The Black-Karasinski Model[a]

- The BK model stipulates that the short rate follows

$$d \ln r = \kappa(t)(\theta(t) - \ln r)\, dt + \sigma(t)\, dW.$$

- This explicitly mean-reverting model depends on time through $\kappa(\cdot)$, $\theta(\cdot)$, and $\sigma(\cdot)$.
- The BK model hence has one more degree of freedom than the BDT model.
- The speed of mean reversion $\kappa(t)$ and the short rate volatility $\sigma(t)$ are independent.

[a]Black and Karasinski (1991).

The Black-Karasinski Model: Discrete Time

- The discrete-time version of the BK model has the same representation as the BDT model.
- To maintain a combining binomial tree, however, requires some manipulations.
- The next plot illustrates the ideas in which

$$
\begin{aligned}
t_2 &\equiv t_1 + \Delta t_1, \\
t_3 &\equiv t_2 + \Delta t_2.
\end{aligned}
$$

$$\begin{array}{ccccc} & & & \nearrow & \\ & & \ln r_{\mathrm{d}}(t_2) & & \\ & \nearrow & & \searrow & \\ \ln r(t_1) & & & & \ln r_{\mathrm{du}}(t_3) = \ln r_{\mathrm{ud}}(t_3) \\ & \searrow & & \nearrow & \\ & & \ln r_{\mathrm{u}}(t_2) & & \\ & & & \searrow & \end{array}$$

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The Black-Karasinski Model: Discrete Time (continued)

- Note that

$$
\begin{aligned}
\ln r_{\mathrm{d}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 - \sigma(t_1)\sqrt{\Delta t_1}\,,\\
\ln r_{\mathrm{u}}(t_2) &= \ln r(t_1) + \kappa(t_1)(\theta(t_1) - \ln r(t_1))\,\Delta t_1 + \sigma(t_1)\sqrt{\Delta t_1}\,.
\end{aligned}
$$

- To ensure that an up move followed by a down move coincides with a down move followed by an up move, impose

$$
\begin{aligned}
& \ln r_{\mathrm{d}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{d}}(t_2))\,\Delta t_2 + \sigma(t_2)\sqrt{\Delta t_2}\,,\\
= \; & \ln r_{\mathrm{u}}(t_2) + \kappa(t_2)(\theta(t_2) - \ln r_{\mathrm{u}}(t_2))\,\Delta t_2 - \sigma(t_2)\sqrt{\Delta t_2}\,.
\end{aligned}
$$

The Black-Karasinski Model: Discrete Time (concluded)

- They imply

$$\kappa(t_2) = \frac{1 - (\sigma(t_2)/\sigma(t_1))\sqrt{\Delta t_2/\Delta t_1}}{\Delta t_2}. \tag{105}$$

- So from Δt_1, we can calculate the Δt_2 that satisfies the combining condition and then iterate.
  - $t_0 \to \Delta t_0 \to t_1 \to \Delta t_1 \to t_2 \to \Delta t_2 \to \cdots \to T$ (roughly).

Problems with Lognormal Models in General

- Lognormal models such as BDT and BK share the problem that $E^{\pi}[\,M(t)\,] = \infty$ for any finite t if they the continuously compounded rate.
- Hence periodic compounding should be used.
- Another issue is computational.
- Lognormal models usually do not give analytical solutions to even basic fixed-income securities.
- As a result, to price short-dated derivatives on long-term bonds, the tree has to be built over the life of the underlying asset instead of the life of the derivative.

Problems with Lognormal Models in General (concluded)

- This problem can be somewhat mitigated by adopting different time steps: Use a fine time step up to the maturity of the short-dated derivative and a coarse time step beyond the maturity.[a]
- A down side of this procedure is that it has to be carried out for each derivative.
- Finally, empirically, interest rates do not follow the lognormal distribution.

[a]Hull and White (1993).

The Extended Vasicek Model[a]

- Hull and White proposed models that extend the Vasicek model and the CIR model.
- They are called the extended Vasicek model and the extended CIR model.
- The extended Vasicek model adds time dependence to the original Vasicek model,

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t)\, dW.$$

- Like the Ho-Lee model, this is a normal model, and the inclusion of $\theta(t)$ allows for an exact fit to the current spot rate curve.

---

[a] Hull and White (1990).

The Extended Vasicek Model (concluded)

- Function $\sigma(t)$ defines the short rate volatility, and $a(t)$ determines the shape of the volatility structure.
- Under this model, many European-style securities can be evaluated analytically, and efficient numerical procedures can be developed for American-style securities.

The Hull-White Model

- The Hull-White model is the following special case,

$$dr = (\theta(t) - ar)\, dt + \sigma\, dW.$$

- When the current term structure is matched,[a]

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

[a]Hull and White (1993).

The Extended CIR Model

- In the extended CIR model the short rate follows

$$dr = (\theta(t) - a(t)\, r)\, dt + \sigma(t) \sqrt{r}\, dW.$$

- The functions $\theta(t)$, $a(t)$, and $\sigma(t)$ are implied from market observables.
- With constant parameters, there exist analytical solutions to a small set of interest-rate-sensitive securities.

The Hull-White Model: Calibration[a]

- We describe a trinomial forward induction scheme to calibrate the Hull-White model given a and σ.
- As with the Ho-Lee model, the set of achievable short rates is evenly spaced.
- Let r_0 be the annualized, continuously compounded short rate at time zero.
- Every short rate on the tree takes on a value $r_0 + j\Delta r$ for some integer j.

[a] Hull and White (1993).

The Hull-White Model: Calibration (continued)

- Time increments on the tree are also equally spaced at Δt apart.
  - Binomial trees should not be used to model mean-reverting interest rates when Δt is a constant.[a]
- Hence nodes are located at times $i\Delta t$ for $i = 0, 1, 2, \ldots$.
- We shall refer to the node on the tree with $t_i \equiv i\Delta t$ and $r_j \equiv r_0 + j\Delta r$ as the (i, j) node.
- The short rate at node (i, j), which equals r_j, is effective for the time period $[\, t_i, t_{i+1})$.

[a]Hull and White (1992).

The Hull-White Model: Calibration (continued)

- Use

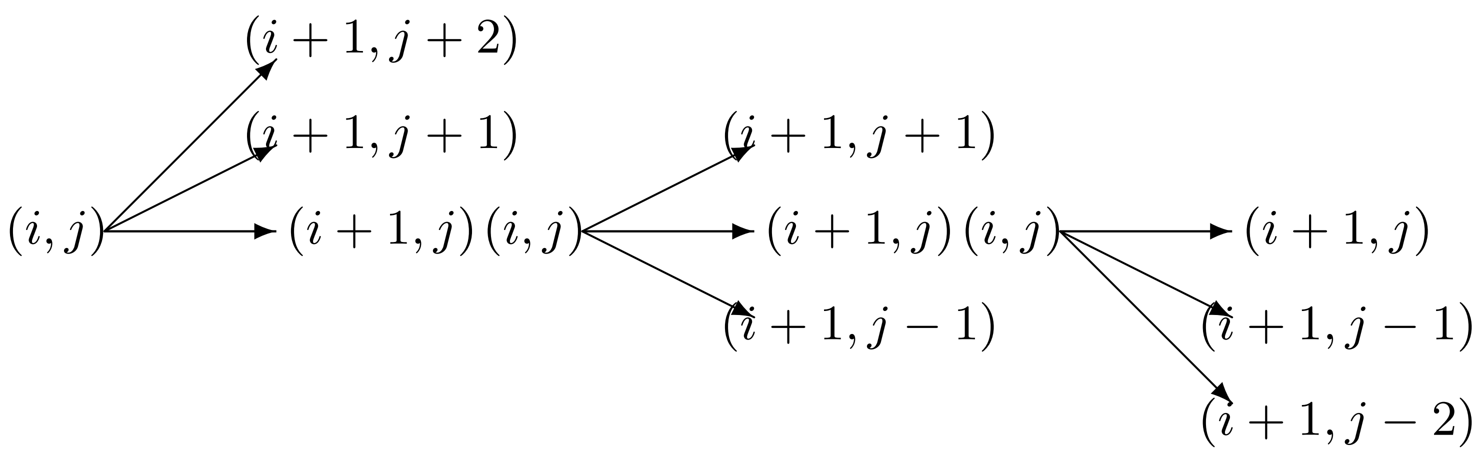
$$\mu_{i,j} \equiv \theta(t_i) - a r_j \tag{106}$$

  to denote the drift rate, or the expected change, of the short rate as seen from node (i, j).

- The three distinct possibilities for node (i, j) with three branches incident from it are displayed on p. 934.

- The interest rate movement described by the middle branch may be an increase of Δr, no change, or a decrease of Δr.

The Hull-White Model: Calibration (continued)



The Hull-White Model: Calibration (continued)

- The upper and the lower branches bracket the middle branch.
- Define

$$
\begin{aligned}
p_1(i,j) &\equiv \text{the probability of following the upper branch from node } (i,j)\\
p_2(i,j) &\equiv \text{the probability of following the middle branch from node } (i,j)\\
p_3(i,j) &\equiv \text{the probability of following the lower branch from node } (i,j)
\end{aligned}
$$

- The root of the tree is set to the current short rate r_0.
- Inductively, the drift $\mu_{i,j}$ at node (i,j) is a function of $\theta(t_i)$.

The Hull-White Model: Calibration (continued)

- Once $\theta(t_i)$ is available, $\mu_{i,j}$ can be derived via Eq. (106) on p. 933.
- This in turn determines the branching scheme at every node (i,j) for each j, as we will see shortly.
- The value of $\theta(t_i)$ must thus be made consistent with the spot rate $r(0, t_{i+2})$.

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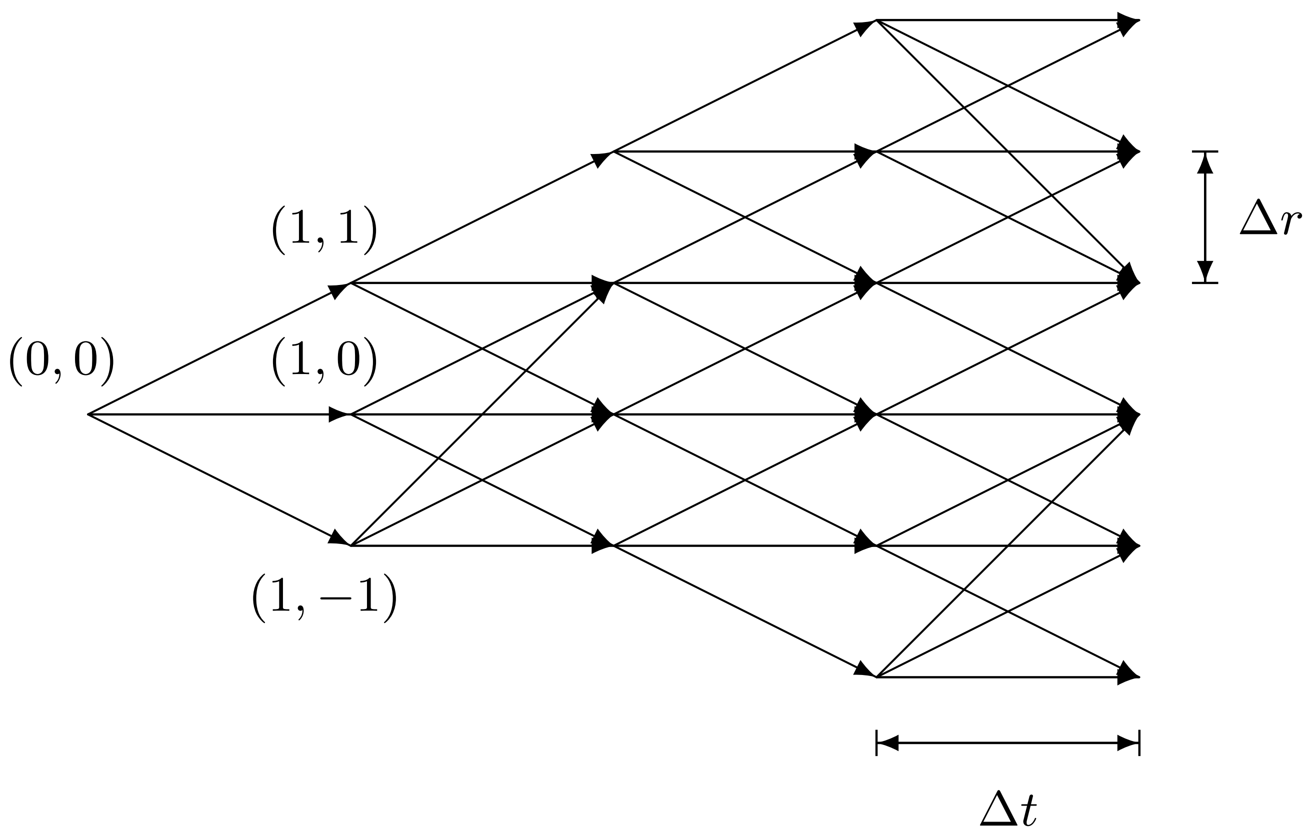
The Hull-White Model: Calibration (continued)

- The branches emanating from node (i, j) with their accompanying probabilities[a] must be chosen to be consistent with $\mu_{i,j}$ and σ.
- This is accomplished by letting the middle node be as close as possible to the current value of the short rate plus the drift.
- Let k be the number among $\{ j-1, j, j+1 \}$ that makes the short rate reached by the middle branch, r_k, closest to $r_j + \mu_{i,j} \Delta t$.

[a] $p_1(i, j)$, $p_2(i, j)$, and $p_3(i, j)$.

The Hull-White Model: Calibration (continued)

- Then the three nodes following node (i, j) are nodes $(i+1, k+1)$, $(i+1, k)$, and $(i+1, k-1)$.
- The resulting tree may have the geometry depicted on p. 939.
- The resulting tree combines because of the constant jump sizes to reach k.



$(0,0)$
$(1,1)$
$(1,0)$
$(1,-1)$
Δr
Δt

The Hull-White Model: Calibration (continued)

- The probabilities for moving along these branches are functions of $\mu_{i,j}$, σ, j, and k:

$$p_1(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} + \frac{\eta}{2\Delta r} \tag{107}$$

$$p_2(i,j) = 1 - \frac{\sigma^2 \Delta t + \eta^2}{(\Delta r)^2} \tag{107$'$}$$

$$p_3(i,j) = \frac{\sigma^2 \Delta t + \eta^2}{2(\Delta r)^2} - \frac{\eta}{2\Delta r} \tag{107$''$}$$

where $\eta \equiv \mu_{i,j} \Delta t + (j-k)\, \Delta r$.

The Hull-White Model: Calibration (continued)

- As trinomial tree algorithms are but explicit methods in disguise, certain relations must hold for Δr and Δt to guarantee stability.
- It can be shown that their values must satisfy

$$\frac{\sigma\sqrt{3\Delta t}}{2} \leq \Delta r \leq 2\sigma\sqrt{\Delta t}$$

  for the probabilities to lie between zero and one.
  - For example, Δr can be set to $\sigma\sqrt{3\Delta t}$.[a]

[a] Hull and White (1988).

The Hull-White Model: Calibration (continued)

- Now it only remains to determine $\theta(t_i)$.
- At this point at time t_i, $r(0, t_1)$, $r(0, t_2)$, $\ldots$, $r(0, t_{i+1})$ have already been matched.
- Let $Q(i, j)$ denote the value of the state contingent claim that pays one dollar at node (i, j) and zero otherwise.
- By construction, the state prices $Q(i, j)$ for all j are known by now.
- We begin with state price $Q(0, 0) = 1$.

The Hull-White Model: Calibration (continued)

- Let $\hat{r}(i)$ refer to the short rate value at time t_i.
- The value at time zero of a zero-coupon bond maturing at time t_{i+2} is then

$$
\begin{aligned}
& e^{-r(0,t_{i+2})(i+2)\,\Delta t} \\
= \; & \sum_j Q(i,j)\, e^{-r_j\,\Delta t}\, E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right]. \qquad (108)
\end{aligned}
$$

- The right-hand side represents the value of \$1 obtained by holding a zero-coupon bond until time t_{i+1} and then reinvesting the proceeds at that time at the prevailing short rate $\hat{r}(i+1)$, which is stochastic.

The Hull-White Model: Calibration (continued)

- The expectation (108) can be approximated by

$$\begin{aligned} & E^{\pi}\left[\, e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] \\ \approx\quad & e^{-r_j\,\Delta t}\left(1 - \mu_{i,j}(\Delta t)^2 + \frac{\sigma^2(\Delta t)^3}{2}\right). \quad (109) \end{aligned}$$

- Substitute Eq. (109) into Eq. (108) and replace $\mu_{i,j}$ with $\theta(t_i) - a r_j$ to obtain

$$\theta(t_i) \approx \frac{\sum_j Q(i,j)\, e^{-2r_j\,\Delta t}\left(1 + a r_j(\Delta t)^2 + \sigma^2(\Delta t)^3/2\right) - e^{-r(0,t_{i+2})(i+2)\,\Delta t}}{(\Delta t)^2 \sum_j Q(i,j)\, e^{-2r_j\,\Delta t}}.$$

The Hull-White Model: Calibration (continued)

- For the Hull-White model, the expectation in Eq. (109) on p. 944 is actually known analytically by Eq. (17) on p. 147:

$$E^{\pi}\left[e^{-\hat{r}(i+1)\,\Delta t} \,\middle|\, \hat{r}(i) = r_j \right] = e^{-r_j\,\Delta t + (-\theta(t_i) + a r_j + \sigma^2 \Delta t/2)(\Delta t)^2}.$$

- Therefore, alternatively,

$$\theta(t_i) = \frac{r(0, t_{i+2})(i+2)}{\Delta t} + \frac{\sigma^2 \Delta t}{2} + \frac{\ln \sum_j Q(i,j)\, e^{-2 r_j\,\Delta t + a r_j\,(\Delta t)^2}}{(\Delta t)^2}.$$

The Hull-White Model: Calibration (concluded)

- With $\theta(t_i)$ in hand, we can compute $\mu_{i,j}$, the probabilities, and finally the state prices at time t_{i+1}:

$$Q(i+1, j) = \sum_{(i,j^*) \text{ is connected to } (i+1,j) \text{ with probability } p_{j^*}} p_{j^*} e^{-r_{j^*} \Delta t} Q(i, j^*).$$

- There are at most 5 choices for j^*.
- The total running time is $O(n^2)$.
- The space requirement is $O(n)$ (why?).

Comments on the Hull-White Model

- One can try different values of a and σ for each option or have an a value common to all options but use a different σ value for each option.
- Either approach can match all the option prices exactly.
- If the demand is for a single set of parameters that replicate all option prices, the Hull-White model can be calibrated to all the observed option prices by choosing a and σ that minimize the mean-squared pricing error.[a]

[a] Hull and White (1995).

The Hull-White Model: Calibration with Irregular Trinomial Trees

- The previous calibration algorithm is quite general.
- For example, it can be modified to apply to cases where the diffusion term has the form σr^b.
- But it has at least two shortcomings.
- First, the resulting trinomial tree is irregular (p. 939).
  - So higher complexity in programming.
- The second shortcoming is again a consequence of the tree's irregular shape.

The Hull-White Model: Calibration with Irregular Trinomial Trees (concluded)

- Recall that the algorithm figured out $\theta(t_i)$ that matches the spot rate $r(0, t_{i+2})$ in order to determine the branching schemes for the nodes at time t_i.
- But without those branches, the tree was not specified, and backward induction on the tree was not possible.
- To avoid this dilemma, the algorithm turned to the continuous-time model to evaluate Eq. (108) on p. 943 that helps derive $\theta(t_i)$ later.
- The resulting $\theta(t_i)$ hence might not yield a tree that matches the spot rates exactly.

The Hull-White Model: Calibration with Regular Trinomial Trees[a]

- We will simplify the previous algorithm to exploit the fact that the Hull-White model has a constant diffusion term σ.
- The resulting trinomial tree will be regular.
- All the $\theta(t_i)$ terms can be chosen by backward induction to match the spot rates exactly.
- The tree is constructed in two phases.

---

[a] Hull and White (1994).

The Hull-White Model: Calibration with Regular Trinomial Trees (continued)

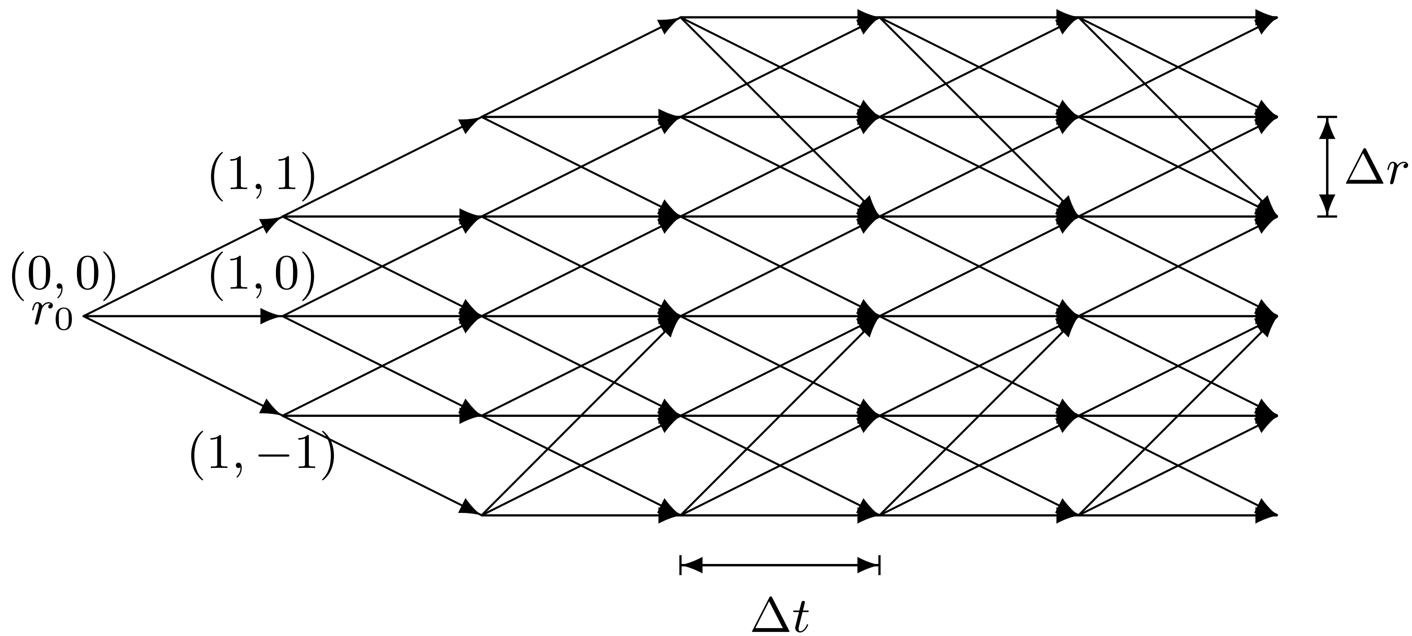
- In the first phase, a tree is built for the $\theta(t) = 0$ case, which is an Ornstein-Uhlenbeck process:

$$dr = -ar\,dt + \sigma\,dW, \quad r(0) = 0.$$

  - The tree is dagger-shaped (p. 953).
  - The number of nodes above the r_0-line, $j_{\max}$, and that below the line, $j_{\min}$, will be picked so that the probabilities (107) on p. 940 are positive for all nodes.
  - The tree's branches and probabilities are in place.

The Hull-White Model: Calibration with Regular Trinomial Trees (concluded)

- Phase two fits the term structure.
  - Backward induction is applied to calculate the β_i to add to the short rates on the tree at time t_i so that the spot rate $r(0, t_{i+1})$ is matched.



The short rate at node $(0, 0)$ equals $r_0 = 0$; here $j_{\max} = 3$ and $j_{\min} = 2$.

The Hull-White Model: Calibration

- Set $\Delta r = \sigma\sqrt{3\Delta t}$ and assume that $a > 0$.
- Node (i, j) is a top node if $j = j_{\max}$ and a bottom node if $j = -j_{\min}$.
- Because the root of the tree has a short rate of $r_0 = 0$, phase one adopts $r_j = j\Delta r$.
- Hence the probabilities in Eqs. (107) on p. 940 use

$$\eta \equiv -aj\Delta r\Delta t + (j - k)\,\Delta r.$$

The Hull-White Model: Calibration (continued)

- The probabilities become

$$
\begin{aligned}
p_1(i,j) &= \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 - aj\Delta t + (j-k)}{2}, && (110)\\
p_2(i,j) &= \frac{2}{3} - \left[\, a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 \,\right], && (111)\\
p_3(i,j) &= \frac{1}{6} + \frac{a^2 j^2 (\Delta t)^2 - 2aj\Delta t(j-k) + (j-k)^2 + aj\Delta t - (j-k)}{2}. && (112)
\end{aligned}
$$

The Hull-White Model: Calibration (continued)

- The dagger shape dictates this:
  - Let $k = j - 1$ if node (i, j) is a top node.
  - Let $k = j + 1$ if node (i, j) is a bottom node.
  - Let $k = j$ for the rest of the nodes.
- Note that the probabilities are identical for nodes (i, j) with the same j.
- Furthermore, $p_1(i, j) = p_3(i, -j)$.

The Hull-White Model: Calibration (continued)

- The inequalities

$$\frac{3-\sqrt{6}}{3} < ja\Delta t < \sqrt{\frac{2}{3}} \tag{113}$$

ensure that all the branching probabilities are positive in the upper half of the tree, that is, $j > 0$ (verify this).

- Similarly, the inequalities

$$-\sqrt{\frac{2}{3}} < ja\Delta t < -\frac{3-\sqrt{6}}{3}$$

ensure that the probabilities are positive in the lower half of the tree, that is, $j < 0$.

The Hull-White Model: Calibration (continued)

- To further make the tree symmetric across the r_0-line, we let $j_{\min} = j_{\max}$.
- As $\frac{3-\sqrt{6}}{3} \approx 0.184$, a good choice is
$$j_{\max} = \lceil 0.184/(a\Delta t) \rceil.$$
- Phase two computes the β_is to fit the spot rates.
- We begin with state price $Q(0,0) = 1$.
- Inductively, suppose that spot rates $r(0,t_1)$, $r(0,t_2)$, $\ldots$, $r(0,t_i)$ have already been matched at time t_i.

The Hull-White Model: Calibration (continued)

- By construction, the state prices $Q(i, j)$ for all j are known by now.
- The value of a zero-coupon bond maturing at time t_{i+1} equals
$$e^{-r(0,t_{i+1})(i+1)\,\Delta t} = \sum_j Q(i, j)\, e^{-(\beta_i + r_j)\Delta t}$$
by risk-neutral valuation.
- Hence
$$\beta_i = \frac{r(0, t_{i+1})(i+1)\,\Delta t + \ln \sum_j Q(i, j)\, e^{-r_j\,\Delta t}}{\Delta t},$$
and the short rate at node (i, j) equals $\beta_i + r_j$.

The Hull-White Model: Calibration (concluded)

- The state prices at time t_{i+1},

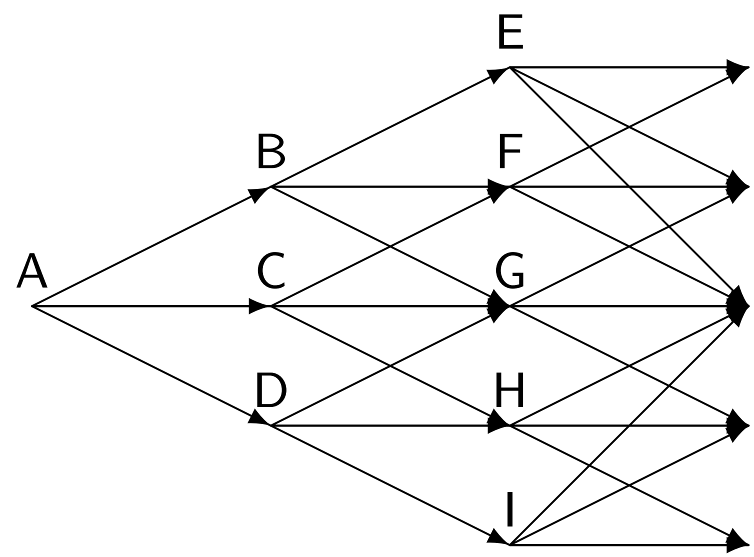
$$Q(i+1, j), \quad -j_{\max} \leq j \leq j_{\max},$$

  can now be calculated as before.

- The total running time is $O(n j_{\max})$.
- The space requirement is $O(n)$.

A Numerical Example

- Assume $a = 0.1$, $\sigma = 0.01$, and $\Delta t = 1$ (year).
- Immediately, $\Delta r = 0.0173205$ and $j_{\max} = 2$.
- The plot on p. 962 illustrates the 3-period trinomial tree after phase one.
- For example, the branching probabilities for node E are calculated by Eqs. (110)–(112) on p. 955 with $j = 2$ and $k = 1$.



Node	A, C, G	B, F	E	D, H	I
r (%)	0.00000	1.73205	3.46410	-1.73205	-3.46410
p_1	0.16667	0.12167	0.88667	0.22167	0.08667
p_2	0.66667	0.65667	0.02667	0.65667	0.02667
p_3	0.16667	0.22167	0.08667	0.12167	0.88667

A Numerical Example (continued)

- Suppose that phase two is to fit the spot rate curve $0.08 - 0.05 \times e^{-0.18 \times t}$.
- The annualized continuously compounded spot rates are

$$r(0,1) = 3.82365\%, r(0,2) = 4.51162\%, r(0,3) = 5.08626\%.$$

- Start with state price $Q(0,0) = 1$ at node A.

A Numerical Example (continued)

- Now,

$$\beta_0 = r(0,1) + \ln Q(0,0)\, e^{-r_0} = r(0,1) = 3.82365\%.$$

- Hence the short rate at node A equals

$$\beta_0 + r_0 = 3.82365\%.$$

- The state prices at year one are calculated as

$$\begin{aligned} Q(1,1) &= p_1(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414, \\ Q(1,0) &= p_2(0,0)\, e^{-(\beta_0 + r_0)} = 0.641657, \\ Q(1,-1) &= p_3(0,0)\, e^{-(\beta_0 + r_0)} = 0.160414. \end{aligned}$$

A Numerical Example (continued)

- The 2-year rate spot rate $r(0,2)$ is matched by picking

$$\beta_1 = r(0,2)\times 2 + \ln\left[\, Q(1,1)\, e^{-\Delta r} + Q(1,0) + Q(1,-1)\, e^{\Delta r} \,\right] = 5.20459\%.$$

- Hence the short rates at nodes B, C, and D equal

$$\beta_1 + r_j,$$

where $j = 1, 0, -1$, respectively.

- They are found to be 6.93664%, 5.20459%, and 3.47254%.

A Numerical Example (continued)

- The state prices at year two are calculated as

$$
\begin{aligned}
Q(2,2) &= p_1(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) = 0.018209, \\
Q(2,1) &= p_2(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) + p_1(1,0)\, e^{-(\beta_1+r_0)} Q(1,0) \\
&= 0.199799, \\
Q(2,0) &= p_3(1,1)\, e^{-(\beta_1+r_1)} Q(1,1) + p_2(1,0)\, e^{-(\beta_1+r_0)} Q(1,0) \\
&\quad + p_1(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) = 0.473597, \\
Q(2,-1) &= p_3(1,0)\, e^{-(\beta_1+r_0)} Q(1,0) + p_2(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) \\
&= 0.203263, \\
Q(2,-2) &= p_3(1,-1)\, e^{-(\beta_1+r_{-1})} Q(1,-1) = 0.018851.
\end{aligned}
$$

A Numerical Example (concluded)

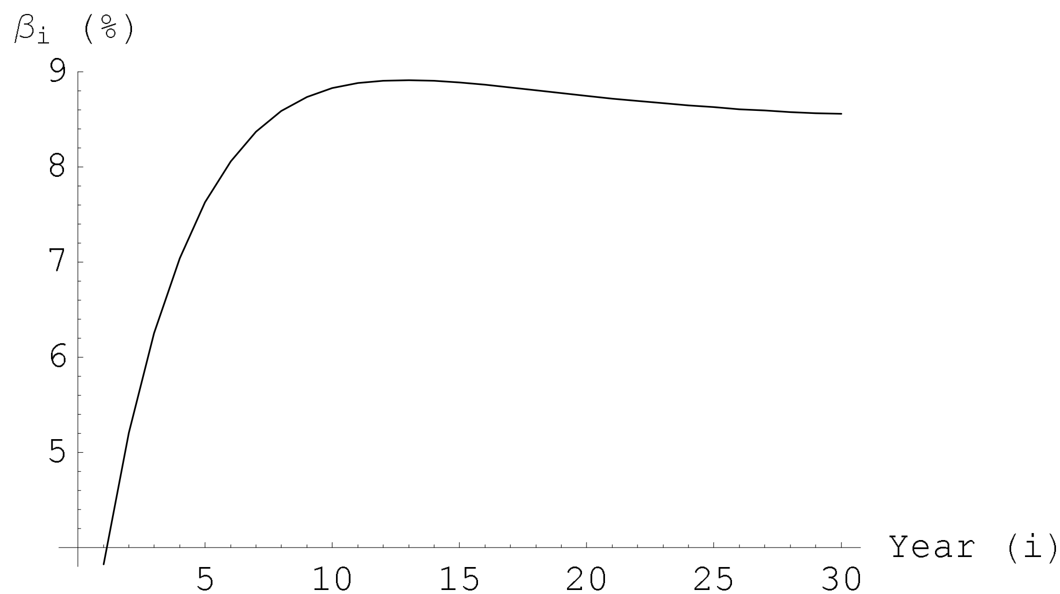
- The 3-year rate spot rate $r(0,3)$ is matched by picking

$$\begin{aligned}\beta_2 &= r(0,3)\times 3+\ln\Big[\,Q(2,2)\,e^{-2\times\Delta r}+Q(2,1)\,e^{-\Delta r}+Q(2,0)\\&\quad +Q(2,-1)\,e^{\Delta r}+Q(2,-2)\,e^{2\times\Delta r}\,\Big]=6.25359\%.\end{aligned}$$

- Hence the short rates at nodes E, F, G, H, and I equal $\beta_2 + r_j$, where $j = 2, 1, 0, -1, -2$, respectively.

- They are found to be 9.71769%, 7.98564%, 6.25359%, 4.52154%, and 2.78949%.

- The figure on p. 968 plots β_i for $i = 0, 1, \ldots, 29$.



β_i (%)
9
8
7
6
5
Year (i)
5
10
15
20
25
30