Pay CMOs (continued)

- This CMO has a total theoretical value of $2,997,326.
- It is slightly less than its par value of $3,000,000.
- See the algorithm in Figure 29.12 in the text for the cash flow generator.

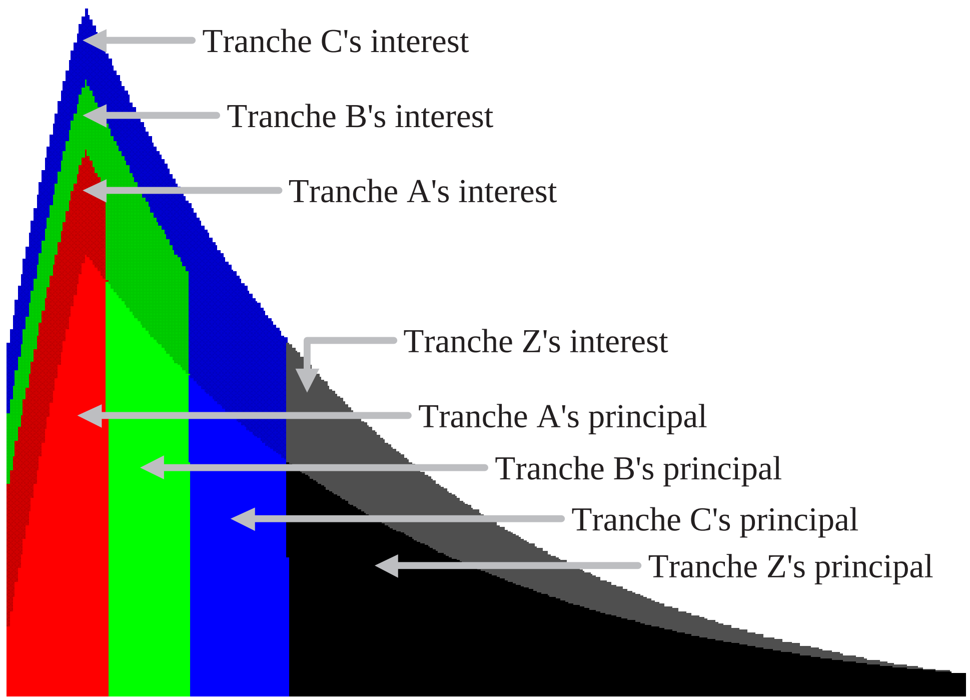
Pricing Sequential-Pay CMOs (continued)

- Suppose we have the interest rate path and the prepayment vector for that interest rate path.
- Then a CMO's cash flow can be calculated and the CMO priced.
- Unfortunately, the remaining principal of a CMO under prepayments is path dependent.
  - For example, a period of high rates before dropping to the current level is not likely to result in the same remaining principal as a period of low rates before rising to the current level.

Pricing Sequential-Pay CMOs (concluded)

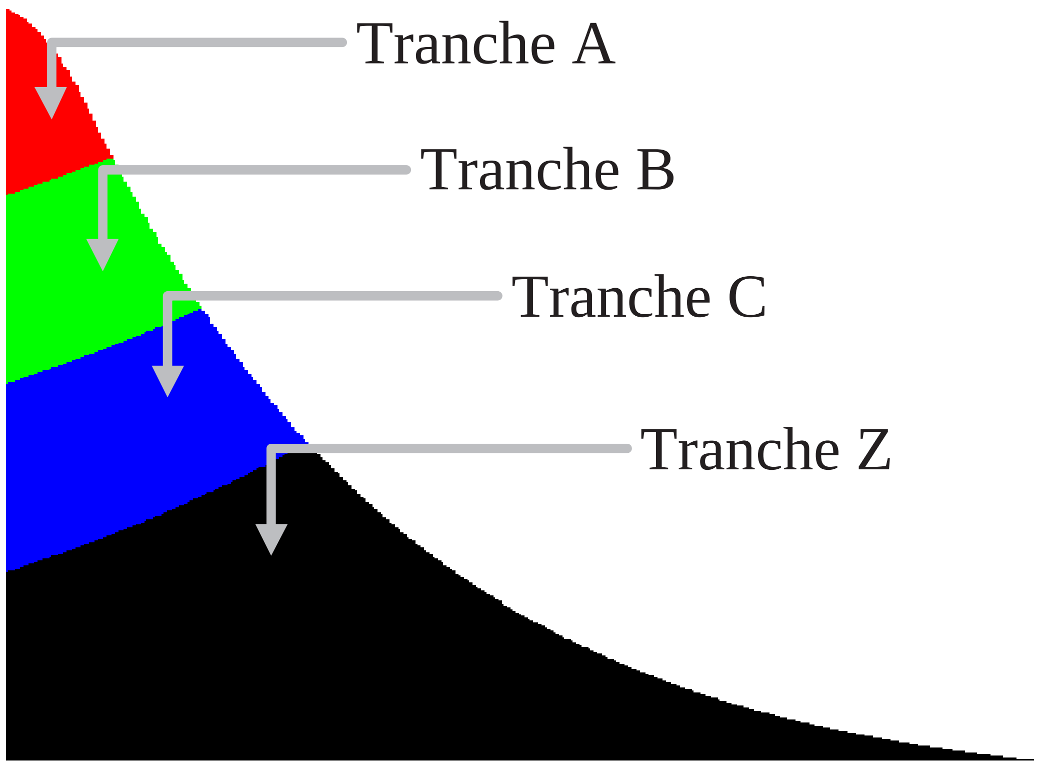
- If we try to price a 30-year CMO on a binomial interest rate model, there will be $2^{360} \approx 2.35 \times 10^{108}$ paths!
- Hence Monte Carlo simulation is the method of choice.
- First, one interest rate path is generated.
- Based on that path, the prepayment model is applied to generate the pool's principal, prepayment, and interest cash flows.
- Now, the cash flows of individual tranches can be generated and their present values derived.
- Repeat the above procedure over many interest rate scenarios and average the present values.

A 4-Tranche Example: Cash Flows



The mortgage rate is 6%, the actual prepayment speed is 150 PSA, and each tranche has an identical original principal amount.

A 4-Tranche Example: Remaining Principals



Collateralized Mortgage Obligations

Capital can be understood only as motion,
not as a thing at rest.
— Karl Marx (1818–1883)

CMOs

- The complexity of a CMO arises from layering different types of payment rules on a prioritized basis.
- In the first-generation CMOs, the sequential-pay CMOs, each class of bond would be retired sequentially.
- A sequential-pay CMO with a large number of tranches will have very narrow cash flow windows for the tranches.
- To further reduce prepayment risk, tranches with a principal repayment schedule were introduced.
- They are called scheduled bonds.

CMOs (continued)

- For example, bonds that guarantee the repayment schedule when the actual prepayment speed lies within a specified range are known as planned amortization class bonds (PACs).
- PACs were introduced in August 1986.
- PACs offer protection against both contraction and extension risks.
- But some investors may desire only protection from one of these risks.
- For them, a bond class known as the targeted amortization class (TAC) was created.

CMOs (continued)

- Scheduled bonds expose certain CMO classes to less prepayment risk.
- However, this can occur only if the redirection in the prepayment risk is absorbed as much as possible by other classes known as the support bonds.
  - Support bonds are a necessary by-product of the creation of scheduled tranches.
- Pro rata bonds provide another means of layering.
- Principal cash flows to these bonds are divided proportionally, but the bonds can have different interest payment rules.

CMOs (continued)

- Suppose the weighted average coupon (WAC) of the collateral is 10%, tranche B1 receives 40% of the principal, and tranche B2 receives 60% of the principal.
- Given this pro rata structure, many choices of interest payment rules are possible for B1 and B2 as long as the interest payments are nonnegative and the WAC does not exceed 10%.
- The coupon rates can even be floating.

CMOs (continued)

- One possibility is for B1 to have a coupon of 5% and for B2 to have a coupon of 13.33%.
- This works because

$$\frac{40}{100} \times 5\% + \frac{60}{100} \times 13.33\% = 10\%.$$

- Bonds with pass-through coupons that are higher and lower than the collateral coupon have thus been created.
- Bonds like B1 are called synthetic discount securities.
- Bonds like B2 are called synthetic premium securities.

CMOs (continued)

- An extreme case is for B1 to receive 99% of the principal and have a 5% coupon and for B2 to receive only 1% of the principal and have a 505% coupon.
- In fact, first-generation IOs took the form of B2 in July 1986.
- IOs have either a nominal principal or a notional principal.
- A nominal principal represents actual principal that will be paid.
- It is called "nominal" because it is extremely small, resulting in an extremely high coupon rate.

CMOs (concluded)

- A case in point is the B2 class with a 505% coupon above.
- A notional principal, in contrast, is the amount on which interest is calculated.
- An IO holder owns none of the notional principal.
- Once the notional principal amount declines to zero, no further payments are made on the IO.

Floating-Rate Tranches

- A form of pro rata bonds are floaters and inverse floaters whose combined coupon does not exceed the collateral coupon.
- A floater is a class whose coupon rate varies directly with the change in the reference rate.
- An inverse floater is a class whose coupon rate changes in the direction opposite to the change in the reference rate.
- When the coupon on the inverse floater changes by x times the amount of the change in the reference rate, this multiple x is called its slope.

Floating-Rate Tranches (continued)

- Because the interest comes from fixed-rate mortgages, floaters must have a coupon cap.
- Similarly, inverse floaters must have a coupon floor.
- Floating-rate classes were created in September 1986.
- Suppose the floater has a principal of P_{f} and the inverse floater has a principal of P_{i}.
- Define $\omega_{\mathrm{f}} \equiv P_{\mathrm{f}}/(P_{\mathrm{f}} + P_{\mathrm{i}})$ and $\omega_{\mathrm{i}} \equiv P_{\mathrm{i}}/(P_{\mathrm{f}} + P_{\mathrm{i}})$.
- The coupon rates of the floater, c_{f}, and the inverse floater, c_{i}, must satisfy $\omega_{\mathrm{f}} \times c_{\mathrm{f}} + \omega_{\mathrm{i}} \times c_{\mathrm{i}} = \mathrm{WAC}$, or

$$c_{\mathrm{i}} = \frac{\mathrm{WAC} - \omega_{\mathrm{f}} \times c_{\mathrm{f}}}{\omega_{\mathrm{i}}}.$$

Floating-Rate Tranches (concluded)

- The slope is clearly $\omega_{\rm f}/\omega_{\rm i}$.
- To make sure that the inverse floater will not encounter a negative coupon, the cap on the floater must be less than $\text{WAC}/\omega_{\rm f}$.
- In fact, caps and floors are related via

$$\mathsf{floor} = \frac{\text{WAC} - \omega_{\rm f} \times \mathsf{cap}}{\omega_{\rm i}}.$$

An Example

- Take a CMO deal that includes a floater with a principal of \$64 million and an inverse floater with a principal of \$16 million.
- The coupon rate for the floating-rate class is $\text{LIBOR} + 0.65$.
- The coupon rate for the inverse floater is $42.4 - 4 \times \text{LIBOR}$.
- The slope is thus four.

An Example (concluded)

- The WAC of the two classes is

$$\frac{64}{80} \times \text{floater coupon rate} + \frac{16}{80} \times \text{inverse floater coupon rate} = 9\%$$

  regardless of the level of the LIBOR.

- Consequently, the coupon rate on the underlying collateral, 9%, can support the aggregate interest payments that must be made to these two classes.

- If we set a floor of 0% for the inverse floater, the cap on the floater is 11.25%.

Superfloaters

- A variant of the floating-rate CMO is the superfloater introduced in 1987.
- In a conventional floating-rate class, the coupon rate moves up or down on a one-to-one basis with the reference rate.
- A superfloater's coupon rate, in comparison, changes by some multiple of the change in the reference rate.
  - It magnifies any changes in the value of the reference rate.
- Superfloater tranches are bearish because their value generally appreciates with rising interest rates.

An Example

- Suppose the initial LIBOR is 7% and the coupon rate for a superfloater is set by

$$(\text{initial LIBOR} - 40 \text{ basis points}) + 2 \times (\text{change in LIBOR}).$$

- The following table shows how the superfloater changes its coupon rate as LIBOR changes.
  - The coupon rates for a conventional floater of LIBOR plus 50 basis points are also listed for comparison.

LIBOR change (basis points)	−300	−200	−100	0	+100	+200	+300
Superfloater	0.6	2.6	4.6	6.6	8.6	10.6	12.6
Conventional floater	4.5	5.5	6.5	7.5	8.5	9.5	10.5

An Example (concluded)

- A superfloater provides a much higher yield than a conventional floater when interest rates rise.
- It provides a much lower yield when interest rates fall or remain stable.
- This can be verified by looking at the above table via spreads in basis points to the LIBOR in the next table.

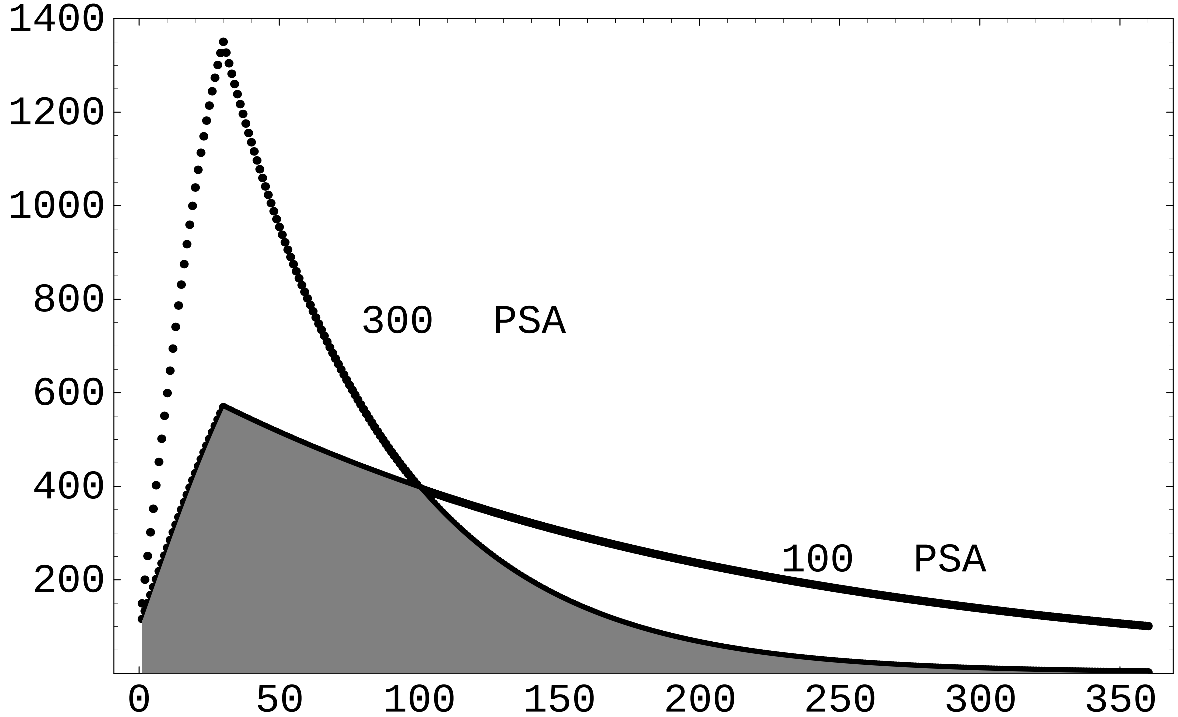
LIBOR change (basis points)	−300	−200	−100	0	+100	+200	+300
Superfloater	−340	−240	−140	−40	60	160	260
Conventional floater	50	50	50	50	50	50	50

PAC Bonds

- PAC bonds are created by calculating the cash flows from the collateral by use of two prepayment speeds: a fast one and a slow one.
- Consider a PAC band of 100 PSA (the lower collar) to 300 PSA (the upper collar).
- The plot on p. 1105 shows the principal payments at the two collars.
- The principal payments under the higher-speed scenario are higher in the earlier years but lower in later years.

PAC Bonds (continued)

- The shaded area represents the principal payment schedule that is guaranteed for every possible prepayment speed between 100% and 300% PSAs.
- It is calculated by taking the minimum of the principal paydowns at the lower collar and those at the upper collar.
- This schedule is called the PAC schedule.
- See Figure 30.2 in the text for a linear-time cash flow generator for a simple CMO containing a PAC bond and a support bond.



The underlying mortgages are 30-year ones with a total original loan amount of \$100,000,000 (the numbers on the y axis are in thousands) and a coupon rate of 6%.

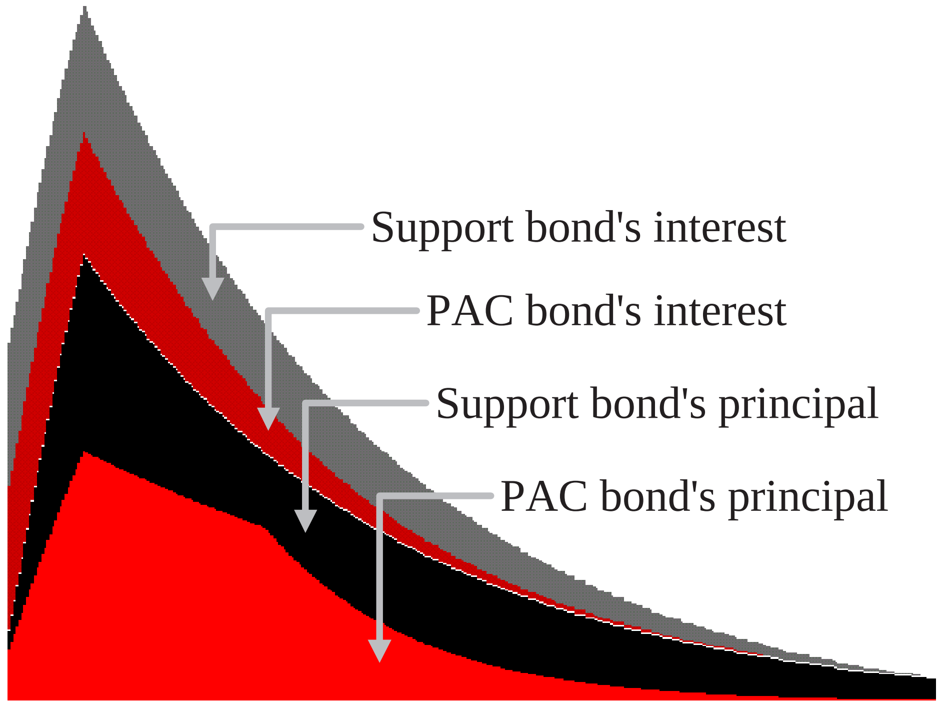
PAC Bonds (continued)

- Adherence to the amortization schedule of the PAC takes priority over those of all other bonds.
- The cash flow of a PAC bond is therefore known as long as its support bonds are not fully paid off.
- Whether this happens depends to a large extent on the CMO structure, such as priority and the relative sizes of PAC and non-PAC classes.
- For example, a relatively small PAC is harder to break than a larger PAC, other things being equal.

PAC Bonds (continued)

- If the actual prepayment speed is 150 PSA, the principal payment pattern of the PAC bond adheres to the PAC schedule.
- The cash flows of the support bond "flow around" the PAC bond (see the plot on p. 1108).
- The cash flows are neither sequential nor pro rata.
- In fact, the support bond pays down *simultaneously* with the PAC bond.
- Because more than one class of bonds may be receiving principal payments at the same time, structures with PAC bonds are simultaneous-pay CMOs.

Cash Flow of a PAC Bond at 150 PSA



The mortgage rate is 6%, the PAC band is 100 PSA to 300 PSA, and the actual prepayment speed is 150 PSA.

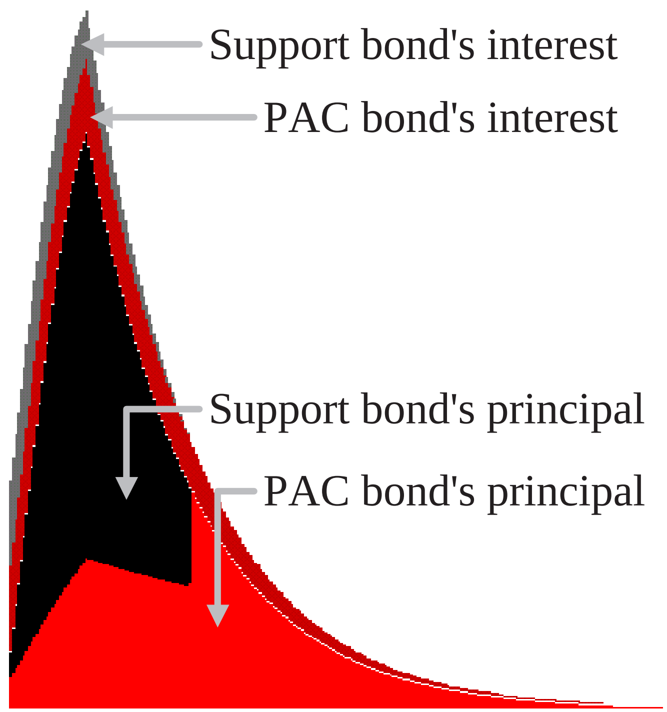
PAC Bonds (continued)

- At the lower prepayment speed of 100 PSA, far less principal cash flow is available in the early years of the CMO.
- As all the principal cash flows go to the PAC bond in the early years, the principal payments on the support bond are deferred and the support bond extends.
- The support bond does receive more interest payments.

PAC Bonds (continued)

- If prepayments move outside the PAC band, the PAC schedule may not be met.
- At 400 PSA, for example, the cash flows to the support bond are accelerated.
- After the support bond is fully paid off, all remaining principal payments go to the PAC bond; its life is shortened.
- See the plot on p. 1111.

Cash Flow of a PAC Bond at 400 PSA



The mortgage rate is 6%, the PAC band is 100 PSA to 300 PSA, and the actual prepayment speed is 400 PSA.

PAC Bonds (concluded)

- The support bond thus absorbs part of the contraction risk.
- Similarly, should the actual prepayment speed fall below the lower collar, then in subsequent periods the PAC bond has priority on the principal payments.
- This reduces the extension risk, which is again absorbed by the support bond.

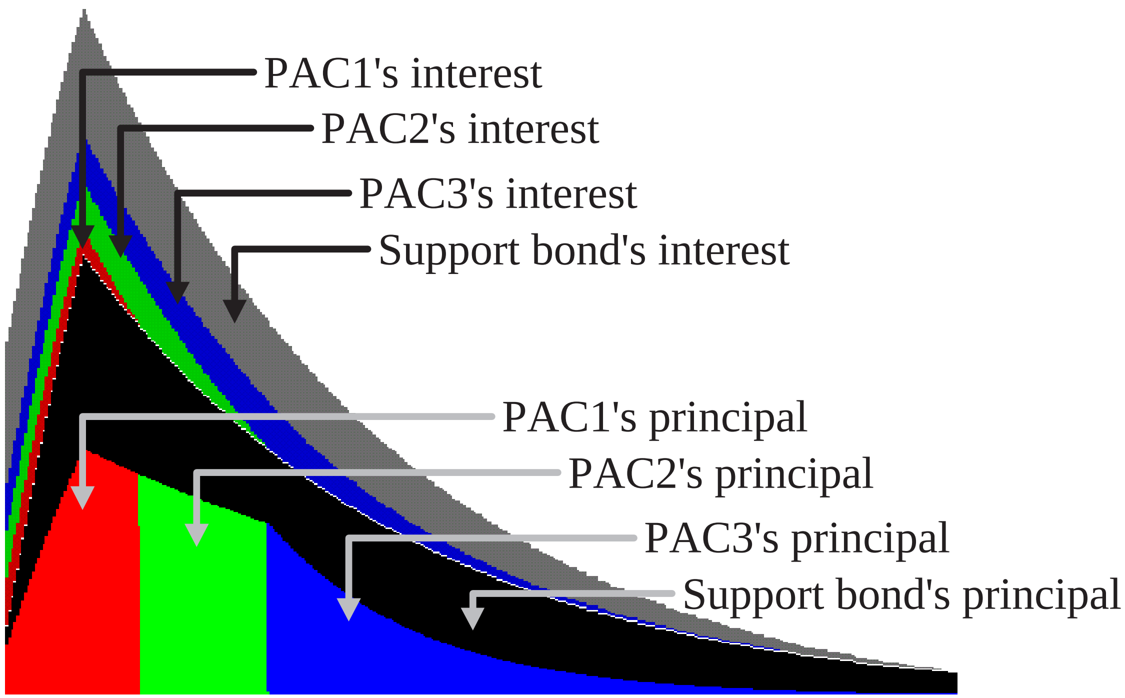
PAC Drift

- The PAC band guarantees that if prepayments occur at any single constant speed within the band *and* stay there, the PAC schedule will be met.
- However, the PAC schedule may not be met even if prepayments on the collateral always vary within the band over time.
- This is because the band that guarantees the original PAC schedule can expand and contract, depending on actual prepayments.
- This phenomenon is known as PAC drift.

Sequential PACs

- PACs can be divided sequentially to provide narrower paydown structures.
- These sequential PACs narrow the range of years over which principal payments occur.
- See the plot on p. 1115.
- Although these bonds are all structured with the same band, the actual range of speeds over which their schedules will be met may differ.

Cash Flow of a Sequential PAC Bond



The mortgage rate is 6%, the PAC band is 100 PSA to 300 PSA, and the actual prepayment speed is 150 PSA. The three PAC bonds have identical original principal amounts.

Sequential PACs (concluded)

- We can take a CMO bond and further structure it.
- For example, the sequential PACs can be split by use of a pro rata structure to create high and low coupon PACs.
- We can also replace the second tranche in a four-tranche ABCZ sequential CMO with a PAC class that amortizes starting in year four, say.
- But note that tranche C may start to receive prepayments that are in excess of the schedule of the PAC bond.
- It may even be retired earlier than tranche B.

Finis