Tracking Error Revisited

- Define the dollar gamma as $S^2\Gamma$.
- The change in value of a delta-hedged *long* option position after a duration of Δt is proportional to the dollar gamma.
- It is about

$$(1/2)S^2\Gamma[\,(\Delta S/S)^2 - \sigma^2\Delta t\,].$$

  - $(\Delta S/S)^2$ is called the daily realized variance.

Tracking Error Revisited (continued)

- Let the rebalancing times be $t_1, t_2, \ldots, t_n$.
- Let $\Delta S_i = S_{i+1} - S_i$.
- The total tracking error at expiration is about

$$\sum_{i=0}^{n-1} e^{r(T-t_i)} \frac{S_i^2 \Gamma_i}{2} \left[\left(\frac{\Delta S_i}{S_i} \right)^2 - \sigma^2 \Delta t \right],$$

- The tracking error is path dependent.

Tracking Error Revisited (concluded)[a]

- The tracking error ϵ_n *over n rebalancing acts* (such as 251,235 on p. 534) has about the same probability of being positive as being negative.
- Subject to certain regularity conditions, the root-mean-square tracking error $\sqrt{E[\,\epsilon_n^2\,]}$ is $O(1/\sqrt{n}\,)$.[b]
- The root-mean-square tracking error increases with σ at first and then decreases.

---

[a] Bertsimas, Kogan, and Lo (2000).
[b] See also Grannan and Swindle (1996).

Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, Δf, due to changes in the stock price, ΔS.
- When ΔS is not small, the second-order term, gamma $\Gamma \equiv \partial^2 f / \partial S^2$, helps (theoretically).
- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.
- To meet this extra condition, one more security needs to be brought in.

Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call f_2 is brought in.
- To set up a delta-gamma hedge, we solve

$$\begin{aligned} -N \times f + n_1 \times S + n_2 \times f_2 - B &= 0 \quad \text{(self-financing)}, \\ -N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 &= 0 \quad \text{(delta neutrality)}, \\ -N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 &= 0 \quad \text{(gamma neutrality)}, \end{aligned}$$

for n_1, n_2, and B.

  - The gammas of the stock and bond are 0.

Other Hedges

- If volatility changes, delta-gamma hedge may not work well.
- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.
- To accomplish this, one more security has to be brought into the process.
- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.

Trees

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

And though the holes were rather small,
they had to count them all.
— The Beatles, *A Day in the Life* (1967)

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 234.
  - In general, it cannot apply to American options.
- We will now apply it to price barrier options.

The Reflection Principle[a]

- Imagine a particle at position $(0, -\boldsymbol{a})$ on the integral lattice that is to reach $(n, -\boldsymbol{b})$.
- Without loss of generality, assume $\boldsymbol{a} > 0$ and $\boldsymbol{b} \geq 0$.
- This particle's movement:

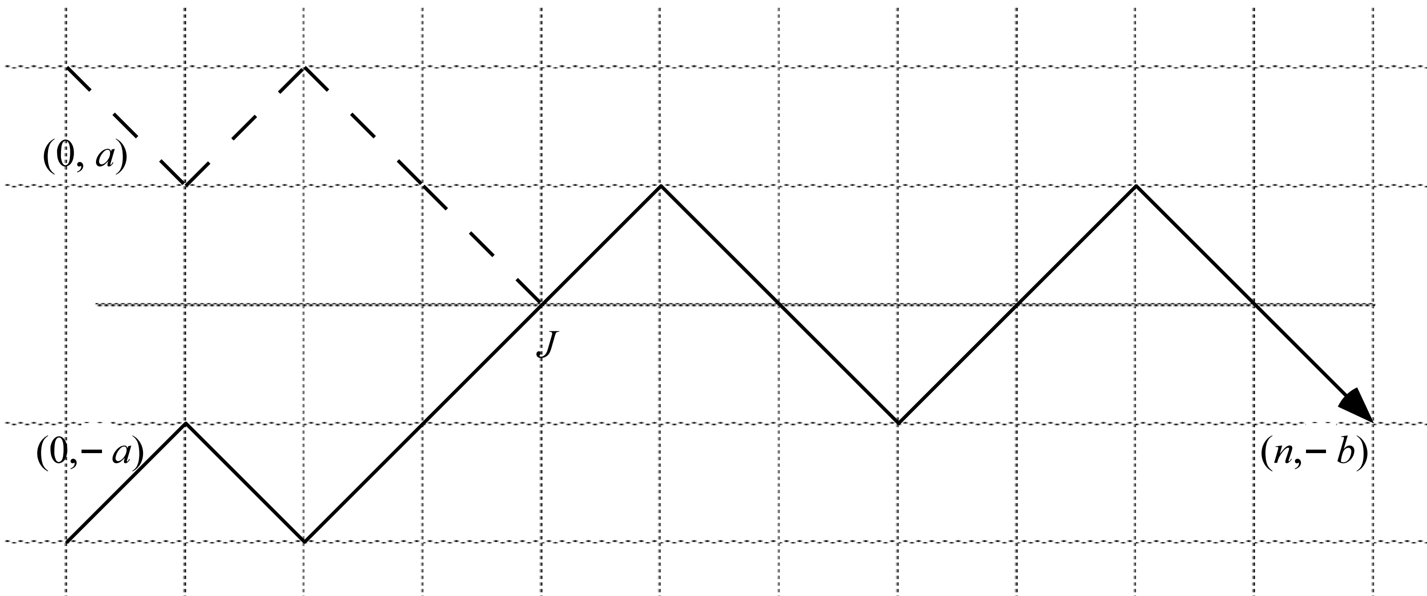
$$(i, j) \nearrow (i+1, j+1) \quad \text{up move } S \to Su$$

$$(i, j) \searrow (i+1, j-1) \quad \text{down move } S \to Sd$$

- How many paths touch the x axis?

---

[a] André (1887).


(0, a)
J
(0, − a)
(n, − b)

The Reflection Principle (continued)

- For a path from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touches the x axis, let J denote the first point this happens.
- Reflect the portion of the path from $(0, -\boldsymbol{a})$ to J.
- A path from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$ is constructed.
- It also hits the x axis at J for the first time.
- The one-to-one mapping shows the number of paths from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touch the x axis equals the number of paths from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$.

The Reflection Principle (concluded)

- A path of this kind has $(n + \boldsymbol{b} + \boldsymbol{a})/2$ down moves and $(n - \boldsymbol{b} - \boldsymbol{a})/2$ up moves.
- Hence there are

$$\binom{n}{\frac{n+\boldsymbol{a}+\boldsymbol{b}}{2}} \tag{57}$$

such paths for even $n + \boldsymbol{a} + \boldsymbol{b}$.

  – Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define

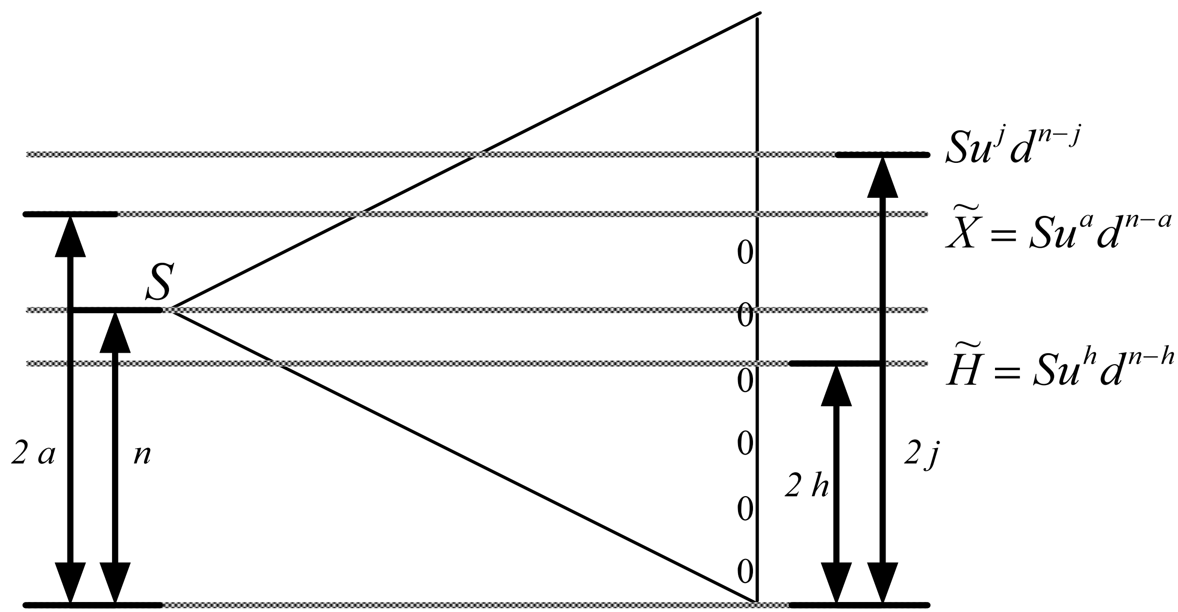
$$a \equiv \left\lceil \frac{\ln(X/(Sd^n))}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil,$$

$$h \equiv \left\lfloor \frac{\ln(H/(Sd^n))}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor.$$

  – h is such that $\tilde{H} \equiv Su^h d^{n-h}$ is the *terminal* price that is closest to, but does not exceed H.
  – a is such that $\tilde{X} \equiv Su^a d^{n-a}$ is the terminal price that is closest to, but is not exceeded by X.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with n moves hence ends up in the money if and only if the number of up moves is at least a.
- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price Sd^n on the binomial tree.
  -
$$Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-2k}. \tag{58}$$


$Su^j d^{n-j}$
$\widetilde{X} = Su^a d^{n-a}$
S
$\widetilde{H} = Su^h d^{n-h}$
0
0
0
0
0
0
2 a
n
2 h
2 j

Pricing Barrier Options (continued)

- The number of paths from S to the terminal price $Su^j d^{n-j}$ is $\binom{n}{j}$, each with probability $p^j (1-p)^{n-j}$.
- With reference to p. 550, the reflection principle can be applied with $\boldsymbol{a} = n - 2h$ and $\boldsymbol{b} = 2j - 2h$ in Eq. (57) on p. 547 by treating the S line as the x axis.
- Therefore,

$$\binom{n}{\frac{n+(n-2h)+(2j-2h)}{2}} = \binom{n}{n-2h+j}$$

paths hit $\tilde{H}$ in the process for $h \leq n/2$.

Pricing Barrier Options (concluded)

- The terminal price $Su^j d^{n-j}$ is reached by a path that hits the effective barrier with probability

$$\binom{n}{n-2h+j} p^j (1-p)^{n-j}.$$

- The option value equals

$$\frac{\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - X\right)}{R^n}. \tag{59}$$

  - $R \equiv e^{r\tau/n}$ is the riskless return per period.

- It implies a linear-time algorithm.

Convergence of BOPM

- Equation (59) results in the sawtooth-like convergence shown on p. 321.
- The reasons are not hard to see.
- The true barrier most likely does not equal the effective barrier.
- The same holds for the strike price and the effective strike price.
- The issue of the strike price is less critical.
- But the issue of the barrier is not negligible.

Convergence of BOPM (continued)

- Convergence is actually good if we limit n to certain values—191, for example.
- These values make the true barrier coincide with or occur just above one of the stock price levels, that is, $H \approx S d^j = S e^{-j\sigma\sqrt{\tau/n}}$ for some integer j.
- The preferred n's are thus

$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

- There is only one minor technicality left.

Convergence of BOPM (continued)

- We picked the effective barrier to be one of the $n+1$ possible terminal stock prices.
- However, the effective barrier above, Sd^j, corresponds to a terminal stock price only when $n-j$ is even.[a]
- To close this gap, we decrement n by one, if necessary, to make $n-j$ an even number.

---

[a]This is because $j = n - 2k$ for some k by Eq. (58) on p. 549. Of course we could have adopted the form Sd^j $(-n \leq j \leq n)$ for the effective barrier.

Convergence of BOPM (concluded)

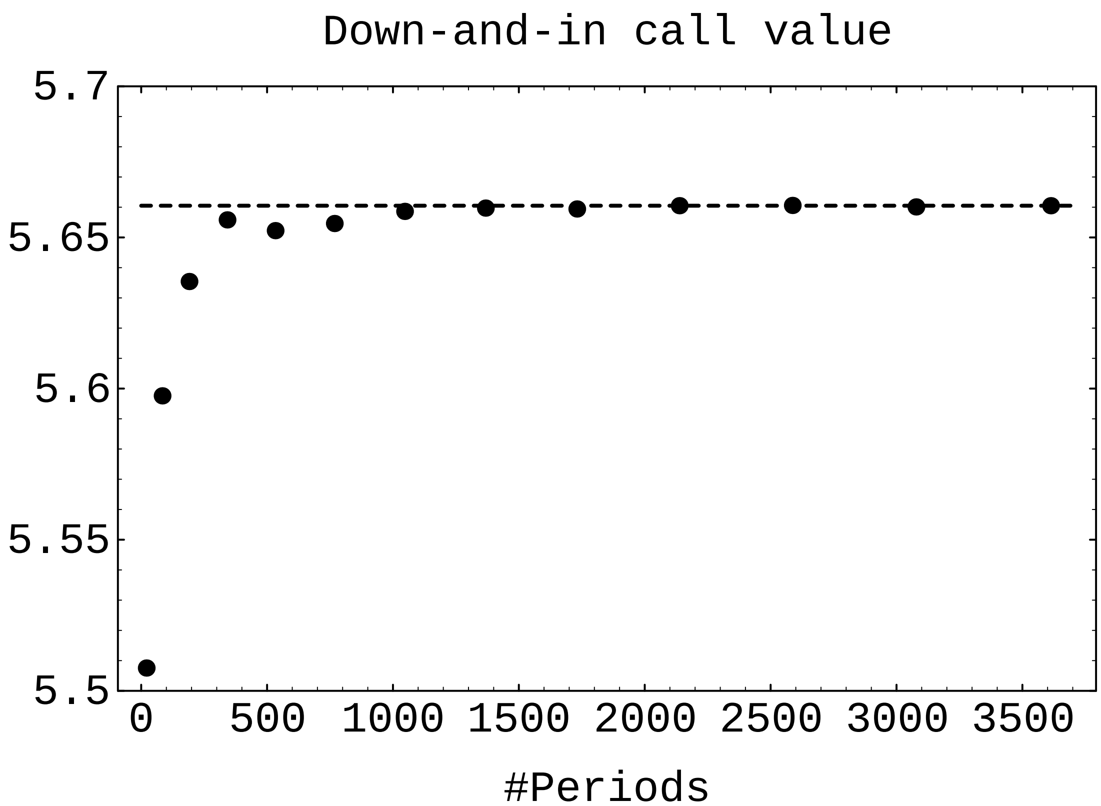
- The preferred n's are now

$$n = \begin{cases} \ell & \text{if } \ell - j \text{ is even} \\ \ell - 1 & \text{otherwise} \end{cases},$$

$j = 1, 2, 3, \ldots,$ where

$$\ell \equiv \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.$$

- Evaluate pricing formula (59) on p. 552 only with the n's above.


Down-and-in call value
5.7
5.65
5.6
5.55
5.5
0 500 1000 1500 2000 2500 3000 3500
#Periods

Practical Implications

- Now that barrier options can be efficiently priced, we can afford to pick very large n's (p. 559).
- This has profound consequences.

n	Combinatorial method	
	Value	Time (milliseconds)
21	5.507548	0.30
84	5.597597	0.90
191	5.635415	2.00
342	5.655812	3.60
533	5.652253	5.60
768	5.654609	8.00
1047	5.658622	11.10
1368	5.659711	15.00
1731	5.659416	19.40
2138	5.660511	24.70
2587	5.660592	30.20
3078	5.660099	36.70
3613	5.660498	43.70
4190	5.660388	44.10
4809	5.659955	51.60
5472	5.660122	68.70
6177	5.659981	76.70
6926	5.660263	86.90
7717	5.660272	97.20

Practical Implications (concluded)

- Pricing is prohibitively time consuming when $S \approx H$ because $n \sim 1/\ln^2(S/H)$.
- This observation is indeed true of standard quadratic-time binomial tree algorithms.
- But it no longer applies to linear-time algorithms (p. 561).
- In fact, this model is $O(1/n)$ convergent.[a]

[a] Lin (2008).

	Barrier at 95.0			Barrier at 99.5			Barrier at 99.9	
n	Value	Time	n	Value	Time	n	Value	Time
	⋮		795	7.47761	8	19979	8.11304	253
2743	2.56095	31.1	3184	7.47626	38	79920	8.11297	1013
3040	2.56065	35.5	7163	7.47682	88	179819	8.11300	2200
3351	2.56098	40.1	12736	7.47661	166	319680	8.11299	4100
3678	2.56055	43.8	19899	7.47676	253	499499	8.11299	6300
4021	2.56152	48.1	28656	7.47667	368	719280	8.11299	8500
True	2.5615			7.4767			8.1130	

(All times in milliseconds.)

Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion $dS/S = r\,dt + \sigma\,dW$.[a]
- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Impose the matching of mean and that of variance:

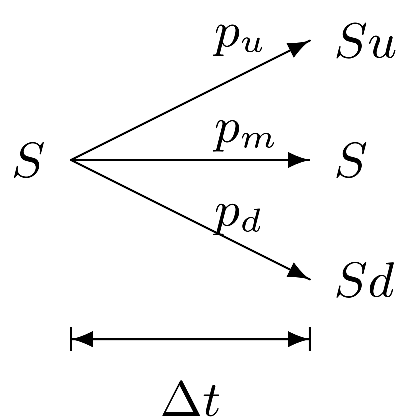
$$\begin{aligned} 1 &= p_u + p_m + p_d, \\ SM &\equiv (p_u u + p_m + (p_d/u))\, S, \\ S^2 V &\equiv p_u (Su - SM)^2 + p_m (S - SM)^2 + p_d (Sd - SM)^2. \end{aligned}$$

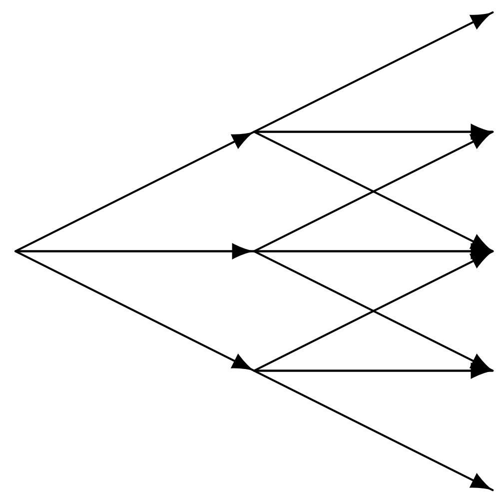
[a]Boyle (1988).

- Above,

$$M \equiv e^{r\Delta t},$$
$$V \equiv M^2(e^{\sigma^2\Delta t} - 1),$$

by Eqs. (18) on p. 149.


p_u
Su
S
p_m
S
p_d
Sd
Δt



Trinomial Tree (continued)

- Use linear algebra to verify that

$$p_u = \frac{u\left(V + M^2 - M\right) - (M-1)}{(u-1)\left(u^2-1\right)},$$

$$p_d = \frac{u^2\left(V + M^2 - M\right) - u^3(M-1)}{(u-1)\left(u^2-1\right)}.$$

  – In practice, must make sure the probabilities lie between 0 and 1.

- Countless variations.

Trinomial Tree (concluded)

- Use $u = e^{\lambda\sigma\sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$
\begin{aligned}
p_u &\to \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}, \\
p_d &\to \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}.
\end{aligned}
$$

- A nice choice for λ is $\sqrt{\pi/2}$.[a]

[a] Omberg (1988).

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting λ so that the barrier is hit exactly.[a]
- It takes

$$h = \frac{\ln(S/H)}{\lambda \sigma \sqrt{\Delta t}}$$

consecutive down moves to go from S to H if h is an integer, which is easy to achieve by adjusting λ.

  - This is because $S e^{-h \lambda \sigma \sqrt{\Delta t}} = H$.

[a] Ritchken (1995).

Barrier Options Revisited (continued)

- Typically, we find the smallest $\lambda \geq 1$ such that h is an integer.
- That is, we find the largest integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ and then let

$$\lambda = \frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}}.$$

  - Such a λ may not exist for very small n's.
  - This is not hard to check.
- This done, one of the layers of the trinomial tree coincides with the barrier.

Barrier Options Revisited (concluded)

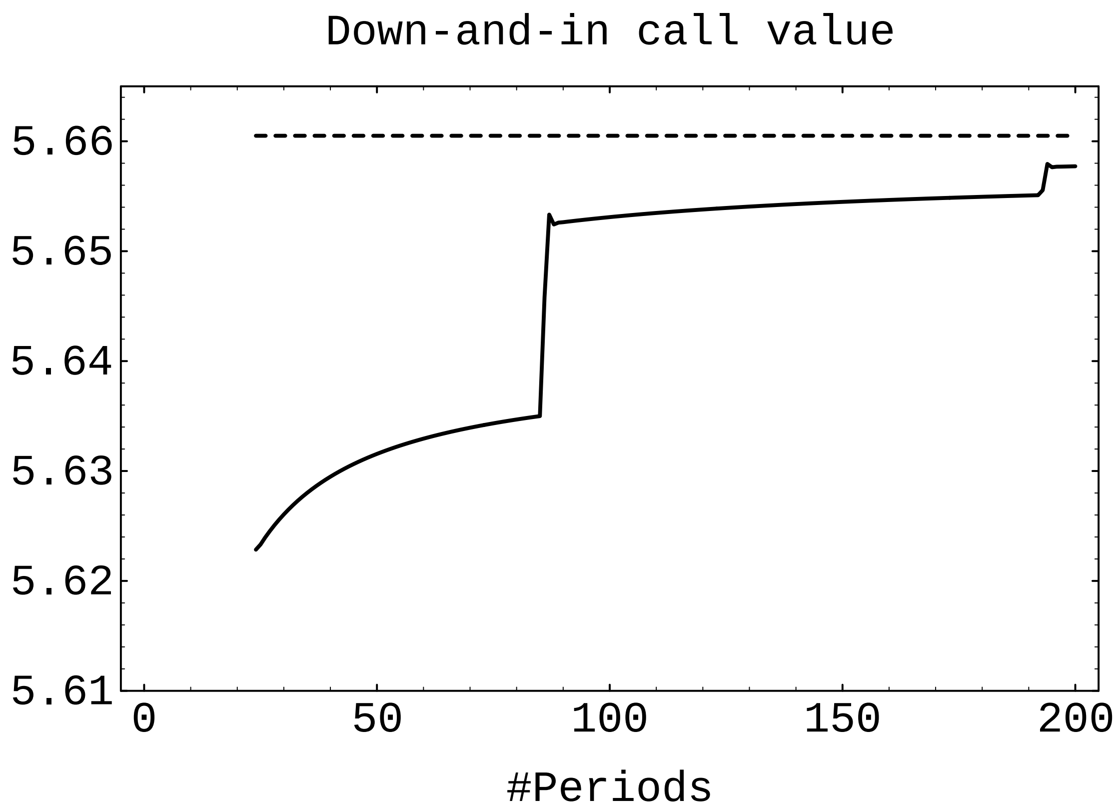
- The following probabilities may be used,

$$p_u = \frac{1}{2\lambda^2} + \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma},$$

$$p_m = 1 - \frac{1}{\lambda^2},$$

$$p_d = \frac{1}{2\lambda^2} - \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}.$$

  - $\mu' \equiv r - \sigma^2/2$.


Down-and-in call value
5.66
5.65
5.64
5.63
5.62
5.61
0
50
100
150
200
#Periods

Algorithms Comparison[a]

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the n value at which they converge.
  - The one with the smallest n wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times.

[a]Lyuu (1998).

Algorithms Comparison (concluded)

- Pages 321 and 570 show the trinomial model converges at a smaller n than BOPM.
- It is in this sense when people say trinomial models converge faster than binomial ones.
- But is the trinomial model better then?
- The linear-time binomial tree algorithm actually performs better than the trinomial one (see next page expanded from p. 559).

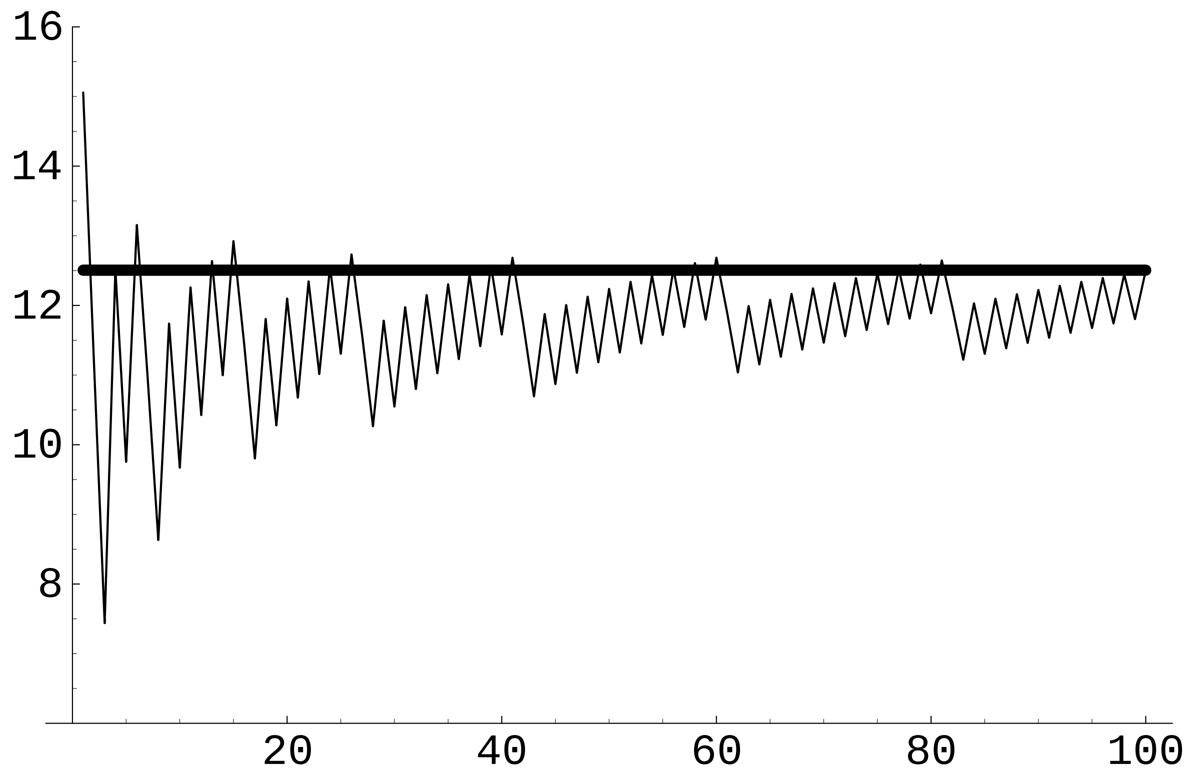
n	Combinatorial method		Trinomial tree algorithm	
	Value	Time	Value	Time
21	5.507548	0.30		
84	5.597597	0.90	5.634936	35.0
191	5.635415	2.00	5.655082	185.0
342	5.655812	3.60	5.658590	590.0
533	5.652253	5.60	5.659692	1440.0
768	5.654609	8.00	5.660137	3080.0
1047	5.658622	11.10	5.660338	5700.0
1368	5.659711	15.00	5.660432	9500.0
1731	5.659416	19.40	5.660474	15400.0
2138	5.660511	24.70	5.660491	23400.0
2587	5.660592	30.20	5.660493	34800.0
3078	5.660099	36.70	5.660488	48800.0
3613	5.660498	43.70	5.660478	67500.0
4190	5.660388	44.10	5.660466	92000.0
4809	5.659955	51.60	5.660454	130000.0
5472	5.660122	68.70		
6177	5.659981	76.70		

(All times in milliseconds.)

Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
- Assume $L < S < H$.
- The binomial model produces oscillating option values (see plot next page).[a]

[a] Chao (1999); Dai and Lyuu (2005);


16
14
12
10
8
20
40
60
80
100

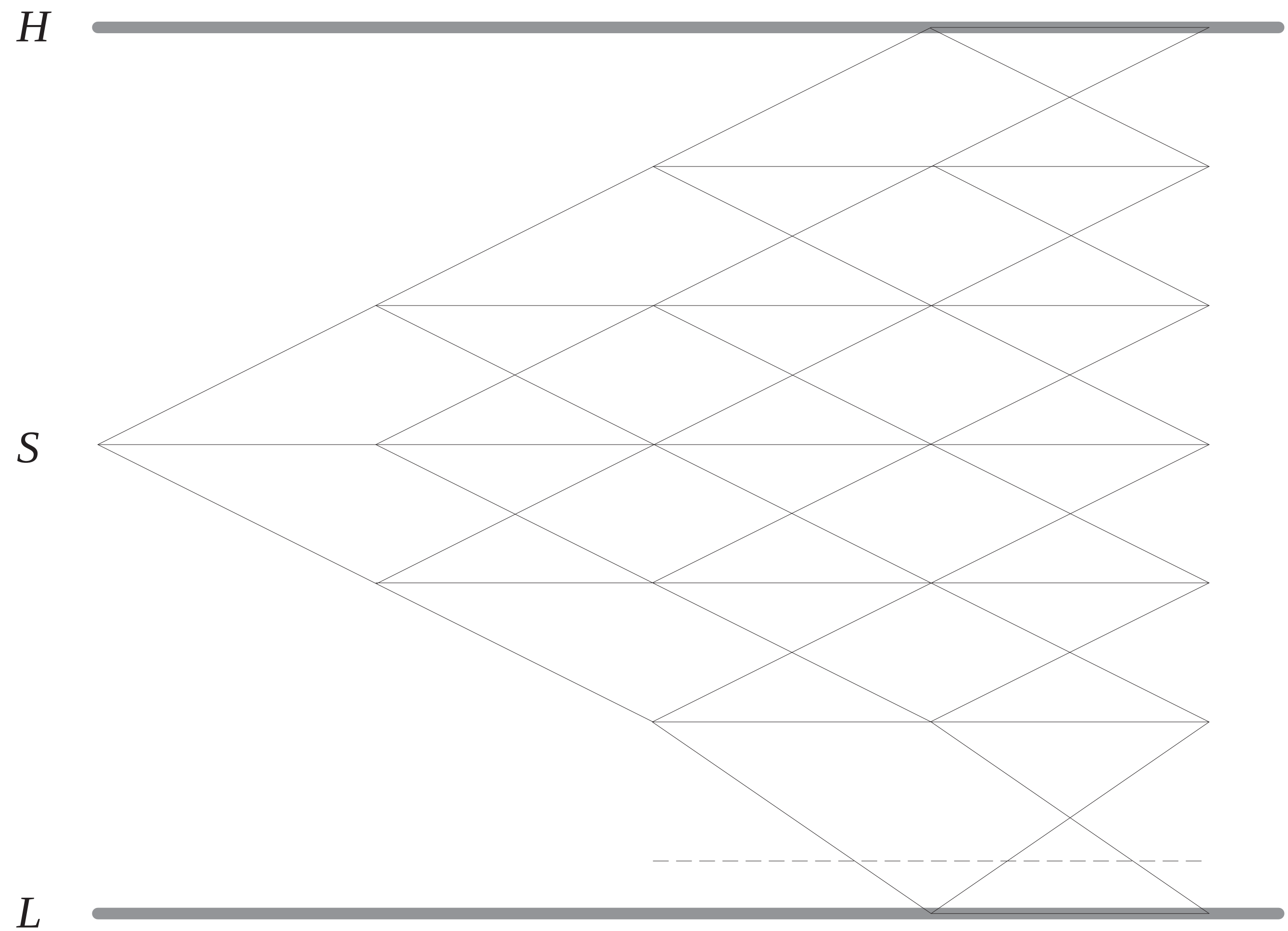
Double-Barrier Knock-Out Options

- We knew how to pick the λ so that one of the layers of the trinomial tree coincides with one barrier, say H.
- This choice, however, does not guarantee that the other barrier, L, is also hit.
- One way to handle this problem is to lower the layer of the tree just above L to coincide with L.[a]
  - More general ways to make the trinomial model hit both barriers are available.[b]

---

[a] Ritchken (1995).

[b] Hsu and Lyuu (2006). Dai and Lyuu (2006) combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options!


H
S
L

Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above L must be adjusted.
- Let ℓ be the positive integer such that

$$Sd^{\ell+1} < L < Sd^{\ell}.$$

- Hence the layer of the tree just above L has price Sd^{ℓ}.

Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying

$$L = Sd^{\ell-1} e^{-\gamma\lambda\sigma\sqrt{\Delta t}}.$$

  - The prices between the barriers are

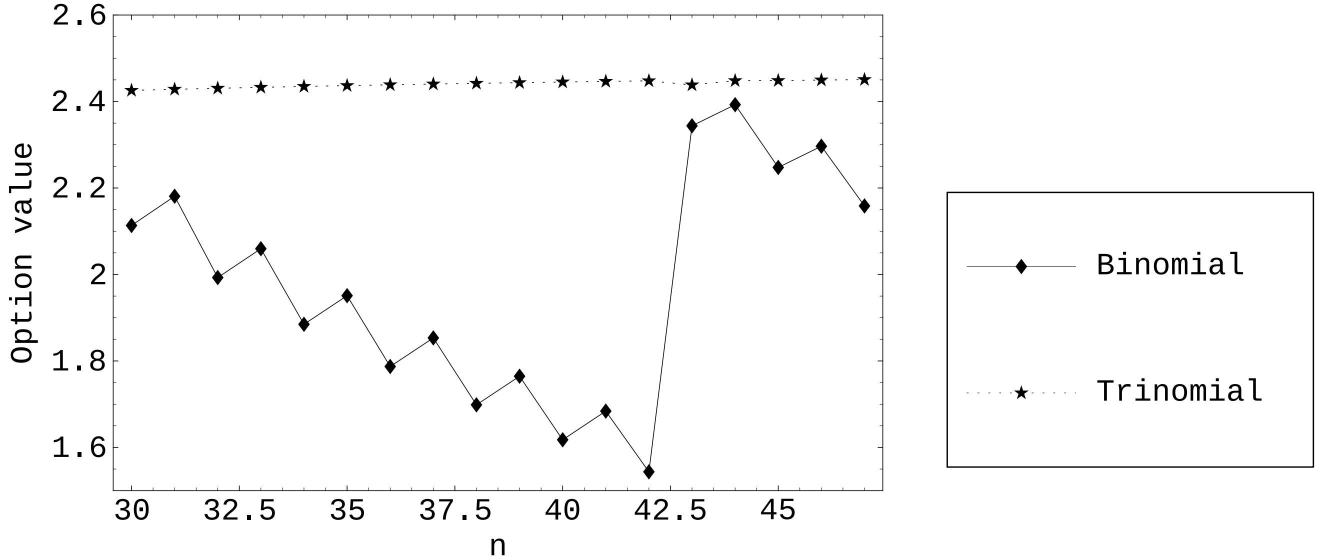
$$L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H.$$

- The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are

$$p'_u = \frac{b + a\gamma}{1+\gamma}, \quad p'_d = \frac{b-a}{\gamma+\gamma^2}, \quad \text{and} \quad p'_m = 1 - p'_u - p'_d,$$

where $a \equiv \mu'\sqrt{\Delta t}/(\lambda\sigma)$ and $b \equiv 1/\lambda^2$.

Convergence: Binomial vs. Trinomial



Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on m assets has the terminal payoff $\max(\sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0)$, where α_i is the percentage of asset i.
- Basket options are essentially options on a portfolio of stocks or index options.
- Option on the best of two risky assets and cash has a terminal payoff of $\max(S_1(\tau), S_2(\tau), X)$.

Correlated Trinomial Model[a]

- Two risky assets S_1 and S_2 follow $dS_i/S_i = r\,dt + \sigma_i\,dW_i$ in a risk-neutral economy, $i = 1, 2$.
- Let

$$
\begin{aligned}
M_i &\equiv e^{r\Delta t}, \\
V_i &\equiv M_i^2 (e^{\sigma_i^2 \Delta t} - 1).
\end{aligned}
$$

  - $S_i M_i$ is the mean of S_i at time Δt.
  - $S_i^2 V_i$ the variance of S_i at time Δt.

[a] Boyle, Evnine, and Gibbs (1989).

Correlated Trinomial Model (continued)

- The value of $S_1 S_2$ at time Δt has a joint lognormal distribution with mean $S_1 S_2 M_1 M_2 e^{\rho \sigma_1 \sigma_2 \Delta t}$, where ρ is the correlation between dW_1 and dW_2.
- Next match the 1st and 2nd moments of the approximating discrete distribution to those of the continuous counterpart.
- At time Δt from now, there are five distinct outcomes.

Correlated Trinomial Model (continued)

- The five-point probability distribution of the asset prices is (as usual, we impose $u_i d_i = 1$)

Probability	Asset 1	Asset 2
p_1	$S_1 u_1$	$S_2 u_2$
p_2	$S_1 u_1$	$S_2 d_2$
p_3	$S_1 d_1$	$S_2 d_2$
p_4	$S_1 d_1$	$S_2 u_2$
p_5	S_1	S_2

Correlated Trinomial Model (continued)

- The probabilities must sum to one, and the means must be matched:

$$\begin{aligned} 1 &= p_1 + p_2 + p_3 + p_4 + p_5, \\ S_1 M_1 &= (p_1 + p_2)\, S_1 u_1 + p_5 S_1 + (p_3 + p_4)\, S_1 d_1, \\ S_2 M_2 &= (p_1 + p_4)\, S_2 u_2 + p_5 S_2 + (p_2 + p_3)\, S_2 d_2. \end{aligned}$$

Correlated Trinomial Model (concluded)

- Let $R \equiv M_1 M_2 e^{\rho\sigma_1\sigma_2\Delta t}$.
- Match the variances and covariance:

$$
\begin{aligned}
S_1^2 V_1 &= (p_1 + p_2)((S_1 u_1)^2 - (S_1 M_1)^2) + p_5(S_1^2 - (S_1 M_1)^2) \\
&\quad + (p_3 + p_4)((S_1 d_1)^2 - (S_1 M_1)^2), \\
S_2^2 V_2 &= (p_1 + p_4)((S_2 u_2)^2 - (S_2 M_2)^2) + p_5(S_2^2 - (S_2 M_2)^2) \\
&\quad + (p_2 + p_3)((S_2 d_2)^2 - (S_2 M_2)^2), \\
S_1 S_2 R &= (p_1 u_1 u_2 + p_2 u_1 d_2 + p_3 d_1 d_2 + p_4 d_1 u_2 + p_5)\, S_1 S_2.
\end{aligned}
$$

- The solutions are complex (see text).

Correlated Trinomial Model Simplified[a]

- Let $\mu'_i \equiv r - \sigma_i^2/2$ and $u_i \equiv e^{\lambda \sigma_i \sqrt{\Delta t}}$ for $i = 1, 2$.
- The following simpler scheme is good enough:

$$
\begin{aligned}
p_1 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu'_1}{\sigma_1} + \frac{\mu'_2}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right], \\
p_2 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu'_1}{\sigma_1} - \frac{\mu'_2}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right], \\
p_3 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu'_1}{\sigma_1} - \frac{\mu'_2}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right], \\
p_4 &= \frac{1}{4}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu'_1}{\sigma_1} + \frac{\mu'_2}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right], \\
p_5 &= 1 - \frac{1}{\lambda^2}.
\end{aligned}
$$

- It cannot price 2-asset 2-barrier options accurately.[b]

---

[a] Madan, Milne, and Shefrin (1989).

[b] See Chang, Hsu, and Lyuu (2006) for a solution.

Extrapolation

- It is a method to speed up numerical convergence.
- Say $f(n)$ converges to an unknown limit f at rate of $1/n$:

$$f(n) = f + \frac{c}{n} + o\left(\frac{1}{n}\right). \tag{60}$$

- Assume c is an unknown constant independent of n.
  - Convergence is basically monotonic and smooth.

Extrapolation (concluded)

- From two approximations $f(n_1)$ and $f(n_2)$ and by ignoring the smaller terms,

$$\begin{aligned} f(n_1) &= f + \frac{c}{n_1}, \\ f(n_2) &= f + \frac{c}{n_2}. \end{aligned}$$

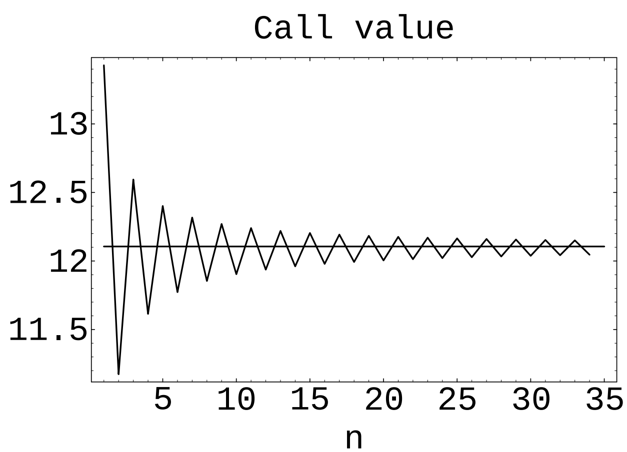
- A better approximation to the desired f is

$$f = \frac{n_1 f(n_1) - n_2 f(n_2)}{n_1 - n_2}. \tag{61}$$

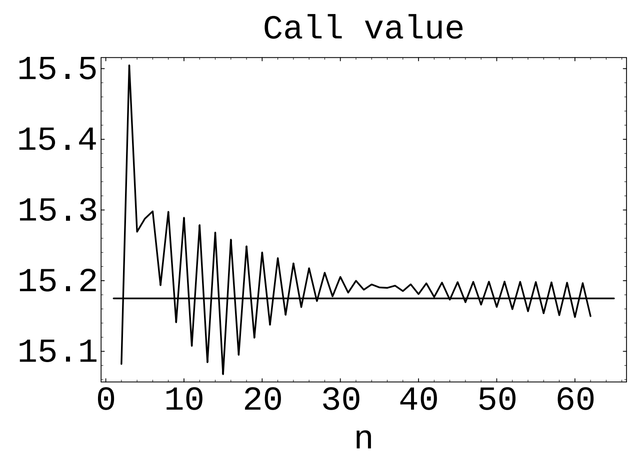
- This estimate should converge faster than $1/n$.
- The Richardson extrapolation uses $n_2 = 2n_1$.

Improving BOPM with Extrapolation

- Consider standard European options.
- Denote the option value under BOPM using n time periods by $f(n)$.
- It is known that BOPM convergences at the rate of $1/n$, consistent with Eq. (60) on p. 588.
- But the plots on p. 249 (redrawn on next page) demonstrate that convergence to the true option value oscillates with n.
- Extrapolation is inapplicable at this stage.



Call value
13
12.5
12
11.5
5
10
15
20
25
30
35
n



Call value
15.5
15.4
15.3
15.2
15.1
0
10
20
30
40
50
60
n

Improving BOPM with Extrapolation (concluded)

- Take the at-the-money option in the left plot on p. 591.
- The sequence with odd n turns out to be monotonic and smooth (see the left plot on p. 593).[a]
- Apply extrapolation (61) on p. 589 with $n_2 = n_1 + 2$, where n_1 is odd.
- Result is shown in the right plot on p. 593.
- The convergence rate is amazing.
- See Exercise 9.3.8 of the text (p. 111) for ideas in the general case.

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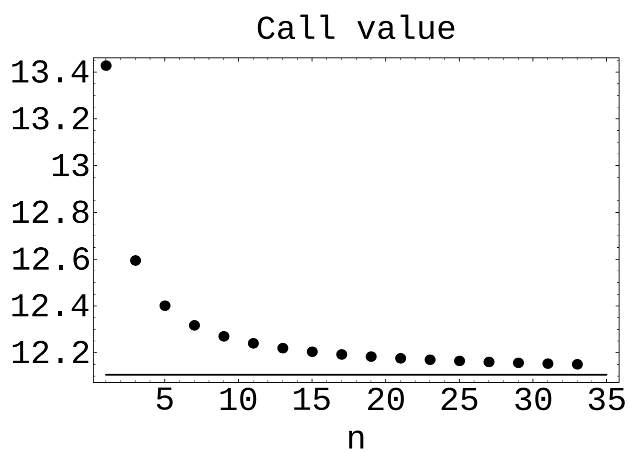
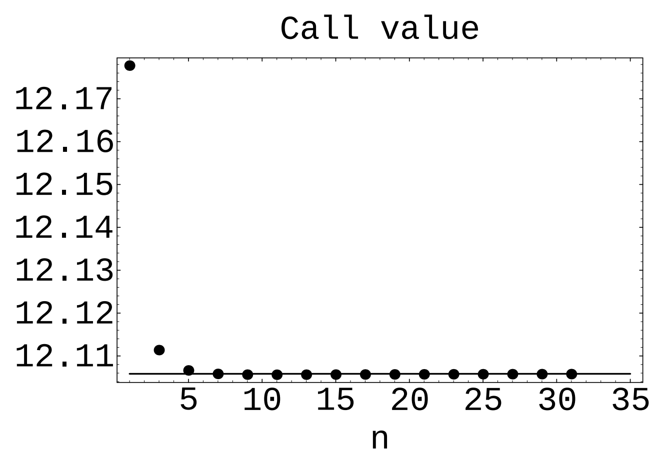
[a]This can be proved; see Chang and Palmer (2007).


Call value
13.4
13.2
13
12.8
12.6
12.4
12.2
5
10
15
20
25
30
35
n

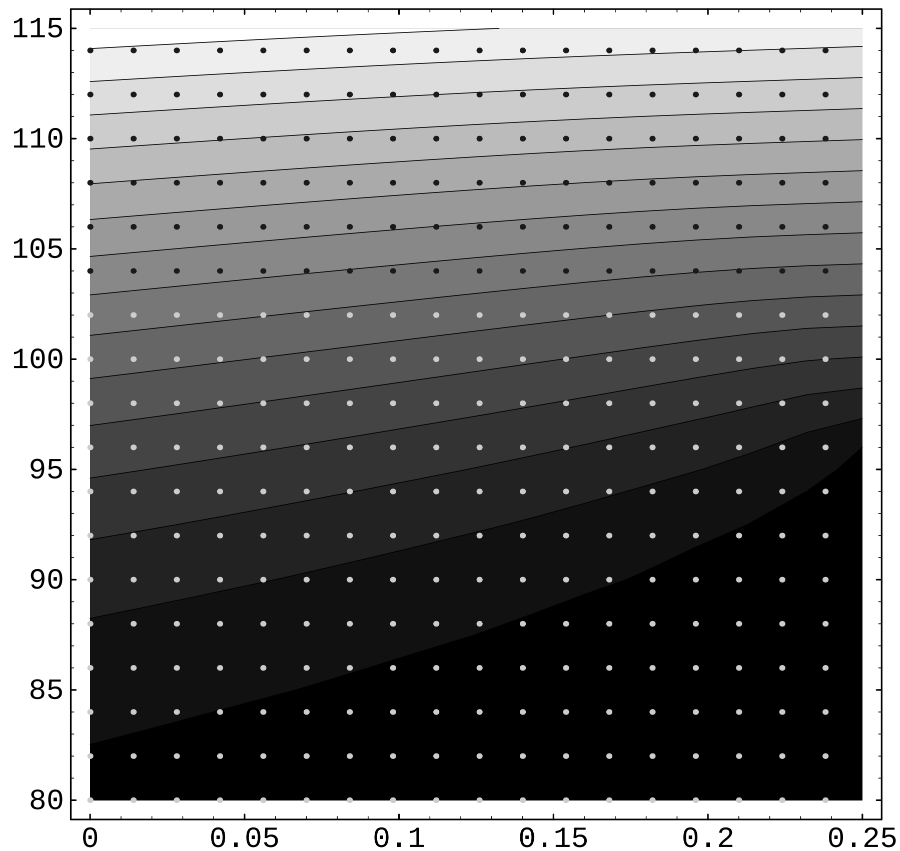

Call value
12.17
12.16
12.15
12.14
12.13
12.12
12.11
5
10
15
20
25
30
35
n

Numerical Methods

All science is dominated
by the idea of approximation.
— Bertrand Russell

Finite-Difference Methods

- Place a grid of points on the space over which the desired function takes value.
- Then approximate the function value at each of these points (p. 597).
- Solve the equation numerically by introducing difference equations in place of derivatives.



115
110
105
100
95
90
85
80
0
0.05
0.1
0.15
0.2
0.25

Example: Poisson's Equation

- It is $\partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 = -\rho(x, y)$.
- Replace second derivatives with finite differences through central difference.
- Introduce evenly spaced grid points with distance of Δx along the x axis and Δy along the y axis.
- The finite difference form is

$$-\rho(x_i, y_j) = \frac{\theta(x_{i+1}, y_j) - 2\theta(x_i, y_j) + \theta(x_{i-1}, y_j)}{(\Delta x)^2} + \frac{\theta(x_i, y_{j+1}) - 2\theta(x_i, y_j) + \theta(x_i, y_{j-1})}{(\Delta y)^2}.$$

Example: Poisson's Equation (concluded)

- In the above, $\Delta x \equiv x_i - x_{i-1}$ and $\Delta y \equiv y_j - y_{j-1}$ for $i, j = 1, 2, \ldots$.
- When the grid points are evenly spaced in both axes so that $\Delta x = \Delta y = h$, the difference equation becomes

$$\begin{aligned} -h^2 \rho(x_i, y_j) &= \theta(x_{i+1}, y_j) + \theta(x_{i-1}, y_j) \\ &\quad + \theta(x_i, y_{j+1}) + \theta(x_i, y_{j-1}) - 4\theta(x_i, y_j). \end{aligned}$$

- Given boundary values, we can solve for the x_is and the y_js within the square $[\pm L, \pm L]$.
- From now on, $\theta_{i,j}$ will denote the finite-difference approximation to the exact $\theta(x_i, y_j)$.

Explicit Methods

- Consider the diffusion equation $D(\partial^2\theta/\partial x^2) - (\partial\theta/\partial t) = 0$.
- Use evenly spaced grid points (x_i, t_j) with distances Δx and Δt, where $\Delta x \equiv x_{i+1} - x_i$ and $\Delta t \equiv t_{j+1} - t_j$.
- Employ central difference for the second derivative and forward difference for the time derivative to obtain

$$\left.\frac{\partial\theta(x,t)}{\partial t}\right|_{t=t_j} = \frac{\theta(x,t_{j+1}) - \theta(x,t_j)}{\Delta t} + \cdots, \tag{62}$$

$$\left.\frac{\partial^2\theta(x,t)}{\partial x^2}\right|_{x=x_i} = \frac{\theta(x_{i+1},t) - 2\theta(x_i,t) + \theta(x_{i-1},t)}{(\Delta x)^2} + \cdots. \tag{63}$$

Explicit Methods (continued)

- Next, assemble Eqs. (62) and (63) into a single equation at (x_i, t_j).
- But we need to decide how to evaluate x in the first equation and t in the second.
- Since central difference around x_i is used in Eq. (63), we might as well use x_i for x in Eq. (62).
- Two choices are possible for t in Eq. (63).
- The first choice uses $t = t_j$ to yield the following finite-difference equation,

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D\,\frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2}. \qquad (64)$$

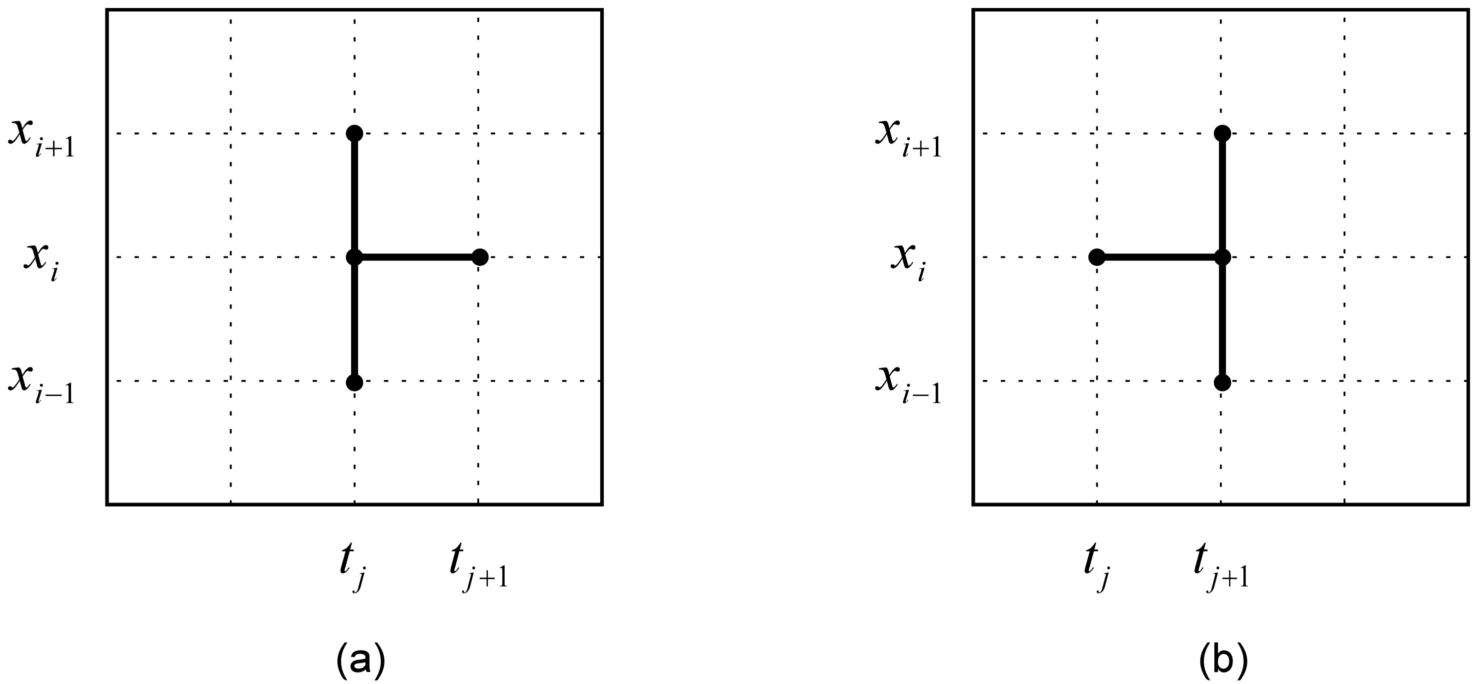
Explicit Methods (continued)

- The stencil of grid points involves four values, $\theta_{i,j+1}$, $\theta_{i,j}$, $\theta_{i+1,j}$, and $\theta_{i-1,j}$.
- Rearrange Eq. (64) on p. 601 as

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- We can calculate $\theta_{i,j+1}$ from $\theta_{i,j}, \theta_{i+1,j}, \theta_{i-1,j}$, at the previous time t_j (see figure (a) on next page).

Stencils



Explicit Methods (concluded)

- Starting from the initial conditions at t_0, that is, $\theta_{i,0} = \theta(x_i, t_0)$, $i = 1, 2, \ldots$, we calculate

$$\theta_{i,1}, \quad i = 1, 2, \ldots .$$

- And then

$$\theta_{i,2}, \quad i = 1, 2, \ldots .$$

- And so on.

Stability

- The explicit method is numerically unstable unless

$$\Delta t \leq (\Delta x)^2 / (2D).$$

  – A numerical method is unstable if the solution is highly sensitive to changes in initial conditions.

- The stability condition may lead to high running times and memory requirements.

- For instance, halving Δx would imply quadrupling $(\Delta t)^{-1}$, resulting in a running time eight times as much.

Explicit Method and Trinomial Tree

- Rearrange Eq. (64) on p. 601 as

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- When the stability condition is satisfied, the three coefficients for $\theta_{i+1,j}$, $\theta_{i,j}$, and $\theta_{i-1,j}$ all lie between zero and one and sum to one.
- They can be interpreted as probabilities.
- So the finite-difference equation becomes identical to backward induction on trinomial trees!
- The freedom in choosing Δx corresponds to similar freedom in the construction of the trinomial trees.