Brownian Motion[a]

- Brownian motion is a stochastic process $\{ X(t), t \geq 0 \}$ with the following properties.

  1. $X(0) = 0$, unless stated otherwise.

  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

  $$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.[b]

  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where μ and $\sigma \neq 0$ are real numbers.

---

[a] Robert Brown (1773–1858).

[b] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

Brownian Motion (concluded)

- Such a process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2.
- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is also called the Wiener process.

[a] Norbert Wiener (1894–1964).

Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then $X(t) - X(s) \sim N(0, t-s)$.
- A (μ, σ) Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \tag{46}$$

- Note that $Y(t+s) - Y(t) \sim N(\mu s, \sigma^2 s)$.

Brownian Motion as Limit of Random Walk

Claim 1 *A* (μ, σ) *Brownian motion is the limiting case of random walk.*

- A particle moves Δx to the left with probability $1 - p$.
- It moves to the right with probability p after Δt time.
- Assume $n \equiv t/\Delta t$ is an integer.
- Its position at time t is

$$Y(t) \equiv \Delta x \left(X_1 + X_2 + \cdots + X_n \right).$$

Brownian Motion as Limit of Random Walk (continued)

- (continued)
  - Here
$$X_i \equiv \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$
  - X_i are independent with
$\text{Prob}[\, X_i = 1\,] = p = 1 - \text{Prob}[\, X_i = -1\,]$.
- Recall $E[\, X_i\,] = 2p - 1$ and $\text{Var}[\, X_i\,] = 1 - (2p-1)^2$.

Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$
$$\mathrm{Var}[\,Y(t)\,] = n(\Delta x)^2\left[\,1-(2p-1)^2\,\right].$$

- With $\Delta x \equiv \sigma\sqrt{\Delta t}$ and $p \equiv [\,1+(\mu/\sigma)\sqrt{\Delta t}\,]/2$,

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \mathrm{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1-(\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t, \end{aligned}$$

as $\Delta t \to 0$.

Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{Y(t), t \geq 0\}$ converges to a (μ, σ) Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Note that

$$\begin{aligned}&\mathrm{Var}[\,Y(t+\Delta t) - Y(t)\,]\\ =&\mathrm{Var}[\,\Delta x\, X_{n+1}\,] = (\Delta x)^2 \times \mathrm{Var}[\,X_{n+1}\,] \to \sigma^2 \Delta t.\end{aligned}$$

- Similarity to the the BOPM: The p is identical to the probability in Eq. (24) on p. 242 and $\Delta x = \ln u$.

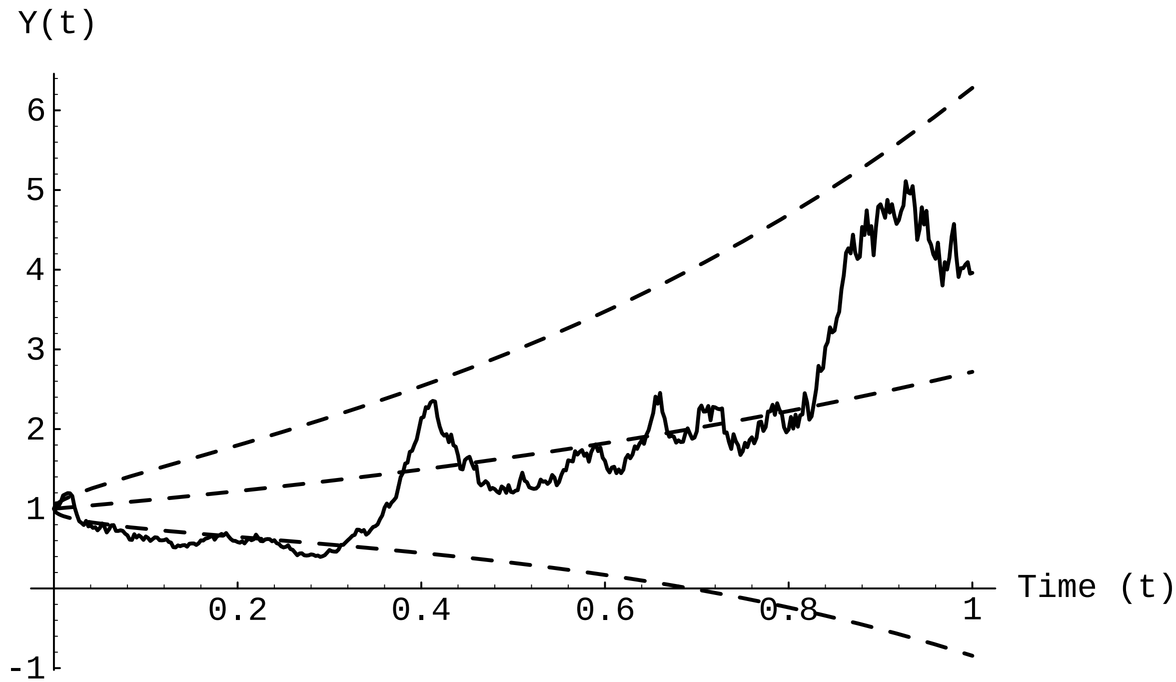
Geometric Brownian Motion

- Let $X \equiv \{ X(t), t \geq 0 \}$ be a Brownian motion process.
- The process
$$\{ Y(t) \equiv e^{X(t)}, t \geq 0 \},$$
is called geometric Brownian motion.
- Suppose further that X is a (μ, σ) Brownian motion.
- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function
$$E\left[e^{sX(t)} \right] = E\left[Y(t)^s \right] = e^{\mu ts + (\sigma^2 ts^2/2)}$$
from Eq. (17) on p 143.

Geometric Brownian Motion (continued)

- In particular,

$$\begin{aligned} E[\,Y(t)\,] &= e^{\mu t+(\sigma^2 t/2)}, \\ \mathrm{Var}[\,Y(t)\,] &= E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2 \\ &= e^{2\mu t+\sigma^2 t}\left(e^{\sigma^2 t} - 1\right). \end{aligned}$$


Y(t)
6
5
4
3
2
1
-1
0.2
0.4
0.6
0.8
1
Time (t)

Geometric Brownian Motion (continued)

- It is useful for situations in which percentage changes are independent and identically distributed.
- Let Y_n denote the stock price at time n and $Y_0 = 1$.
- Assume relative returns

$$X_i \equiv \frac{Y_i}{Y_{i-1}}$$

are independent and identically distributed.

Geometric Brownian Motion (concluded)

- Then

$$\ln Y_n = \sum_{i=1}^{n} \ln X_i$$

is a sum of independent, identically distributed random variables.

- Thus $\{ \ln Y_n, n \geq 0 \}$ is approximately Brownian motion.
  - And $\{ Y_n, n \geq 0 \}$ is approximately geometric Brownian motion.

Continuous-Time Financial Mathematics

A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
Science and the Modern World

Stochastic Integrals

- Use $W \equiv \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of X from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \equiv \int_0^t X\, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of X with respect to W.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X\, dW$.

[a] Kiyoshi Ito (1915–2008).

Stochastic Integrals (concluded)

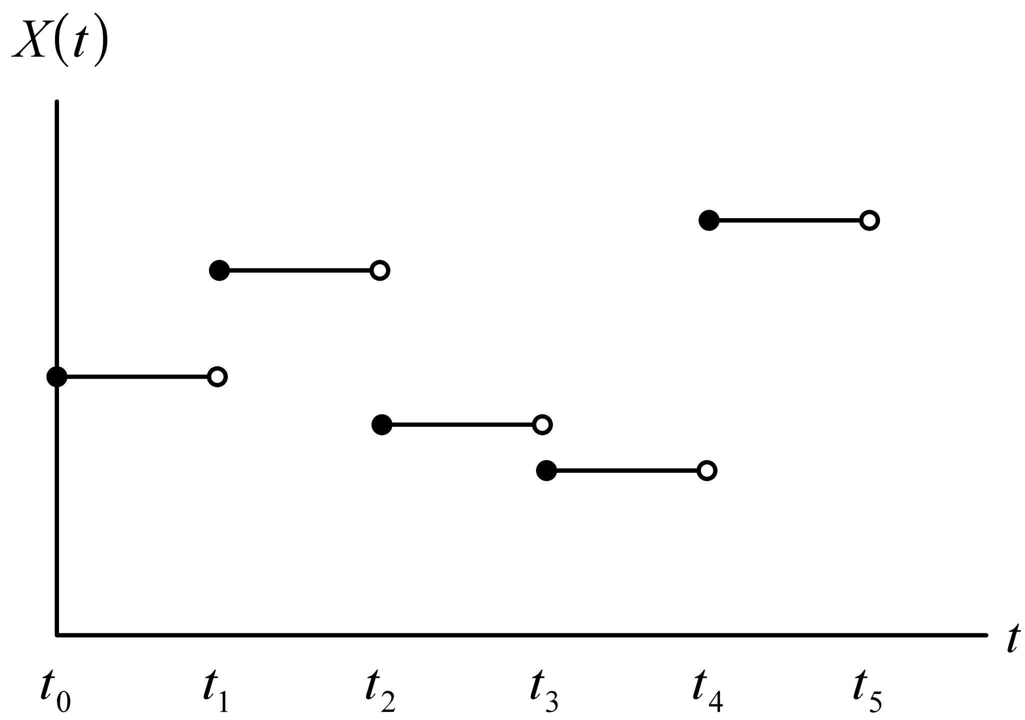
- Typical requirements for X in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\, ds < \infty$.
  - The information set at time t includes the history of X and W up to that point in time.
  - But it contains nothing about the evolution of X or W after t (nonanticipating, so to speak).
  - The future cannot influence the present.
- $\{X(s), 0 \leq s \leq t\}$ is independent of $\{W(t+u) - W(t), u > 0\}$.

Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist $0 = t_0 < t_1 < \cdots$ such that

$$X(t) = X(t_{k-1}) \text{ for } t \in [t_{k-1}, t_k), \; k = 1, 2, \ldots$$

  for any realization (see figure on next page).



$X(t)$
t
t_0
t_1
t_2
t_3
t_4
t_5

Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \equiv \sum_{k=0}^{n-1} X(t_k)[\,W(t_{k+1}) - W(t_k)\,], \tag{47}$$

where $t_n = t$.

  - The integrand X is evaluated at t_k, not t_{k+1}.

- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (continued)

- Let $X = \{ X(t), t \geq 0 \}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that X_n converges in probability to X.
- If X is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as $\delta_n \equiv \max_{1 \leq k \leq n}(t_k - t_{k-1})$ goes to zero.

Ito Integral (concluded)

- It is a fundamental fact that $\int X\, dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.
  - A corollary is the mean value formula

$$E\left[\int_a^b X\, dW\right] = 0.$$

Theorem 15 *The Ito integral* $\int X\, dW$ *is a martingale.*

Discrete Approximation

- Recall Eq. (47) on p. 471.
- The following simple stochastic process $\{ \widehat{X}(t) \}$ can be used in place of X to approximate the stochastic integral $\int_0^t X \, dW$,

$$\widehat{X}(s) \equiv X(t_{k-1}) \;\; \text{for} \;\; s \in [\, t_{k-1}, t_k), \, k = 1, 2, \ldots, n.$$

- Note the nonanticipating feature of $\widehat{X}$.
  - The information up to time s,

$$\{ \widehat{X}(t), W(t), 0 \leq t \leq s \},$$

    cannot determine the future evolution of X or W.

Discrete Approximation (concluded)

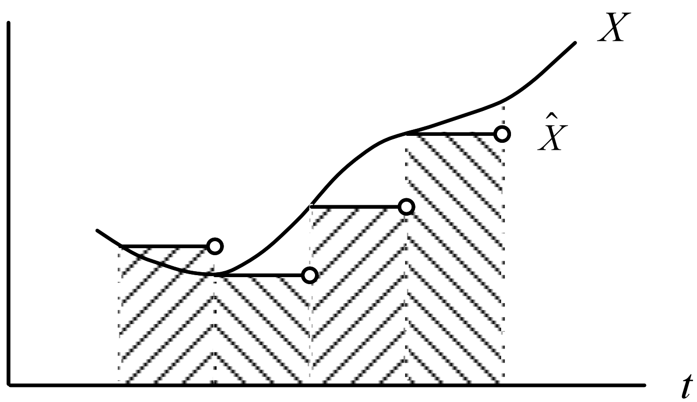
- Suppose we defined the stochastic integral as

$$\sum_{k=0}^{n-1} X(t_{k+1})[\,W(t_{k+1}) - W(t_k)\,].$$

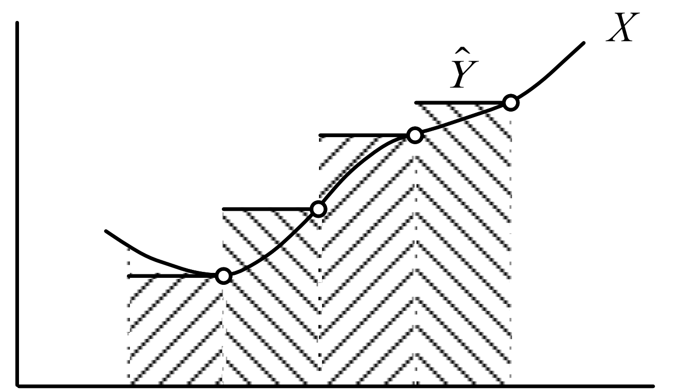
- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \equiv X(t_k) \;\text{ for }\; s \in [\,t_{k-1}, t_k),\; k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of X.



(a)



(b)

Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

is called an Ito process.

  - X_0 is a scalar starting point.
  - $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.

- The terms $a(X_t, t)$ and $b(X_t, t)$ are the drift and the diffusion, respectively.

Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{48}$$

  - Or simply $dX_t = a_t\, dt + b_t\, dW_t$.

- This is Brownian motion with an instantaneous drift a_t and an instantaneous variance b_t^2.

- X is a martingale if the drift a_t is zero by Theorem 15 (p. 473).

[a]Paul Langevin (1904).

Ito Process (concluded)

- dW is normally distributed with mean zero and variance dt.
- An equivalent form to Eq. (48) is

$$dX_t = a_t \, dt + b_t \sqrt{dt} \; \xi, \tag{49}$$

where $\xi \sim N(0,1)$.

Euler Approximation

- The following approximation follows from Eq. (49),

$$\begin{aligned} &\widehat{X}(t_{n+1}) \\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\,\Delta W(t_n), \end{aligned} \tag{50}$$

  where $t_n \equiv n\Delta t$.

- It is called the Euler or Euler-Maruyama method.
- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.
- Recall that $\Delta W(t_n)$ should be interpreted as $W(t_{n+1}) - W(t_n)$ instead of $W(t_n) - W(t_{n-1})$.

More Discrete Approximations

- Under fairly loose regularity conditions, approximation (50) on p. 480 can be replaced by

$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\sqrt{\Delta t}\,Y(t_n).\end{aligned}$$

  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

$$\begin{aligned}&\widehat{X}(t_{n+1})\\=&\widehat{X}(t_n)+a(\widehat{X}(t_n),t_n)\,\Delta t+b(\widehat{X}(t_n),t_n)\sqrt{\Delta t}\,\xi.\end{aligned}$$

  - $\text{Prob}[\,\xi=1\,]=\text{Prob}[\,\xi=-1\,]=1/2$.
  - Note that $E[\,\xi\,]=0$ and $\text{Var}[\,\xi\,]=1$.
- This clearly defines a binomial model.
- As Δt goes to zero, $\widehat{X}$ converges to X.

Trading and the Ito Integral

- Consider an Ito process $d\boldsymbol{S}_t = \mu_t \, dt + \sigma_t \, dW_t$.
  - $\boldsymbol{S}_t$ is the vector of security prices at time t.
- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time t.
  - Hence the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time t.
- $\boldsymbol{\phi}_t \, d\boldsymbol{S}_t \equiv \boldsymbol{\phi}_t(\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time t.

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \equiv \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

measures the gains realized by the trading strategy over the period $[\,0, T\,]$.

Ito's Lemma

A smooth function of an Ito process is itself an Ito process.

Theorem 16 *Suppose* $f : R \to R$ *is twice continuously differentiable and* $dX = a_t\, dt + b_t\, dW$*. Then* $f(X)$ *is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \quad & f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int_0^t f'(X_s)\, b_s\, dW \\ & + \frac{1}{2} \int_0^t f''(X_s)\, b_s^2\, ds \end{aligned}$$

for $t \geq 0$.

Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$df(X) = f'(X)\, a\, dt + f'(X)\, b\, dW + \frac{1}{2}\, f''(X)\, b^2\, dt. \tag{51}$$

- Compared with calculus, the interesting part is the third term on the right-hand side.

- A convenient formulation of Ito's lemma is

$$df(X) = f'(X)\, dX + \frac{1}{2}\, f''(X)(dX)^2.$$

Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

$\times$	dW	dt
dW	dt	0
dt	0	0

  - The $(dW)^2 = dt$ entry is justified by a known result.

- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (continued)

Theorem 17 (Higher-Dimensional Ito's Lemma) *Let* $W_1, W_2, \ldots, W_n$ *be independent Wiener processes and* $X \equiv (X_1, X_2, \ldots, X_m)$ *be a vector process. Suppose* $f : R^m \to R$ *is twice continuously differentiable and* X_i *is an Ito process with* $dX_i = a_i\, dt + \sum_{j=1}^n b_{ij}\, dW_j$*. Then* $df(X)$ *is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^m f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^m \sum_{k=1}^m f_{ik}(X)\, dX_i\, dX_k,$$

where $f_i \equiv \partial f / \partial x_i$ *and* $f_{ik} \equiv \partial^2 f / \partial x_i \partial x_k$*.*

Ito's Lemma (continued)

- The multiplication table for Theorem 17 is

$\times$	dW_i	dt
dW_k	$\delta_{ik}\, dt$	0
dt	0	0

in which

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Ito's Lemma (continued)

Theorem 18 (Alternative Ito's Lemma) *Let $W_1, W_2, \ldots, W_m$ be Wiener processes and $X \equiv (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : R^m \to R$ is twice continuously differentiable and X_i is an Ito process with $dX_i = a_i\, dt + b_i\, dW_i$. Then $df(X)$ is the following Ito process,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k.$$

Ito's Lemma (concluded)

- The multiplication table for Theorem 18 is

$\times$	dW_i	dt
dW_k	$\rho_{ik}\, dt$	0
dt	0	0

- Here, ρ_{ik} denotes the correlation between dW_i and dW_k.

Geometric Brownian Motion

- Consider the geometric Brownian motion process $Y(t) \equiv e^{X(t)}$
  - $X(t)$ is a (μ, σ) Brownian motion.
  - Hence $dX = \mu\, dt + \sigma\, dW$ by Eq. (46) on p. 455.
- As $\partial Y / \partial X = Y$ and $\partial^2 Y / \partial X^2 = Y$, Ito's formula (51) on p. 486 implies

$$
\begin{aligned}
dY &= Y\, dX + (1/2)\, Y\, (dX)^2 \\
&= Y\, (\mu\, dt + \sigma\, dW) + (1/2)\, Y\, (\mu\, dt + \sigma\, dW)^2 \\
&= Y\, (\mu\, dt + \sigma\, dW) + (1/2)\, Y \sigma^2\, dt.
\end{aligned}
$$

Geometric Brownian Motion (concluded)

- Hence
$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right) dt + \sigma\, dW.$$
- The annualized instantaneous rate of return is $\mu + \sigma^2/2$ not μ.

Product of Geometric Brownian Motion Processes

- Let

$$\begin{aligned} dY/Y &= a\,dt + b\,dW_Y, \\ dZ/Z &= f\,dt + g\,dW_Z. \end{aligned}$$

- Consider the Ito process $U \equiv YZ$.
- Apply Ito's lemma (Theorem 18 on p. 490):

$$\begin{aligned} dU &= Z\,dY + Y\,dZ + dY\,dZ \\ &= ZY(a\,dt + b\,dW_Y) + YZ(f\,dt + g\,dW_Z) \\ &\quad + YZ(a\,dt + b\,dW_Y)(f\,dt + g\,dW_Z) \\ &= U(a + f + bg\rho)\,dt + Ub\,dW_Y + Ug\,dW_Z. \end{aligned}$$

Product of Geometric Brownian Motion Processes (continued)

- The product of two (or more) correlated geometric Brownian motion processes thus remains geometric Brownian motion.
- Note that

$$
\begin{aligned}
Y &= \exp\left[\left(a - b^2/2\right) dt + b\, dW_Y\right], \\
Z &= \exp\left[\left(f - g^2/2\right) dt + g\, dW_Z\right], \\
U &= \exp\left[\left(a + f - \left(b^2 + g^2\right)/2\right) dt + b\, dW_Y + g\, dW_Z\right].
\end{aligned}
$$

Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.
- This holds even if Y and Z are correlated.
- Finally, $\ln Y$ and $\ln Z$ have correlation ρ.

Quotients of Geometric Brownian Motion Processes

- Suppose Y and Z are drawn from p. 494.
- Let $U \equiv Y/Z$.
- We now show that[a]

$$\frac{dU}{U} = (a - f + g^2 - bg\rho)\, dt + b\, dW_Y - g\, dW_Z. \tag{52}$$

- Keep in mind that dW_Y and dW_Z have correlation ρ.

[a] Exercise 14.3.6 of the textbook is erroneous.

Quotients of Geometric Brownian Motion Processes (concluded)

- The multidimensional Ito's lemma (Theorem 18 on p. 490) can be employed to show that

$$
\begin{aligned}
& dU \\
= \;& (1/Z)\, dY - (Y/Z^2)\, dZ - (1/Z^2)\, dY\, dZ + (Y/Z^3)\, (dZ)^2 \\
= \;& (1/Z)(aY\, dt + bY\, dW_Y) - (Y/Z^2)(fZ\, dt + gZ\, dW_Z) \\
& -(1/Z^2)(bgYZ\rho\, dt) + (Y/Z^3)(g^2 Z^2\, dt) \\
= \;& U(a\, dt + b\, dW_Y) - U(f\, dt + g\, dW_Z) \\
& -U(bg\rho\, dt) + U(g^2\, dt) \\
= \;& U(a - f + g^2 - bg\rho)\, dt + Ub\, dW_Y - Ug\, dW_Z.
\end{aligned}
$$

Ornstein-Uhlenbeck Process

- The Ornstein-Uhlenbeck process:

$$dX = -\kappa X\, dt + \sigma\, dW,$$

where $\kappa, \sigma \geq 0$.

- It is known that

$$\begin{aligned}
E[\, X(t)\,] &= e^{-\kappa(t-t_0)}\, E[\, x_0\,], \\
\mathrm{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-t_0)}\right) + e^{-2\kappa(t-t_0)}\, \mathrm{Var}[\, x_0\,], \\
\mathrm{Cov}[\, X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa(t-s)}\left[\, 1 - e^{-2\kappa(s-t_0)}\,\right] \\
&\quad + e^{-\kappa(t+s-2t_0)}\, \mathrm{Var}[\, x_0\,],
\end{aligned}$$

for $t_0 \leq s \leq t$ and $X(t_0) = x_0$.

Ornstein-Uhlenbeck Process (continued)

- $X(t)$ is normally distributed if x_0 is a constant or normally distributed.
- X is said to be a normal process.
- $E[\, x_0 \,] = x_0$ and $\text{Var}[\, x_0 \,] = 0$ if x_0 is a constant.
- The Ornstein-Uhlenbeck process has the following mean reversion property.
  - When $X > 0$, X is pulled toward zero.
  - When $X < 0$, it is pulled toward zero again.

Ornstein-Uhlenbeck Process (continued)

- Another version:

$$dX = \kappa(\mu - X)\,dt + \sigma\,dW,$$

where $\sigma \geq 0$.

- Given $X(t_0) = x_0$, a constant, it is known that

$$E[\,X(t)\,] = \mu + (x_0 - \mu)\,e^{-\kappa(t-t_0)}, \tag{53}$$

$$\mathrm{Var}[\,X(t)\,] = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(t-t_0)}\right],$$

for $t_0 \leq t$.

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly μ and $\sigma/\sqrt{2\kappa}$, respectively.
- For large t, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$.
- The process is mean-reverting.
  - X tends to move toward μ.
  - Useful for modeling term structure, stock price volatility, and stock price return.

Continuous-Time Derivatives Pricing

I have hardly met a mathematician
who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Toward the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation.
- The key step is recognizing that the same random process drives both securities.
- As their prices are perfectly correlated, we figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio's return to be the riskless rate.

Assumptions

- The stock price follows $dS = \mu S\,dt + \sigma S\,dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at r.
- There is unlimited riskless borrowing and lending.
- t is the current time, T is the expiration time, and $\tau \equiv T - t$.

Black-Scholes Differential Equation

- Let C be the price of a derivative on S.
- From Ito's lemma (p. 488),

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW.$$

  - The same W drives both C and S.
- Short one derivative and long $\partial C / \partial S$ shares of stock (call it Π).
- By construction,

$$\Pi = -C + S(\partial C / \partial S).$$

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is[a]

$$d\Pi = -dC + \frac{\partial C}{\partial S}\, dS.$$

- Substitute the formulas for dC and dS into the partial differential equation to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\,\sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r\Pi\, dt$.

[a]Mathematically speaking, it is not quite right (Bergman, 1982).

Black-Scholes Differential Equation (concluded)

- So

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt = r \left(C - S\, \frac{\partial C}{\partial S} \right) dt.$$

- Equate the terms to finally obtain

$$\frac{\partial C}{\partial t} + rS\, \frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} = rC.$$

- When there is a dividend yield q,

$$\frac{\partial C}{\partial t} + (r - q)\, S\, \frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} = rC.$$

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC. \tag{54}$$

- Identity (54) leads to an alternative way of computing Θ numerically from Δ and Γ.
- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2}\,\sigma^2 S^2 \Gamma = rC.$$

  - A definite relation thus exists between Γ and Θ.

PDEs for Asian Options

- Add the new variable $A(t) \equiv \int_0^t S(u)\, du$.
- Then the value V of the Asian option satisfies this two-dimensional PDE:[a]

$$\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + S\,\frac{\partial V}{\partial A} = rV.$$

- The terminal conditions are

$$\begin{aligned} V(T, S, A) &= \max\left(\frac{A}{T} - X, 0\right) \quad \text{for call,} \\ V(T, S, A) &= \max\left(X - \frac{A}{T}, 0\right) \quad \text{for put.} \end{aligned}$$

[a] Kemna and Vorst (1990).

PDEs for Asian Options (continued)

- The two-dimensional PDE produces algorithms similar to that on pp. 340ff.
- But one-dimensional PDEs are available for Asian options.[a]
- For example, Večeř (2001) derives the following PDE for Asian calls:

$$\frac{\partial u}{\partial t} + r\left(1 - \frac{t}{T} - z\right)\frac{\partial u}{\partial z} + \frac{\left(1 - \frac{t}{T} - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

with the terminal condition $u(T, z) = \max(z, 0)$.

[a] Rogers and Shi (1995); Večeř (2001); Dubois and Lelièvre (2005).

PDEs for Asian Options (concluded)

- For Asian puts:

$$\frac{\partial u}{\partial t} + r\left(\frac{t}{T} - 1 - z\right)\frac{\partial u}{\partial z} + \frac{\left(\frac{t}{T} - 1 - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

  with the same terminal condition.

- One-dimensional PDEs lead to highly efficient numerical methods.

Heston's Stochastic-Volatility Model[a]

- Heston assumes the stock price follows

$$\frac{dS}{S} = (\mu - q)\,dt + \sqrt{V}\,dW_1, \tag{55}$$

$$dV = \kappa(\theta - V)\,dt + \sigma\sqrt{V}\,dW_2. \tag{56}$$

  - V is the instantaneous variance, which follows a square-root process.
  - dW_1 and dW_2 have correlation ρ.
  - The riskless rate r is constant.
- It may be the most popular continuous-time stochastic-volatility model.

[a] Heston (1993).

Heston's Stochastic-Volatility Model (continued)

- Heston assumes the market price of risk is $b_2 \sqrt{V}$.
- So $\mu = r + b_2 V$.
- Define

$$\begin{aligned} dW_1^* &= dW_1 + b_2 \sqrt{V}\, dt, \\ dW_2^* &= dW_2 + \rho b_2 \sqrt{V}\, dt, \\ \kappa^* &= \kappa + \rho b_2 \sigma, \\ \theta^* &= \frac{\theta \kappa}{\kappa + \rho b_2 \sigma}. \end{aligned}$$

- dW_1^* and dW_2^* have correlation ρ.
- Under the risk-neutral probability measure Q, both W_1^* and W_2^* are Wiener processes.

Heston's Stochastic-Volatility Model (continued)

- Heston's model becomes, under probability measure Q,

$$\begin{aligned}
\frac{dS}{S} &= (r-q)\,dt+\sqrt{V}\,dW_1^*,\\
dV &= \kappa^*(\theta^*-V)\,dt+\sigma\sqrt{V}\,dW_2^*.
\end{aligned}$$

Heston's Stochastic-Volatility Model (continued)

- Define

$$
\begin{aligned}
\phi(u,\tau) &= \exp\Big\{\, \imath u(\ln S + (r-q)\,\tau) \\
&\quad + \theta^*\kappa^*\sigma^{-2}\left[(\kappa^* - \rho\sigma u\imath - d)\,\tau - 2\ln\frac{1-ge^{-d\tau}}{1-g}\right] \\
&\quad + \frac{v\sigma^{-2}(\kappa^* - \rho\sigma u\imath - d)\left(1-e^{-d\tau}\right)}{1-ge^{-d\tau}}\Big\}, \\
d &= \sqrt{(\rho\sigma u\imath - \kappa^*)^2 - \sigma^2(-\imath u - u^2)}, \\
g &= (\kappa^* - \rho\sigma u\imath - d)/(\kappa^* - \rho\sigma u\imath + d).
\end{aligned}
$$

Heston's Stochastic-Volatility Model (concluded)

The formulas are[a]

$$
\begin{aligned}
C &= S\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u-\imath,\tau)}{\imath u S e^{r\tau}}\right)\,du\right] \\
&\quad -Xe^{-r\tau}\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u,\tau)}{\imath u}\right)\,du\right], \\
P &= Xe^{-r\tau}\left[\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u,\tau)}{\imath u}\right)\,du\right], \\
&\quad -S\left[\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{X^{-\imath u}\phi(u-\imath,\tau)}{\imath u S e^{r\tau}}\right)\,du\right],
\end{aligned}
$$

where $\imath = \sqrt{-1}$ and $\mathrm{Re}(x)$ denotes the real part of the complex number x.

[a]Contributed by Mr. Chen, Jung-Ying (D95723006) on August 17, 2008 and Mr. Liou, Yan-Fu (R92723060) on August 26, 2008.

Stochastic-Volatility Models and Further Extensions[a]

- How to explain the October 1987 crash?
- Stochastic-volatility models require an implausibly high-volatility level prior to *and* after the crash.
- Merton (1976) proposed jump models.
- Discontinuous jump models *in the asset price* can alleviate the problem somewhat.

[a]Eraker (2004).

Stochastic-Volatility Models and Further Extensions (continued)

- But if the jump intensity is a constant, it cannot explain the tendency of large movements to cluster over time.
- This assumption also has no impacts on option prices.
- Jump-diffusion models combine both.
  - E.g., add a jump process to Eq. (55) on p. 514.

Stochastic-Volatility Models and Further Extensions (concluded)

- But they still do not adequately describe the systematic variations in option prices.[a]
- Jumps in volatility are alternatives.[b]
  - E.g., add correlated jump processes to Eqs. (55) and Eq. (56) on p. 514.
- Such models allow high level of volatility caused by a jump to volatility.[c]

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[a] Bates (2000) and Pan (2002).

[b] Duffie, Pan, and Singleton (2000).

[c] Eraker, Johnnes, and Polson (2000).

Hedging

When Professors Scholes and Merton and I
invested in warrants,
Professor Merton lost the most money.
And I lost the least.
— Fischer Black (1938–1995)

Delta Hedge

- The delta (hedge ratio) of a derivative f is defined as $\Delta \equiv \partial f / \partial S$.
- Thus $\Delta f \approx \Delta \times \Delta S$ for relatively small changes in the stock price, ΔS.
- A delta-neutral portfolio is hedged in the sense that it is immunized against small changes in the stock price.
- A trading strategy that dynamically maintains a delta-neutral portfolio is called delta hedge.

Delta Hedge (concluded)

- Delta changes with the stock price.
- A delta hedge needs to be rebalanced periodically in order to maintain delta neutrality.
- In the limit where the portfolio is adjusted continuously, perfect hedge is achieved and the strategy becomes self-financing.

Implementing Delta Hedge

- We want to hedge N *short* derivatives.
- Assume the stock pays no dividends.
- The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus B borrowed dollars such that

$$-N \times f + N \times \Delta \times S - B = 0.$$

- At next rebalancing point when the delta is Δ', buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares with a total borrowing of $B' = N \times \Delta' \times S' - N \times f'$.
- Delta hedge is the discrete-time analog of the continuous-time limit and will rarely be self-financing.

Example

- A hedger is *short* 10,000 European calls.
- $\sigma = 30\%$ and $r = 6\%$.
- This call's expiration is four weeks away, its strike price is \$50, and each call has a current value of $f = 1.76791$.
- As an option covers 100 shares of stock, $N = 1{,}000{,}000$.
- The trader adjusts the portfolio weekly.
- The calls are replicated[a] well if the cumulative cost of trading *stock* is close to the call premium's FV.

[a]This example takes the replication viewpoint.

Example (continued)

- As $\Delta = 0.538560$, $N \times \Delta = 538,560$ shares are purchased for a total cost of $538{,}560 \times 50 = 26{,}928{,}000$ dollars to make the portfolio delta-neutral.
- The trader finances the purchase by borrowing

$$B = N \times \Delta \times S - N \times f = 25{,}160{,}090$$

dollars net.[a]

- The portfolio has zero net value now.

---

[a] This takes the hedging viewpoint — an alternative. See an exercise in the text.

Example (continued)

- At 3 weeks to expiration, the stock price rises to \$51.
- The new call value is $f' = 2.10580$.
- So the portfolio is worth

$$-N \times f' + 538{,}560 \times 51 - Be^{0.06/52} = 171,622$$

before rebalancing.

Example (continued)

- A delta hedge does not replicate the calls perfectly; it is not self-financing as \$171,622 can be withdrawn.
- The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.
- In fact, the tracking error *over one rebalancing act* is positive about 68% of the time, but its expected value is essentially zero.[a]
- It is furthermore proportional to vega.

[a] Boyle and Emanuel (1980).

Example (continued)

- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.
- With a higher delta $\Delta' = 0.640355$, the trader buys $N \times (\Delta' - \Delta) = 101,795$ shares for \$5,191,545.
- The number of shares is increased to $N \times \Delta' = 640,355$.

Example (continued)

- The cumulative cost is

$$26{,}928{,}000 \times e^{0.06/52} + 5{,}191{,}545 = 32{,}150{,}634.$$

- The total borrowed amount is

$$B' = 640{,}355 \times 51 - N \times f' = 30{,}552{,}305.$$

- The portfolio is again delta-neutral with zero value.

		Option value	Delta	Change in delta	No. shares bought	Cost of shares	Cumulative cost
τ	S	f	Δ		$N \times (5)$	$(1) \times (6)$	FV(8')+(7)
	(1)	(2)	(3)	(5)	(6)	(7)	(8)
4	50	1.7679	0.53856	—	538,560	26,928,000	26,928,000
3	51	2.1058	0.64036	0.10180	101,795	5,191,545	32,150,634
2	53	3.3509	0.85578	0.21542	215,425	11,417,525	43,605,277
1	52	2.2427	0.83983	−0.01595	−15,955	−829,660	42,825,960
0	54	4.0000	1.00000	0.16017	160,175	8,649,450	51,524,853

The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).

Example (concluded)

- At expiration, the trader has 1,000,000 shares.
- They are exercised against by the in-the-money calls for $50,000,000.
- The trader is left with an obligation of

$$51{,}524{,}853 - 50{,}000{,}000 = 1{,}524{,}853,$$

which represents the replication cost.

- Compared with the FV of the call premium,

$$1{,}767{,}910 \times e^{0.06 \times 4/52} = 1{,}776{,}088,$$

the net gain is $1{,}776{,}088 - 1{,}524{,}853 = 251{,}235$.