Equilibrium Term Structure Models

8. What's your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*

Introduction

- This chapter surveys equilibrium models.
- Since the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t},$$

the discount function $P(t,T)$ suffices to establish the spot rate curve.

- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\, dt + \sigma\, dW.$$

- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\, dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\, r(T) \,|\, r(t) = r\,] = \mu + (r - \mu)\, e^{-\beta(T-t)}$$

from Eq. (54) on p. 503.

[a]Vasicek (1977).

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{100}$$

where

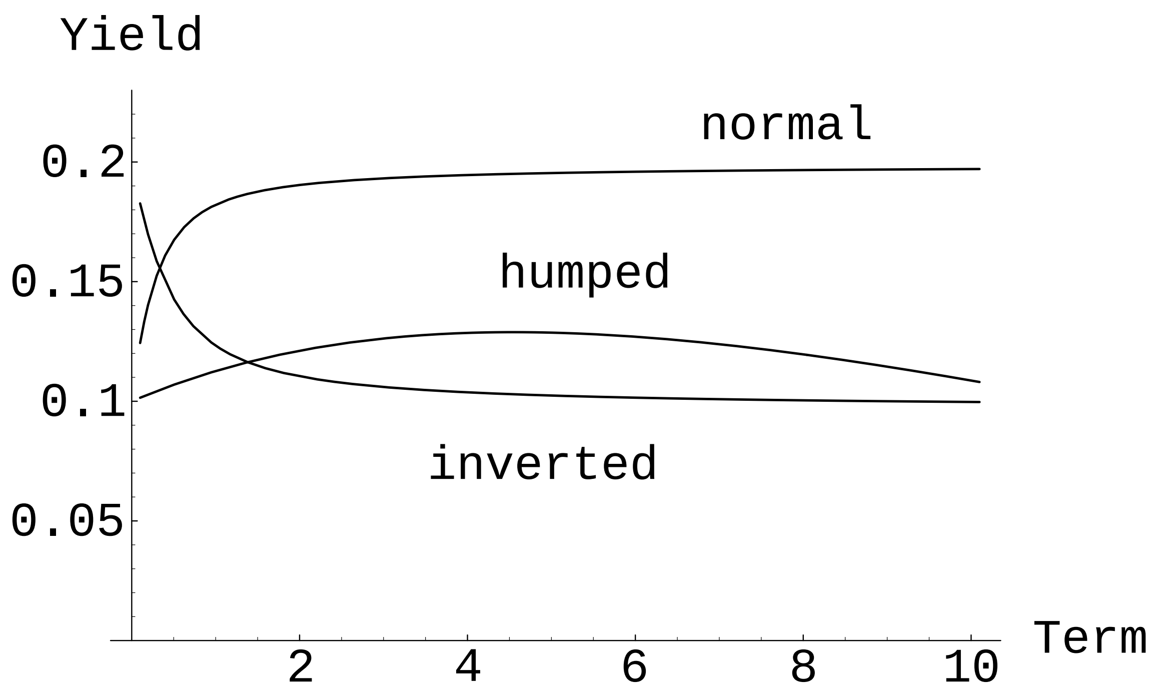
$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta}\right] & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2(T-t)^3}{6}\right] & \text{if } \beta = 0. \end{cases}$$

and

$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T-t & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (concluded)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, P may exceed one for a finite T.
- The spot rate volatility structure is the curve $(\partial r(t, T)/\partial r)\, \sigma = \sigma B(t, T)/(T - t)$.
- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, β, controls the shape of the curve.
- Indeed, higher β leads to greater attenuation of volatility with maturity.



Yield
normal
0.2
humped
0.15
0.1
inverted
0.05
2
4
6
8
10
Term

The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.

- Its price is given by

$$P(t, s)\, N(x) - X P(t, T)\, N(x - \sigma_v).$$

[a] Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$
\begin{aligned}
x &\equiv \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X}\right) + \frac{\sigma_v}{2}, \\
\sigma_v &\equiv v(t,T)\,B(T,s), \\
v(t,T)^2 &\equiv \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}.
\end{aligned}
$$

- By the put-call parity, the price of a European put is

$$
XP(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).
$$

Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[0, T]$ divided into n identical pieces.
- Let $\Delta t \equiv T/n$ and

$$p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[a]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\,\xi(k), \quad 0 \leq k < n.$$

[a]Nelson and Ramaswamy (1990).

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases}.$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.
- For a general process Y with nonconstant volatility, the resulting binomial tree may not combine.

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\,dt + \sigma\sqrt{r}\,dW. \tag{101}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.
- See text for the bond pricing formula.

[a] Cox, Ingersoll, and Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into n periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process

$$x(r) \equiv 2\sqrt{r}/\sigma.$$

- It follows

$$dx = m(x)\, dt + dW,$$

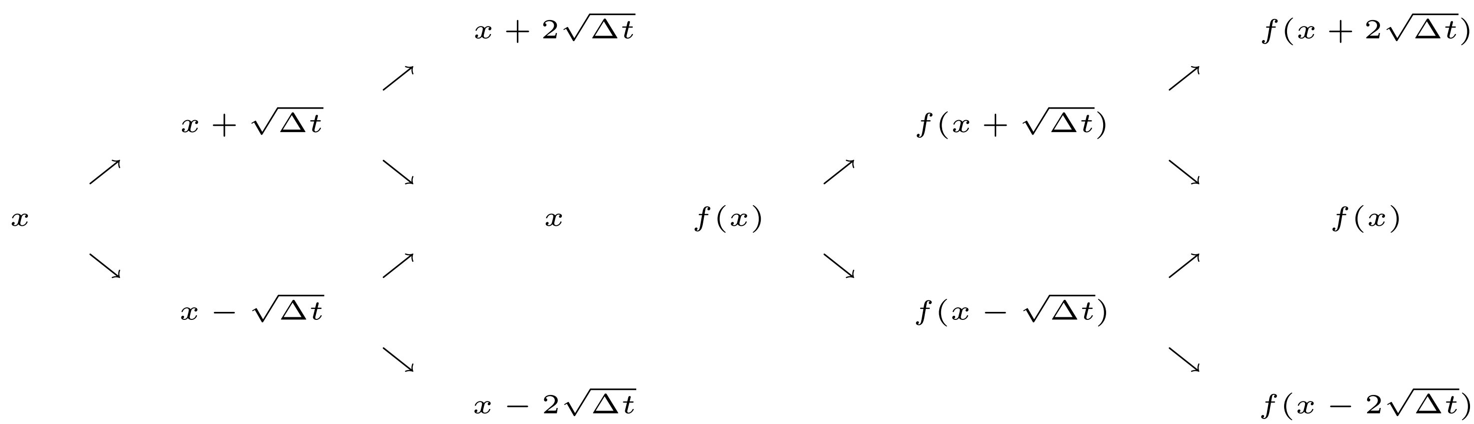
where

$$m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- Since this new process has a constant volatility, its associated binomial tree combines.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation $r = f(x) \equiv x^2\sigma^2/4$ (p. 890).


x
$x + \sqrt{\Delta t}$
$x - \sqrt{\Delta t}$
$x + 2\sqrt{\Delta t}$
x
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$f(x)$
$f(x + \sqrt{\Delta t})$
$f(x - \sqrt{\Delta t})$
$f(x + 2\sqrt{\Delta t})$
$f(x)$
$f(x - 2\sqrt{\Delta t})$

Binomial CIR (concluded)

- The probability of an up move at each node r is

$$p(r) \equiv \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}. \tag{102}$$

  - $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Numerical Examples

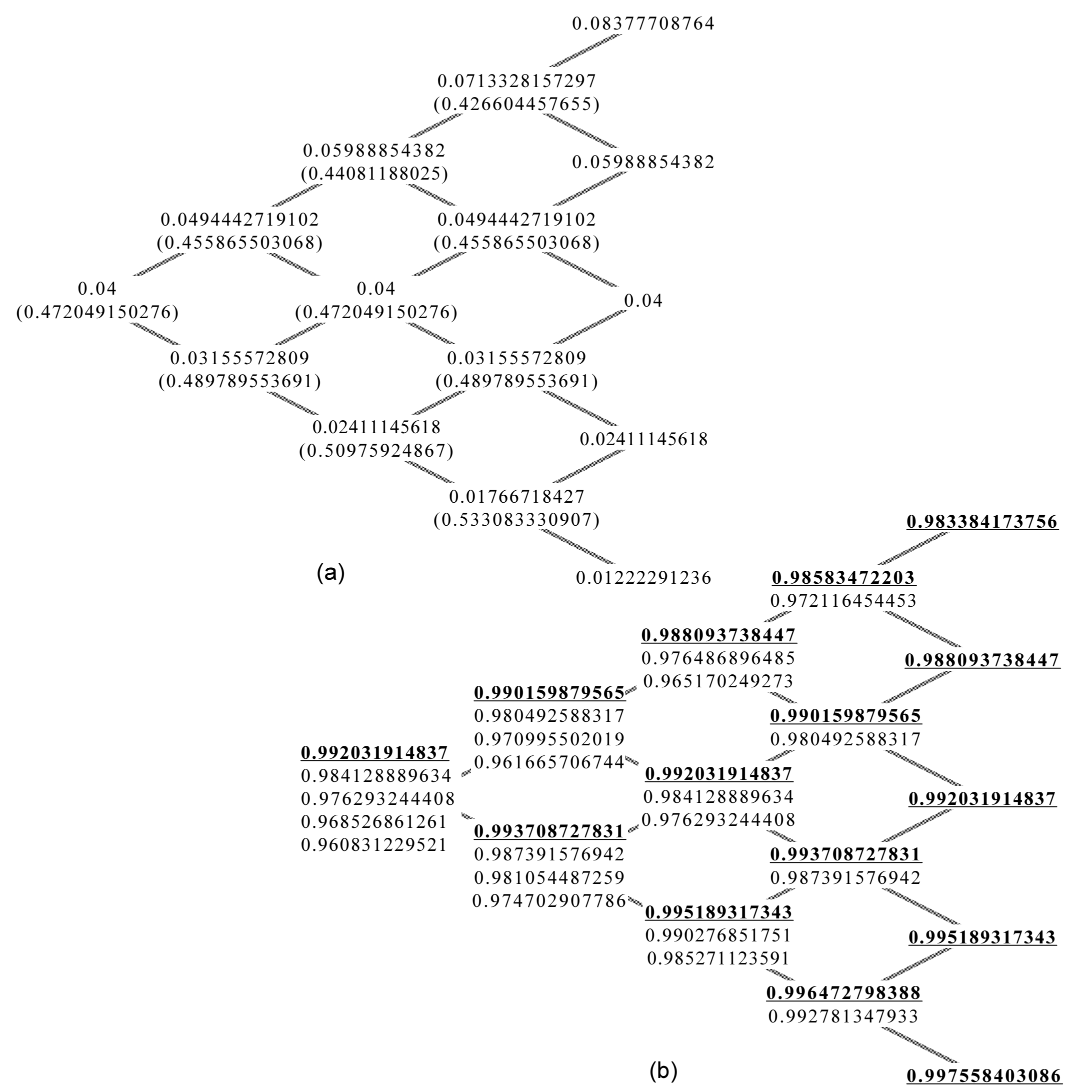
- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.

- See p. 893(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.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(b)

Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$.

Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process $dy = \alpha(y,t)\,dt + \sigma(y,t)\,dW$.
- Make sure the binomial model's drift and diffusion converge to the above process by setting the probability of an up move to

$$\frac{\alpha(y,t)\,\Delta t + y - y_{\mathrm{d}}}{y_{\mathrm{u}} - y_{\mathrm{d}}}.$$

- Here $y_{\mathrm{u}} \equiv y + \sigma(y,t)\sqrt{\Delta t}$ and $y_{\mathrm{d}} \equiv y - \sigma(y,t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.
- The displacements are identical, at $\sigma(y,t)\sqrt{\Delta t}$.

[a]Nelson and Ramaswamy (1990).

A General Method (continued)

- But the binomial tree may not combine:

$$\sigma(y,t)\sqrt{\Delta t} - \sigma(y_{\rm u},t)\sqrt{\Delta t} \neq -\sigma(y,t)\sqrt{\Delta t} + \sigma(y_{\rm d},t)\sqrt{\Delta t}$$

  in general.

- When $\sigma(y,t)$ is a constant independent of y, equality holds and the tree combines.
- To achieve this, define the transformation

$$x(y,t) \equiv \int^{y} \sigma(z,t)^{-1}\, dz.$$

- Then x follows $dx = m(y,t)\, dt + dW$ for some $m(y,t)$ (see text).

A General Method (continued)

- The key is that the diffusion term is now a constant, and the binomial tree for x combines.
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.
- Note that $y_{\mathrm{u}}(x,t) \equiv y(x+\sqrt{\Delta t}, t+\Delta t)$ and $y_{\mathrm{d}}(x,t) \equiv y(x-\sqrt{\Delta t}, t+\Delta t)$.

A General Method (concluded)

- The transformation is

$$\int^{r} (\sigma\sqrt{z})^{-1}\, dz = 2\sqrt{r}/\sigma$$

  for the CIR model.

- The transformation is

$$\int^{S} (\sigma z)^{-1}\, dz = (1/\sigma) \ln S$$

  for the Black-Scholes model.

- The familiar binomial option pricing model in fact discretizes $\ln S$ not S.

Model Calibration

- In the time-series approach, the time series of short rates is used to estimate the parameters of the process.
- This approach may help in validating the proposed interest rate process.
- But it alone cannot be used to estimate the risk premium parameter λ.
- The model prices based on the estimated parameters may also deviate a lot from those in the market.

Model Calibration (concluded)

- The cross-sectional approach uses a cross section of bond prices observed at the same time.
- The parameters are to be such that the model prices closely match those in the market.
- After this procedure, the calibrated model can be used to price interest rate derivatives.
- Unlike the time-series approach, the cross-sectional approach is unable to separate out the interest rate risk premium from the model parameters.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two-factor ones.

Options on Coupon Bonds[a]

- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time T on a bond with par value \$1.
- Let X denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all i.
- The payoff for the option is

$$\max\left(\sum_{i=1}^{n} c_i P(r(T), T, t_i) - X, 0\right).$$

[a] Jamshidian (1989).

Options on Coupon Bonds (continued)

- At time T, there is a unique value r^* for $r(T)$ that renders the coupon bond's price equal the strike price X.
- This r^* can be obtained by solving $X = \sum_{i=1}^{n} c_i P(r, T, t_i)$ numerically for r.
- The solution is also unique for one-factor models whose bond price is a monotonically decreasing function of r.
- Let $X_i \equiv P(r^*, T, t_i)$, the value at time T of a zero-coupon bond with par value \$1 and maturing at time t_i if $r(T) = r^*$.

Options on Coupon Bonds (concluded)

- Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$.
- As $X = \sum_i c_i X_i$, the option's payoff equals

$$\max\left(\sum_{i=1}^{n} c_i P(r(T), T, t_i) - \sum_i c_i X_i, 0\right)$$
$$= \sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

- Thus the call is a package of n options on the underlying zero-coupon bond.
- Why can't we do the same thing for Asian options?[a]

[a] Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.

No-Arbitrage Term Structure Models

How much of the structure of our theories
really tells us about things in nature,
and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)

Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.

No-Arbitrage Models[a]

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

[a] Ho and Lee (1986).

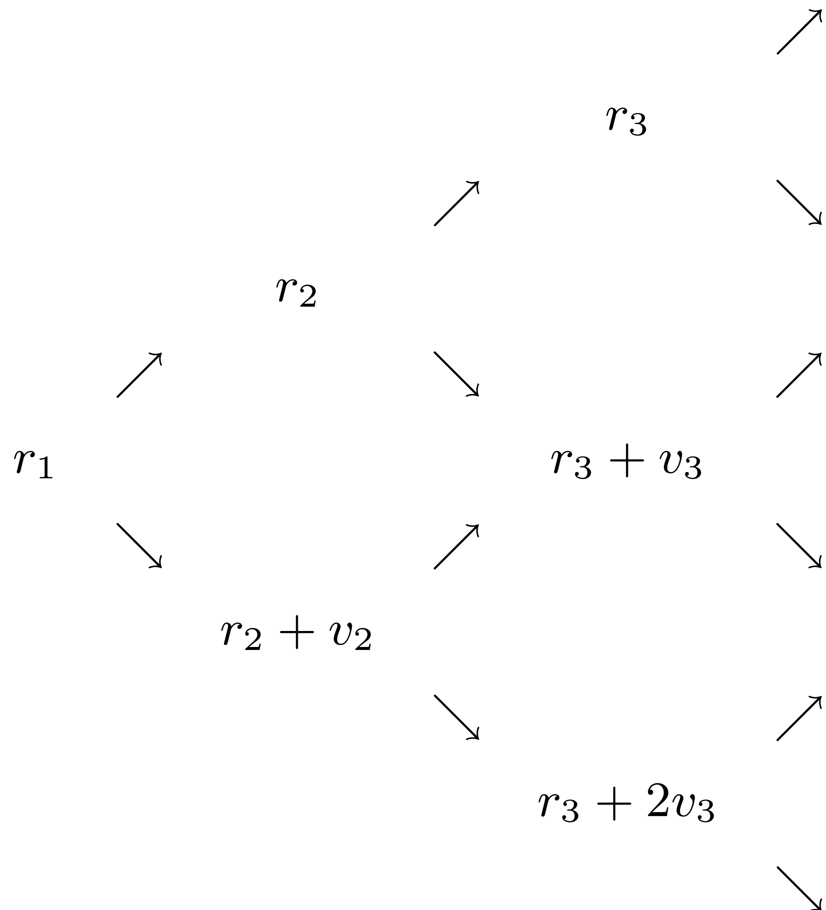
No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.

The Ho-Lee Model[a]

- The short rates at any given time are evenly spaced.
- Let p denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

[a]Ho and Lee (1986).


r_3
r_2
r_1
$r_3 + v_3$
$r_2 + v_2$
$r_3 + 2v_3$

The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t+1), P(t, t+2), \ldots$ at time t identified with the root of the tree.
- Let the discount factors in the next period be

$$\begin{array}{ll} P_{\mathrm{d}}(t+1, t+2), P_{\mathrm{d}}(t+1, t+3), \ldots & \text{if short rate moves down} \\ P_{\mathrm{u}}(t+1, t+2), P_{\mathrm{u}}(t+1, t+3), \ldots & \text{if short rate moves up} \end{array}$$

- By backward induction, it is not hard to see that for $n \geq 2$,

$$P_{\mathrm{u}}(t+1, t+n) = P_{\mathrm{d}}(t+1, t+n)\, e^{-(v_2 + \cdots + v_n)} \tag{103}$$

(see text).

The Ho-Lee Model (continued)

- It is also not hard to check that the n-period zero-coupon bond has yields

$$\begin{aligned} y_{\mathrm{d}}(n) &\equiv -\frac{\ln P_{\mathrm{d}}(t+1,t+n)}{n-1} \\ y_{\mathrm{u}}(n) &\equiv -\frac{\ln P_{\mathrm{u}}(t+1,t+n)}{n-1} = y_{\mathrm{d}}(n) + \frac{v_2 + \cdots + v_n}{n-1} \end{aligned}$$

- The volatility of the yield to maturity for this bond is therefore

$$\begin{aligned} \kappa_n &\equiv \sqrt{p y_{\mathrm{u}}(n)^2 + (1-p)\, y_{\mathrm{d}}(n)^2 - [\, p y_{\mathrm{u}}(n) + (1-p)\, y_{\mathrm{d}}(n)\,]^2} \\ &= \sqrt{p(1-p)}\, (y_{\mathrm{u}}(n) - y_{\mathrm{d}}(n)) \\ &= \sqrt{p(1-p)}\, \frac{v_2 + \cdots + v_n}{n-1}. \end{aligned}$$

The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)}\, v_2. \tag{104}$$

- The variance of the short rate therefore equals $p(1-p)(r_{\mathrm{u}} - r_{\mathrm{d}})^2$, where r_{u} and r_{d} are the two successor rates.[a]

[a] Contrast this with the lognormal model.

The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of $\kappa_2, \kappa_3, \ldots$.
  - It is independent of the r_i.
- It is easy to compute the v_is from the volatility structure, and vice versa.
- The r_is can be computed by forward induction.
- The volatility structure is supplied by the market.

The Ho-Lee Model: Bond Price Process

- In a risk-neutral economy, the initial discount factors satisfy

$$P(t, t+n) = (pP_{\mathrm{u}}(t+1, t+n) + (1-p)\, P_{\mathrm{d}}(t+1, t+n))\, P(t, t+1)$$

- Combine the above with Eq. (103) on p. 915 and assume $p = 1/2$ to obtain[a]

$$P_{\mathrm{d}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2 \times \exp[\, v_2 + \cdots + v_n\,]}{1 + \exp[\, v_2 + \cdots + v_n\,]}, \tag{105}$$

$$P_{\mathrm{u}}(t+1, t+n) = \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1 + \exp[\, v_2 + \cdots + v_n\,]}. \tag{105$'$}$$

[a]In the limit, only the volatility matters.

The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.
- Suppose all v_i equal some constant v and $\delta \equiv e^v > 0$.
- Then

$$\begin{aligned} P_{\mathrm{d}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2\delta^{n-1}}{1+\delta^{n-1}}, \\ P_{\mathrm{u}}(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1+\delta^{n-1}}. \end{aligned}$$

- Short rate volatility σ equals $v/2$ by Eq. (104) on p. 917.
- Price derivatives by taking expectations under the risk-neutral probability.

The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an n-period zero-coupon bond is

$$r(t, t+n) \equiv \ln\left(\frac{P(t+1, t+n)}{P(t, t+n)}\right).$$

- Its value is either $\ln \frac{P_{\text{d}}(t+1,t+n)}{P(t,t+n)}$ or $\ln \frac{P_{\text{u}}(t+1,t+n)}{P(t,t+n)}$.
- Thus the variance of return is

$$\text{Var}[\, r(t, t+n)\,] = p(1-p)((n-1)\, v)^2 = (n-1)^2\sigma^2.$$

The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t, t+n)$ and $r(t, t+m)$ is $(n-1)(m-1)\,\sigma^2$ (see text).
- As a result, the correlation between any two one-period rates of return is unity.
- Strong correlation between rates is inherent in all one-factor Markovian models.

The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is

$$dr = \theta(t)\, dt + \sigma\, dW.$$

- This is Vasicek's model with the mean-reverting drift replaced by a deterministic, time-dependent drift.
- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying, i.e., $dr = \theta(t)\, dt + \sigma(t)\, dW$.
- This corresponds to the discrete-time model in which v_i are not all identical.

The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.

Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model's state variables (factors) not its parameters.
- Model *parameters*, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.

Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model's intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.

The Black-Derman-Toy Model[a]

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 767ff (repeated on next page).
- The volatility structure is given by the market.
- From it, the short rate volatilities (thus v_i) are determined together with r_i.

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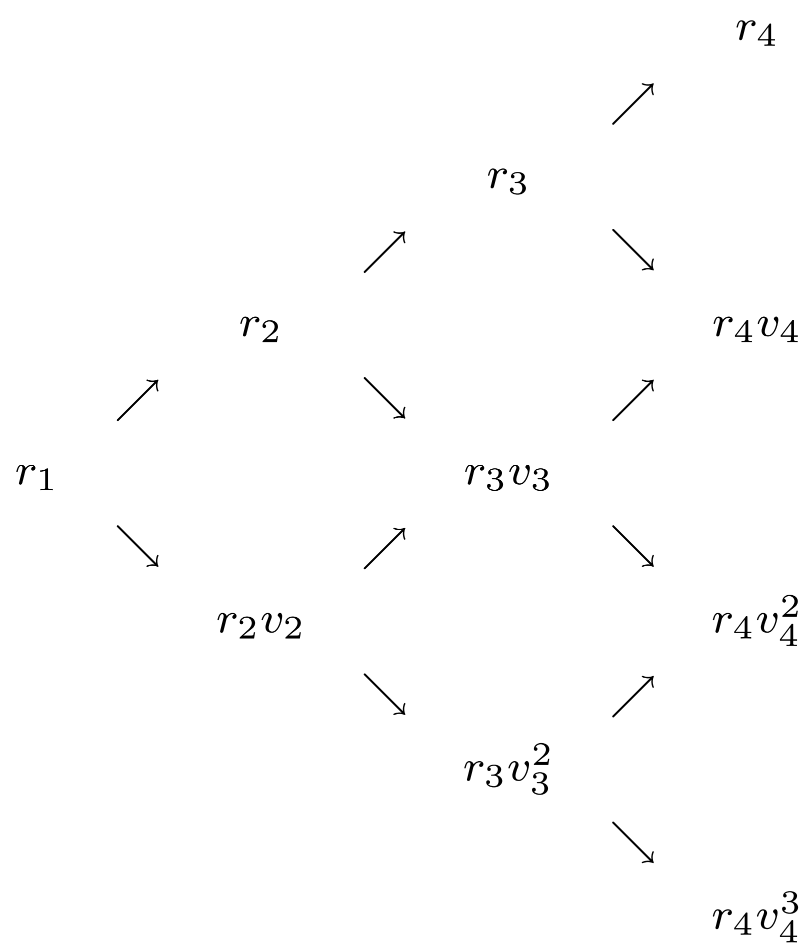
[a] Black, Derman, and Toy (BDT) (1990).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$r_3 v_3^2$
$r_4 v_4^3$

The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes v_i are given a priori.
  - A related model of Salomon Brothers takes v_i to be constants.
- Lognormal models preclude negative short rates.

The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.
- Let the yield volatility of the i-period zero-coupon bond be denoted by κ_i.
- P_{u} is the price of the i-period zero-coupon bond one period from now if the short rate makes an up move.

The BDT Model: Volatility Structure (concluded)

- P_{d} is the price of the i-period zero-coupon bond one period from now if the short rate makes a down move.
- Corresponding to these two prices are the following yields to maturity,

$$
\begin{aligned}
y_{\mathrm{u}} &\equiv P_{\mathrm{u}}^{-1/(i-1)} - 1, \\
y_{\mathrm{d}} &\equiv P_{\mathrm{d}}^{-1/(i-1)} - 1.
\end{aligned}
$$

- The yield volatility is defined as $\kappa_i \equiv (1/2) \ln(y_{\mathrm{u}}/y_{\mathrm{d}})$.

The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

$$(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1}).$$

  - They define the binomial tree up to period $i - 1$.
- We now proceed to calculate r_i and v_i to extend the tree to period i.

The BDT Model: Calibration (continued)

- Assume the price of the i-period zero can move to $P_{\rm u}$ or $P_{\rm d}$ one period from now.
- Let y denote the current i-period spot rate, which is known.
- In a risk-neutral economy,

$$\frac{P_{\rm u} + P_{\rm d}}{2(1 + r_1)} = \frac{1}{(1 + y)^i}. \tag{106}$$

- Obviously, $P_{\rm u}$ and $P_{\rm d}$ are functions of the unknown r_i and v_i.

The BDT Model: Calibration (continued)

- Viewed from now, the future $(i-1)$-period spot rate at time one is uncertain.
- Recall that y_{u} and y_{d} represent the spot rates at the up node and the down node, respectively (p. 931).
- With κ^2 denoting their variance, we have

$$\kappa_i = \frac{1}{2} \ln \left(\frac{{P_{\mathrm{u}}}^{-1/(i-1)} - 1}{{P_{\mathrm{d}}}^{-1/(i-1)} - 1} \right). \tag{107}$$

The BDT Model: Calibration (continued)

- We will employ forward induction to derive a quadratic-time calibration algorithm.[a]
- Recall that forward induction inductively figures out, by moving forward in time, how much \$1 at a node contributes to the price (review p. 792(a)).
- This number is called the state price and is the price of the claim that pays \$1 at that node and zero elsewhere.

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[a]Chen and Lyuu (1997); Lyuu (1999).

The BDT Model: Calibration (continued)

- Let the unknown baseline rate for period i be $r_i = r$.
- Let the unknown multiplicative ratio be $v_i = v$.
- Let the state prices at time $i-1$ be $P_1, P_2, \ldots, P_i$, corresponding to rates $r, rv, \ldots, rv^{i-1}$, respectively.
- One dollar at time i has a present value of

$$f(r, v) \equiv \frac{P_1}{1+r} + \frac{P_2}{1+rv} + \frac{P_3}{1+rv^2} + \cdots + \frac{P_i}{1+rv^{i-1}}.$$

The BDT Model: Calibration (continued)

- The yield volatility is

$$g(r,v) \equiv \frac{1}{2}\,\ln\left(\frac{\left(\frac{P_{\text{u},1}}{1+rv}+\frac{P_{\text{u},2}}{1+rv^2}+\cdots+\frac{P_{\text{u},i-1}}{1+rv^{i-1}}\right)^{-1/(i-1)}-1}{\left(\frac{P_{\text{d},1}}{1+r}+\frac{P_{\text{d},2}}{1+rv}+\cdots+\frac{P_{\text{d},i-1}}{1+rv^{i-2}}\right)^{-1/(i-1)}-1}\right).$$

- Above, $P_{\text{u},1}, P_{\text{u},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the up node (like $r_2 v_2$ on p. 928).
- And $P_{\text{d},1}, P_{\text{d},2}, \ldots$ denote the state prices at time $i-1$ of the subtree rooted at the down node (like r_2 on p. 928).

The BDT Model: Calibration (concluded)

- Now solve

$$f(r,v) = \frac{1}{(1+y)^i} \quad \text{and} \quad g(r,v) = \kappa_i$$

for $r = r_i$ and $v = v_i$.

- This $O(n^2)$-time algorithm appears in the text.