Equilibrium Term Structure Models
8. What’s your problem? Any moron can understand bond pricing models.
— *Top Ten Lies Finance Professors Tell Their Students*
Introduction

• This chapter surveys equilibrium models.
• Since the spot rates satisfy

\[ r(t, T) = -\frac{\ln P(t, T)}{T - t}, \]

the discount function \( P(t, T) \) suffices to establish the spot rate curve.
• All models to follow are short rate models.
• Unless stated otherwise, the processes are risk-neutral.
The Vasicek Model\textsuperscript{a}

- The short rate follows

\[ dr = \beta(\mu - r) \, dt + \sigma \, dW. \]

- The short rate is pulled to the long-term mean level \( \mu \) at rate \( \beta \).

- Superimposed on this “pull” is a normally distributed stochastic term \( \sigma \, dW \).

- Since the process is an Ornstein-Uhlenbeck process,

\[ E[r(T) \mid r(t) = r] = \mu + (r - \mu) e^{-\beta(T-t)} \]

from Eq. (54) on p. 503.

\textsuperscript{a}Vasicek (1977).
The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

\[
P(t, T) = A(t, T) e^{-B(t, T) r(t)}, \tag{100}
\]

where

\[
A(t, T) = \begin{cases} 
\exp \left[ \frac{(B(t, T) - T + t)(\beta^2 \mu - \sigma^2/2) - \sigma^2 B(t, T)^2}{4 \beta} \right] & \text{if } \beta \neq 0, \\
\exp \left[ \frac{\sigma^2 (T - t)^3}{6} \right] & \text{if } \beta = 0. 
\end{cases}
\]

and

\[
B(t, T) = \begin{cases} 
\frac{1 - e^{-\beta (T - t)}}{\beta} & \text{if } \beta \neq 0, \\
T - t & \text{if } \beta = 0. 
\end{cases}
\]
The Vasicek Model (concluded)

- If $\beta = 0$, then $P$ goes to infinity as $T \to \infty$.
- Sensibly, $P$ goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, $P$ may exceed one for a finite $T$.
- The spot rate volatility structure is the curve
  \[
  (\partial r(t, T)/\partial r) \sigma = \sigma B(t, T)/(T - t).
  \]
- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, $\beta$, controls the shape of the curve.
- Indeed, higher $\beta$ leads to greater attenuation of volatility with maturity.
The Vasicek Model: Options on Zeros\textsuperscript{a}

- Consider a European call with strike price $X$ expiring at time $T$ on a zero-coupon bond with par value $1$ and maturing at time $s > T$.

- Its price is given by

$$P(t, s) N(x) - XP(t, T) N(x - \sigma_v).$$

\textsuperscript{a}Jamshidian (1989).
The Vasicek Model: Options on Zeros (concluded)

- Above

\[
x = \frac{1}{\sigma_v} \ln \left( \frac{P(t, s)}{P(t, T) X} \right) + \frac{\sigma_v}{2},
\]

\[\sigma_v \equiv v(t, T) B(T, s),\]

\[v(t, T)^2 \equiv \begin{cases} 
\frac{\sigma^2 [1 - e^{-2\beta(T-t)}]}{2\beta}, & \text{if } \beta \neq 0 \\
\sigma^2(T - t), & \text{if } \beta = 0
\end{cases}.\]

- By the put-call parity, the price of a European put is

\[XP(t, T) N(-x + \sigma_v) - P(t, s) N(-x).\]
Binomial Vasicek

- Consider a binomial model for the short rate in the time interval \([0, T]\) divided into \(n\) identical pieces.

- Let \(\Delta t \equiv T/n\) and

\[
p(r) \equiv \frac{1}{2} + \frac{\beta (\mu - r) \sqrt{\Delta t}}{2\sigma}.
\]

- The following binomial model converges to the Vasicek model,\(^a\)

\[
r(k + 1) = r(k) + \sigma \sqrt{\Delta t} \xi(k), \quad 0 \leq k < n.
\]

\(^a\)Nelson and Ramaswamy (1990).
Binomial Vasicek (continued)

• Above, $\xi(k) = \pm 1$ with

\[
\text{Prob}[\xi(k) = 1] = \begin{cases} 
  p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\
  0 & \text{if } p(r(k)) < 0 \\
  1 & \text{if } 1 < p(r(k))
\end{cases}
\]

• Observe that the probability of an up move, $p$, is a decreasing function of the interest rate $r$.

• This is consistent with mean reversion.
Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its constant volatility, \( \sigma \).
- For a general process \( Y \) with nonconstant volatility, the resulting binomial tree may not combine.
The Cox-Ingersoll-Ross Model\textsuperscript{a}

• It is the following square-root short rate model:

\[ dr = \beta(\mu - r) \, dt + \sigma\sqrt{r} \, dW. \]  \hspace{1cm} (101)

• The diffusion differs from the Vasicek model by a multiplicative factor \( \sqrt{r} \).

• The parameter \( \beta \) determines the speed of adjustment.

• The short rate can reach zero only if \( 2\beta\mu < \sigma^2 \).

• See text for the bond pricing formula.

\textsuperscript{a}Cox, Ingersoll, and Ross (1985).
Binomial CIR

• We want to approximate the short rate process in the time interval \([0, T]\).
• Divide it into \(n\) periods of duration \(\Delta t \equiv T/n\).
• Assume \(\mu, \beta \geq 0\).
• A direct discretization of the process is problematic because the resulting binomial tree will not combine.
Binomial CIR (continued)

• Instead, consider the transformed process

\[ x(r) \equiv 2\sqrt{r}/\sigma. \]

• It follows

\[ dx = m(x) \, dt + dW, \]

where

\[ m(x) \equiv 2\beta\mu/(\sigma^2x) - (\beta x/2) - 1/(2x). \]

• Since this new process has a constant volatility, its associated binomial tree combines.
Binomial CIR (continued)

- Construct the combining tree for \( r \) as follows.
- First, construct a tree for \( x \).
- Then transform each node of the tree into one for \( r \) via the inverse transformation \( r = f(x) \equiv x^2\sigma^2/4 \) (p. 890).
Binomial CIR (concluded)

- The probability of an up move at each node $r$ is
  \[ p(r) \equiv \frac{\beta(\mu - r) \Delta t + r - r^-}{r^+ - r^-}. \] (102)

- $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.

- $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

- Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.
Numerical Examples

• Consider the process,

\[ 0.2 (0.04 - r) \, dt + 0.1 \sqrt{r} \, dW, \]

for the time interval \([0, 1]\) given the initial rate \(r(0) = 0.04\).

• We shall use \(\Delta t = 0.2\) (year) for the binomial approximation.

• See p. 893(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.
Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.

- Since the root has $x = 2\sqrt{r(0)/\sigma} = 4$, this particular node’s $x$ value equals $4 + \sqrt{\Delta t} = 4.4472135955$.

- Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$. 
Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).
A General Method for Constructing Binomial Models

- We are given a continuous-time process
  \[ dy = \alpha(y, t) \, dt + \sigma(y, t) \, dW. \]

- Make sure the binominal model’s drift and diffusion converge to the above process by setting the probability of an up move to
  \[ \frac{\alpha(y, t) \Delta t + y - y_d}{y_u - y_d}. \]

- Here \( y_u \equiv y + \sigma(y, t) \sqrt{\Delta t} \) and \( y_d \equiv y - \sigma(y, t) \sqrt{\Delta t} \) represent the two rates that follow the current rate \( y \).

- The displacements are identical, at \( \sigma(y, t) \sqrt{\Delta t} \).

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\(^a\)Nelson and Ramaswamy (1990).
A General Method (continued)

- But the binomial tree may not combine:
  \[ \sigma(y, t) \sqrt{\Delta t} - \sigma(y_u, t) \sqrt{\Delta t} \neq -\sigma(y, t) \sqrt{\Delta t} + \sigma(y_d, t) \sqrt{\Delta t} \]
  in general.
- When \( \sigma(y, t) \) is a constant independent of \( y \), equality holds and the tree combines.
- To achieve this, define the transformation
  \[ x(y, t) \equiv \int_y^y \sigma(z, t)^{-1} \, dz. \]
- Then \( x \) follows \( dx = m(y, t) \, dt + dW \) for some \( m(y, t) \) (see text).
A General Method (continued)

- The key is that the diffusion term is now a constant, and the binomial tree for $x$ combines.

- The probability of an up move remains

$$\frac{\alpha(y(x, t), t) \Delta t + y(x, t) - y_d(x, t)}{y_u(x, t) - y_d(x, t)},$$

where $y(x, t)$ is the inverse transformation of $x(y, t)$ from $x$ back to $y$.

- Note that $y_u(x, t) \equiv y(x + \sqrt{\Delta t}, t + \Delta t)$ and $y_d(x, t) \equiv y(x - \sqrt{\Delta t}, t + \Delta t)$. 
A General Method (concluded)

• The transformation is
  \[ \int^r (\sigma \sqrt{z})^{-1} \, dz = 2\sqrt{r}/\sigma \]
  for the CIR model.

• The transformation is
  \[ \int^S (\sigma z)^{-1} \, dz = (1/\sigma) \ln S \]
  for the Black-Scholes model.

• The familiar binomial option pricing model in fact discretizes \( \ln S \) not \( S \).
Model Calibration

- In the time-series approach, the time series of short rates is used to estimate the parameters of the process.
- This approach may help in validating the proposed interest rate process.
- But it alone cannot be used to estimate the risk premium parameter $\lambda$.
- The model prices based on the estimated parameters may also deviate a lot from those in the market.
Model Calibration (concluded)

- The cross-sectional approach uses a cross section of bond prices observed at the same time.
- The parameters are to be such that the model prices closely match those in the market.
- After this procedure, the calibrated model can be used to price interest rate derivatives.
- Unlike the time-series approach, the cross-sectional approach is unable to separate out the interest rate risk premium from the model parameters.
On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate levels only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.
On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.

- Derivatives whose values depend on the correlation structure will be mispriced.

- The calibrated models may not generate term structures as concave as the data suggest.

- The term structure empirically changes in slope and curvature as well as makes parallel moves.

- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.
On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two-factor ones.
Options on Coupon Bonds

- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time $T$ on a bond with par value $1$.
- Let $X$ denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all $i$.
- The payoff for the option is
  \[ \max \left( \sum_{i=1}^{n} c_i P(r(T), T, t_i) - X, 0 \right) . \]

\[ \text{Jamshidian (1989).} \]
Options on Coupon Bonds (continued)

- At time $T$, there is a unique value $r^*$ for $r(T)$ that renders the coupon bond’s price equal the strike price $X$.
- This $r^*$ can be obtained by solving 
  \[ X = \sum_{i=1}^{n} c_i P(r, T, t_i) \] numerically for $r$.
- The solution is also unique for one-factor models whose bond price is a monotonically decreasing function of $r$.
- Let $X_i \equiv P(r^*, T, t_i)$, the value at time $T$ of a zero-coupon bond with par value $1$ and maturing at time $t_i$ if $r(T) = r^*$. 
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Options on Coupon Bonds (concluded)

• Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$.  

• As $X = \sum_i c_i X_i$, the option’s payoff equals

$$\max \left( \sum_{i=1}^n c_i P(r(T), T, t_i) - \sum_i c_i X_i, 0 \right)$$

$$= \sum_{i=1}^n c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

• Thus the call is a package of $n$ options on the underlying zero-coupon bond.

• Why can’t we do the same thing for Asian options?\(^a\)

\(^a\)Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.
No-Arbitrage Term Structure Models
How much of the structure of our theories really tells us about things in nature, and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)
Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.
No-Arbitrage Models\textsuperscript{a}

- No-arbitrage models utilize the full information of the term structure.

- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.

- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.

- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

\textsuperscript{a}Ho and Lee (1986).
No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.
- Bond price and forward rate models are usually non-Markovian (path dependent).
- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).
- Markovian models are easier to handle computationally.
The Ho-Lee Model$^a$

- The short rates at any given time are evenly spaced.
- Let $p$ denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

$^a$Ho and Lee (1986).
\[ r_1 \quad r_2 \quad r_3 + v_3 \]
\[ r_2 + v_2 \quad r_3 + 2v_3 \]
The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t + 1), P(t, t + 2), \ldots$ at time $t$ identified with the root of the tree.

- Let the discount factors in the next period be
  
  \[ P_d(t + 1, t + 2), P_d(t + 1, t + 3), \ldots \]  
  \[ \text{if short rate moves down} \]

  \[ P_u(t + 1, t + 2), P_u(t + 1, t + 3), \ldots \]  
  \[ \text{if short rate moves up} \]

- By backward induction, it is not hard to see that for $n \geq 2$,

  \[ P_u(t + 1, t + n) = P_d(t + 1, t + n) e^{-(v_2 + \cdots + v_n)} \]  

  (103)

(see text).
The Ho-Lee Model (continued)

• It is also not hard to check that the \( n \)-period zero-coupon bond has yields

\[
y_d(n) \equiv -\frac{\ln P_d(t + 1, t + n)}{n - 1}
\]
\[
y_u(n) \equiv -\frac{\ln P_u(t + 1, t + n)}{n - 1} = y_d(n) + \frac{v_2 + \cdots + v_n}{n - 1}
\]

• The volatility of the yield to maturity for this bond is therefore

\[
\kappa_n \equiv \sqrt{py_u(n)^2 + (1 - p) y_d(n)^2 - [py_u(n) + (1 - p) y_d(n)]^2}
\]
\[
= \sqrt{p(1 - p)} \ (y_u(n) - y_d(n))
\]
\[
= \sqrt{p(1 - p)} \, \frac{v_2 + \cdots + v_n}{n - 1}.
\]
The Ho-Lee Model (concluded)

- In particular, the short rate volatility is determined by taking \( n = 2 \):

\[
\sigma = \sqrt{p(1 - p)} \nu_2. \tag{104}
\]

- The variance of the short rate therefore equals 
\( p(1 - p)(r_u - r_d)^2 \), where \( r_u \) and \( r_d \) are the two successor rates.\(^a\)

\(^a\)Contrast this with the lognormal model.
The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of $\kappa_2, \kappa_3, \ldots$.
  - It is independent of the $r_i$.

- It is easy to compute the $v_i$s from the volatility structure, and vice versa.

- The $r_i$s can be computed by forward induction.

- The volatility structure is supplied by the market.
The Ho-Lee Model: Bond Price Process

• In a risk-neutral economy, the initial discount factors satisfy

\[ P(t, t+n) = (pP_u(t+1, t+n) + (1-p) P_d(t+1, t+n)) \ P(t, t+1) \]

• Combine the above with Eq. (103) on p. 915 and assume \( p = 1/2 \) to obtain\(^a\)

\[
P_d(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \ \frac{2 \times \exp[v_2 + \cdots + v_n]}{1 + \exp[v_2 + \cdots + v_n]}, \quad (105)
\]

\[
P_u(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \ \frac{2}{1 + \exp[v_2 + \cdots + v_n]}. \quad (105')
\]

\(^a\)In the limit, only the volatility matters.
The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.

- Suppose all $v_i$ equal some constant $v$ and $\delta \equiv e^v > 0$.

- Then

  $$P_d(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2\delta^{n-1}}{1 + \delta^{n-1}};$$

  $$P_u(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2}{1 + \delta^{n-1}}.$$

- Short rate volatility $\sigma$ equals $v/2$ by Eq. (104) on p. 917.

- Price derivatives by taking expectations under the risk-neutral probability.
The Ho-Lee Model: Yields and Their Covariances

• The one-period rate of return of an \( n \)-period zero-coupon bond is

\[
    r(t, t + n) \equiv \ln \left( \frac{P(t + 1, t + n)}{P(t, t + n)} \right).
\]

• Its value is either \( \ln \frac{P_d(t+1,t+n)}{P(t,t+n)} \) or \( \ln \frac{P_u(t+1,t+n)}{P(t,t+n)} \).

• Thus the variance of return is

\[
    \text{Var}[r(t, t + n)] = p(1 - p)((n - 1) \nu)^2 = (n - 1)^2 \sigma^2.
\]
The Ho-Lee Model: Yields and Their Covariances (concluded)

• The covariance between \( r(t, t + n) \) and \( r(t, t + m) \) is \( (n - 1)(m - 1) \sigma^2 \) (see text).

• As a result, the correlation between any two one-period rates of return is unity.

• Strong correlation between rates is inherent in all one-factor Markovian models.
The Ho-Lee Model: Short Rate Process

- The continuous-time limit of the Ho-Lee model is

\[ dr = \theta(t) \, dt + \sigma \, dW. \]

- This is Vasicek’s model with the mean-reverting drift replaced by a deterministic, time-dependent drift.

- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying, i.e.,

\[ dr = \theta(t) \, dt + \sigma(t) \, dW. \]

- This corresponds to the discrete-time model in which \( v_i \) are not all identical.
The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
Problems with No-Arbitrage Models in General

- Interest rate movements should reflect shifts in the model’s state variables (factors) not its parameters.
- Model parameters, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.
- But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.
Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.
- Consequently, a model’s intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.
The Black-Derman-Toy Model\textsuperscript{a}

- This model is extensively used by practitioners.
- The BDT short rate process is the lognormal binomial interest rate process described on pp. 767ff (repeated on next page).
- The volatility structure is given by the market.
- From it, the short rate volatilities (thus $v_i$) are determined together with $r_i$.

\textsuperscript{a}Black, Derman, and Toy (BDT) (1990).
The Black-Derman-Toy Model (concluded)

- Our earlier binomial interest rate tree, in contrast, assumes $v_i$ are given a priori.
  - A related model of Salomon Brothers takes $v_i$ to be constants.

- Lognormal models preclude negative short rates.
The BDT Model: Volatility Structure

- The volatility structure defines the yield volatilities of zero-coupon bonds of various maturities.

- Let the yield volatility of the $i$-period zero-coupon bond be denoted by $\kappa_i$.

- $P_u$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes an up move.
The BDT Model: Volatility Structure (concluded)

• $P_d$ is the price of the $i$-period zero-coupon bond one period from now if the short rate makes a down move.

• Corresponding to these two prices are the following yields to maturity,

$$y_u \equiv P_u^{-(i-1)} - 1,$$
$$y_d \equiv P_d^{-(i-1)} - 1.$$

• The yield volatility is defined as $\kappa_i \equiv (1/2) \ln(y_u/y_d)$. 
The BDT Model: Calibration

- The inputs to the BDT model are riskless zero-coupon bond yields and their volatilities.
- For economy of expression, all numbers are period based.
- Suppose inductively that we have calculated

\[(r_1, v_1), (r_2, v_2), \ldots, (r_{i-1}, v_{i-1})\].

- They define the binomial tree up to period \(i - 1\).
- We now proceed to calculate \(r_i\) and \(v_i\) to extend the tree to period \(i\).
The BDT Model: Calibration (continued)

- Assume the price of the $i$-period zero can move to $P_u$ or $P_d$ one period from now.
- Let $y$ denote the current $i$-period spot rate, which is known.
- In a risk-neutral economy,

$$\frac{P_u + P_d}{2(1 + r_1)} = \frac{1}{(1 + y)^i}. \quad (106)$$

- Obviously, $P_u$ and $P_d$ are functions of the unknown $r_i$ and $v_i$. 
The BDT Model: Calibration (continued)

• Viewed from now, the future $(i-1)$-period spot rate at time one is uncertain.

• Recall that $y_u$ and $y_d$ represent the spot rates at the up node and the down node, respectively (p. 931).

• With $\kappa^2$ denoting their variance, we have

$$\kappa_i = \frac{1}{2} \ln \left( \frac{P_u^{-1/(i-1)} - 1}{P_d^{-1/(i-1)} - 1} \right).$$  (107)
The BDT Model: Calibration (continued)

- We will employ forward induction to derive a quadratic-time calibration algorithm.\(^a\)

- Recall that forward induction inductively figures out, by moving forward in time, how much $1 at a node contributes to the price (review p. 792(a)).

- This number is called the state price and is the price of the claim that pays $1 at that node and zero elsewhere.

\(^a\)Chen and Lyuu (1997); Lyuu (1999).
The BDT Model: Calibration (continued)

• Let the unknown baseline rate for period $i$ be $r_i = r$.
• Let the unknown multiplicative ratio be $v_i = v$.
• Let the state prices at time $i - 1$ be $P_1, P_2, \ldots, P_i$, corresponding to rates $r, rv, \ldots, rv^{i-1}$, respectively.
• One dollar at time $i$ has a present value of

$$f(r, v) \equiv \frac{P_1}{1 + r} + \frac{P_2}{1 + rv} + \frac{P_3}{1 + rv^2} + \cdots + \frac{P_i}{1 + rv^{i-1}}.$$
The BDT Model: Calibration (continued)

- The yield volatility is

\[ g(r, v) \equiv \frac{1}{2} \ln \left( \frac{P_{u,1}}{1+rv} \cdot \frac{P_{u,2}}{(1+rv)^2} \cdot \ldots \cdot \frac{P_{u,i-1}}{(1+rv)^{i-1}} \right)^{-1/(i-1)} - 1 \] 

\[ \left( \frac{P_{d,1}}{1+r} + \frac{P_{d,2}}{1+rv} + \ldots + \frac{P_{d,i-1}}{1+rv^{i-2}} \right)^{-1/(i-1)} - 1 \]

- Above, \( P_{u,1}, P_{u,2}, \ldots \) denote the state prices at time \( i - 1 \) of the subtree rooted at the up node (like \( r_2v_2 \) on p. 928).

- And \( P_{d,1}, P_{d,2}, \ldots \) denote the state prices at time \( i - 1 \) of the subtree rooted at the down node (like \( r_2 \) on p. 928).
The BDT Model: Calibration (concluded)

• Now solve

\[ f(r, v) = \frac{1}{(1 + y)^i} \quad \text{and} \quad g(r, v) = \kappa_i \]

for \( r = r_i \) and \( v = v_i \).

• This \( O(n^2) \)-time algorithm appears in the text.