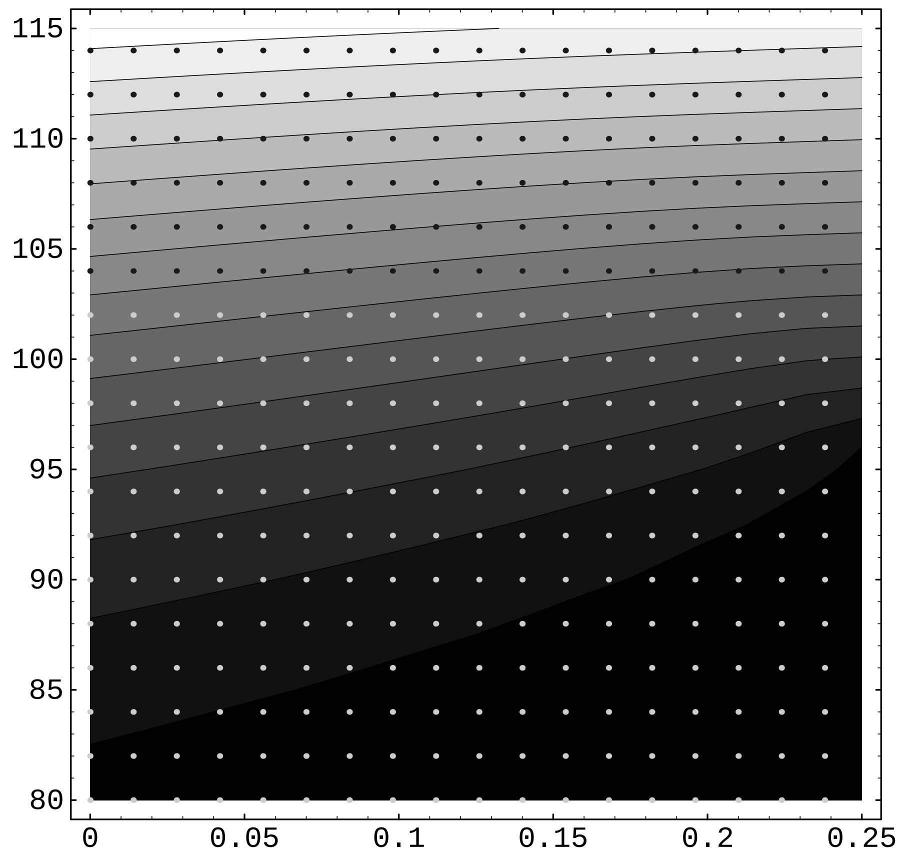
*Numerical Methods*

All science is dominated
by the idea of approximation.
— Bertrand Russell

## Finite-Difference Methods

- Place a grid of points on the space over which the desired function takes value.
- Then approximate the function value at each of these points (p. 582).
- Solve the equation numerically by introducing difference equations in place of derivatives.



115
110
105
100
95
90
85
80
0
0.05
0.1
0.15
0.2
0.25

## Example: Poisson's Equation

- It is $\partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 = -\rho(x, y)$.
- Replace second derivatives with finite differences through central difference.
- Introduce evenly spaced grid points with distance of $\Delta x$ along the $x$ axis and $\Delta y$ along the $y$ axis.
- The finite difference form is

$$\begin{aligned}
-\rho(x_i, y_j) = {} & \frac{\theta(x_{i+1}, y_j) - 2\theta(x_i, y_j) + \theta(x_{i-1}, y_j)}{(\Delta x)^2} \\
& + \frac{\theta(x_i, y_{j+1}) - 2\theta(x_i, y_j) + \theta(x_i, y_{j-1})}{(\Delta y)^2}.
\end{aligned}$$

## Example: Poisson's Equation (concluded)

- In the above, $\Delta x \equiv x_i - x_{i-1}$ and $\Delta y \equiv y_j - y_{j-1}$ for $i, j = 1, 2, \ldots$.
- When the grid points are evenly spaced in both axes so that $\Delta x = \Delta y = h$, the difference equation becomes

$$-h^2 \rho(x_i, y_j) = \theta(x_{i+1}, y_j) + \theta(x_{i-1}, y_j) + \theta(x_i, y_{j+1}) + \theta(x_i, y_{j-1}) - 4\theta(x_i, y_j).$$

- Given boundary values, we can solve for the $x_i$s and the $y_j$s within the square $[\pm L, \pm L]$.
- From now on, $\theta_{i,j}$ will denote the finite-difference approximation to the exact $\theta(x_i, y_j)$.

# Explicit Methods

- Consider the diffusion equation $D(\partial^2\theta/\partial x^2) - (\partial\theta/\partial t) = 0$.
- Use evenly spaced grid points $(x_i, t_j)$ with distances $\Delta x$ and $\Delta t$, where $\Delta x \equiv x_{i+1} - x_i$ and $\Delta t \equiv t_{j+1} - t_j$.
- Employ central difference for the second derivative and forward difference for the time derivative to obtain

$$\left.\frac{\partial\theta(x,t)}{\partial t}\right|_{t=t_j} = \frac{\theta(x,t_{j+1}) - \theta(x,t_j)}{\Delta t} + \cdots, \quad (60)$$

$$\left.\frac{\partial^2\theta(x,t)}{\partial x^2}\right|_{x=x_i} = \frac{\theta(x_{i+1},t) - 2\theta(x_i,t) + \theta(x_{i-1},t)}{(\Delta x)^2} + \cdots. \quad (61)$$

# Explicit Methods (continued)

- Next, assemble Eqs. (60) and (61) into a single equation at $(x_i, t_j)$.
- But we need to decide how to evaluate $x$ in the first equation and $t$ in the second.
- Since central difference around $x_i$ is used in Eq. (61), we might as well use $x_i$ for $x$ in Eq. (60).
- Two choices are possible for $t$ in Eq. (61).
- The first choice uses $t = t_j$ to yield the following finite-difference equation,

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D\, \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2}. \tag{62}$$

# Explicit Methods (concluded)

- The stencil of grid points involves four values, $\theta_{i,j+1}$, $\theta_{i,j}$, $\theta_{i+1,j}$, and $\theta_{i-1,j}$.
- We can calculate $\theta_{i,j+1}$ from $\theta_{i,j}, \theta_{i+1,j}, \theta_{i-1,j}$, at the previous time $t_j$ (see figure (a) on next page).
- Starting from the initial conditions at $t_0$, that is, $\theta_{i,0} = \theta(x_i, t_0)$, $i = 1, 2, \ldots$, we calculate

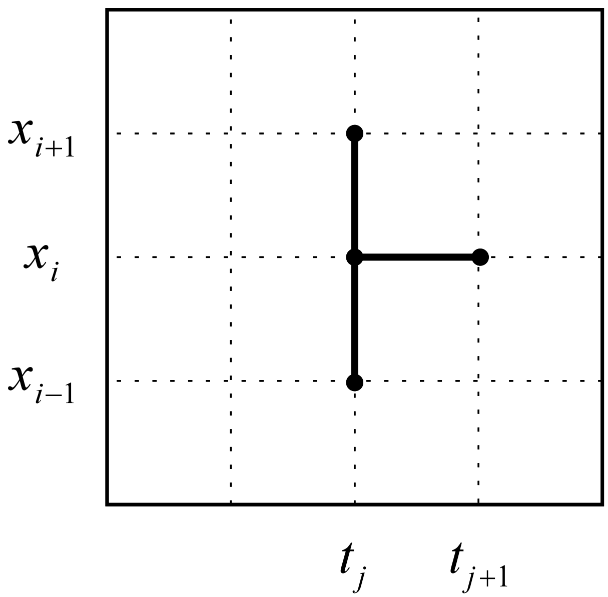
$$\theta_{i,1}, \quad i = 1, 2, \ldots\,,$$

and then

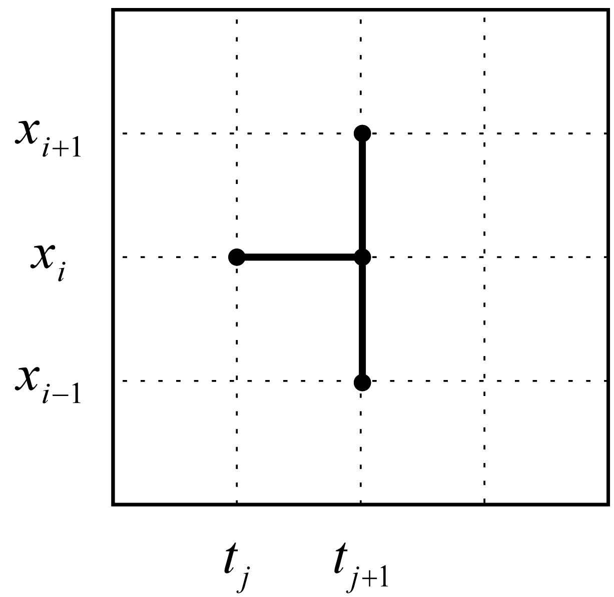
$$\theta_{i,2}, \quad i = 1, 2, \ldots\,,$$

and so on.

# Stencils




(a)




(b)

## Stability

- The explicit method is numerically unstable unless

$$\Delta t \leq (\Delta x)^2/(2D).$$

  - A numerical method is unstable if the solution is highly sensitive to changes in initial conditions.

- The stability condition may lead to high running times and memory requirements.

- For instance, halving $\Delta x$ would imply quadrupling $(\Delta t)^{-1}$, resulting in a running time eight times as much.

# Explicit Method and Trinomial Tree

- Rearrange Eq. (62) on p. 586 as

$$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- When the stability condition is satisfied, the three coefficients for $\theta_{i+1,j}$, $\theta_{i,j}$, and $\theta_{i-1,j}$ all lie between zero and one and sum to one.
- They can be interpreted as probabilities.
- So the finite-difference equation becomes identical to backward induction on trinomial trees!
- The freedom in choosing $\Delta x$ corresponds to similar freedom in the construction of the trinomial trees.

# Implicit Methods

- Suppose we use $t = t_{j+1}$ in Eq. (61) on p. 585 instead.
- The finite-difference equation becomes

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D\,\frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2}. \tag{63}$$

- The stencil involves $\theta_{i,j}$, $\theta_{i,j+1}$, $\theta_{i+1,j+1}$, and $\theta_{i-1,j+1}$.
- This method is implicit:
  - The value of any one of the three quantities at $t_{j+1}$ cannot be calculated unless the other two are known.
  - See exhibit (b) on p. 588.

# Implicit Methods (continued)

- Equation (63) can be rearranged as

$$\theta_{i-1,j+1} - (2+\gamma)\,\theta_{i,j+1} + \theta_{i+1,j+1} = -\gamma\theta_{i,j},$$

  where $\gamma \equiv (\Delta x)^2/(D\Delta t)$.

- This equation is unconditionally stable.
- Suppose the boundary conditions are given at $x = x_0$ and $x = x_{N+1}$.
- After $\theta_{i,j}$ has been calculated for $i = 1, 2, \ldots, N$, the values of $\theta_{i,j+1}$ at time $t_{j+1}$ can be computed as the solution to the following tridiagonal linear system,

# Implicit Methods (continued)

$$\begin{bmatrix} a & 1 & 0 & \cdots & \cdots & 0 \\ 1 & a & 1 & 0 & \cdots & 0 \\ 0 & 1 & a & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & a & 1 \\ 0 & \cdots & \cdots & \cdots & 1 & a \end{bmatrix} \begin{bmatrix} \theta_{1,j+1} \\ \theta_{2,j+1} \\ \theta_{3,j+1} \\ \vdots \\ \vdots \\ \vdots \\ \theta_{N,j+1} \end{bmatrix} = \begin{bmatrix} -\gamma\theta_{1,j} - \theta_{0,j+1} \\ -\gamma\theta_{2,j} \\ -\gamma\theta_{3,j} \\ \vdots \\ \vdots \\ -\gamma\theta_{N-1,j} \\ -\gamma\theta_{N,j} - \theta_{N+1,j+1} \end{bmatrix},$$

where $a \equiv -2 - \gamma$.

## Implicit Methods (concluded)

- Tridiagonal systems can be solved in $O(N)$ time and $O(N)$ space.
- The matrix above is nonsingular when $\gamma \geq 0$.
  - A square matrix is nonsingular if its inverse exists.

# Crank-Nicolson Method

- Take the average of explicit method (62) on p. 586 and implicit method (63) on p. 591:

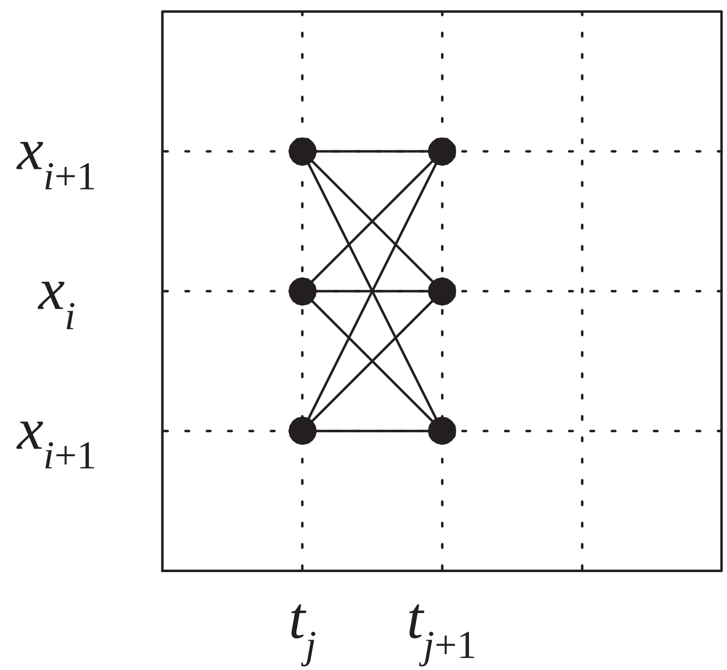
$$\frac{\theta_{i,j+1}-\theta_{i,j}}{\Delta t} = \frac{1}{2}\left(D\,\frac{\theta_{i+1,j}-2\theta_{i,j}+\theta_{i-1,j}}{(\Delta x)^2}+D\,\frac{\theta_{i+1,j+1}-2\theta_{i,j+1}+\theta_{i-1,j+1}}{(\Delta x)^2}\right).$$

- After rearrangement,

$$\gamma\theta_{i,j+1}-\frac{\theta_{i+1,j+1}-2\theta_{i,j+1}+\theta_{i-1,j+1}}{2}=\gamma\theta_{i,j}+\frac{\theta_{i+1,j}-2\theta_{i,j}+\theta_{i-1,j}}{2}.$$

- This is an unconditionally stable implicit method with excellent rates of convergence.

# Stencil



## Numerically Solving the Black-Scholes PDE

- See text.

## Monte Carlo Simulation[a]

- Monte Carlo simulation is a sampling scheme.
- In many important applications within finance and without, Monte Carlo is one of the few feasible tools.
- When the time evolution of a stochastic process is not easy to describe analytically, Monte Carlo may very well be the only strategy that succeeds consistently.

[a]A top 10 algorithm according to Dongarra and Sullivan (2000).

## The Big Idea

- Assume $X_1, X_2, \ldots, X_n$ have a joint distribution.
- $\theta \equiv E[\,g(X_1, X_2, \ldots, X_n)\,]$ for some function $g$ is desired.
- We generate
$$\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}\right), \quad 1 \leq i \leq N$$
independently with the same joint distribution as $(X_1, X_2, \ldots, X_n)$.
- Set
$$Y_i \equiv g\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}\right).$$

# The Big Idea (concluded)

- $Y_1, Y_2, \ldots, Y_N$ are independent and identically distributed random variables.
- Each $Y_i$ has the same distribution as

$$Y \equiv g(X_1, X_2, \ldots, X_n).$$

- Since the average of these $N$ random variables, $\overline{Y}$, satisfies $E[\,\overline{Y}\,] = \theta$, it can be used to estimate $\theta$.
- The strong law of large numbers says that this procedure converges almost surely.
- The number of replications (or independent trials), $N$, is called the sample size.

## Accuracy

- The Monte Carlo estimate and true value may differ owing to two reasons:
  1. Sampling variation.
  2. The discreteness of the sample paths.[a]
- The first can be controlled by the number of replications.
- The second can be controlled by the number of observations along the sample path.

[a]This may not be an issue if the derivative only requires discrete sampling along the time dimension.

## Accuracy and Number of Replications

- The statistical error of the sample mean $\overline{Y}$ of the random variable $Y$ grows as $1/\sqrt{N}$.
  - Because $\text{Var}[\overline{Y}] = \text{Var}[Y]/N$.
- In fact, this convergence rate is asymptotically optimal by the Berry-Esseen theorem.
- So the variance of the estimator $\overline{Y}$ can be reduced by a factor of $1/N$ by doing $N$ times as much work.
- This is amazing because the same order of convergence holds independently of the dimension $n$.

## Accuracy and Number of Replications (concluded)

- In contrast, classic numerical integration schemes have an error bound of $O(N^{-c/n})$ for some constant $c > 0$.
  - $n$ is the dimension.
- The required number of evaluations thus grows exponentially in $n$ to achieve a given level of accuracy.
  - The curse of dimensionality.
- The Monte Carlo method, for example, is more efficient than alternative procedures for securities depending on more than one asset, the multivariate derivatives.

## Variance Reduction

- The statistical efficiency of Monte Carlo simulation can be measured by the variance of its output.
- If this variance can be lowered without changing the expected value, fewer replications are needed.
- Methods that improve efficiency in this manner are called variance-reduction techniques.
- Such techniques become practical when the added costs are outweighed by the reduction in sampling.

# Monte Carlo Option Pricing

- For the pricing of European options on a dividend-paying stock, we may proceed as follows.
- Stock prices $S_1, S_2, S_3, \ldots$ at times $\Delta t, 2\Delta t, 3\Delta t, \ldots$ can be generated via

$$S_{i+1} = S_i e^{(\mu - \sigma^2/2)\,\Delta t + \sigma\sqrt{\Delta t}\,\xi}, \quad \xi \sim N(0,1) \tag{64}$$

when $dS/S = \mu\, dt + \sigma\, dW$.

## Monte Carlo Option Pricing (concluded)

- Non-dividend-paying stock prices in a risk-neutral economy can be generated by setting $\mu = r$.
- Pricing Asian options is easy (see text).

## Pricing American Options

- Standard Monte Carlo simulation is inappropriate for American options because of early exercise.
- It is difficult to determine the early-exercise point based on one single path.
- Monte Carlo simulation can be modified to price American options with small biases (see p. 649).[a]

[a] Longstaff and Schwartz (2001).

# Delta and Common Random Numbers

- In estimating delta, it is natural to start with the finite-difference estimate

$$e^{-r\tau}\,\frac{E[\,P(S+\epsilon)\,]-E[\,P(S-\epsilon)\,]}{2\epsilon}.$$

  - $P(x)$ is the terminal payoff of the derivative security when the underlying asset's initial price equals $x$.

- Use simulation to estimate $E[\,P(S+\epsilon)\,]$ first.

- Use another simulation to estimate $E[\,P(S-\epsilon)\,]$.

- Finally, apply the formula to approximate the delta.

## Delta and Common Random Numbers (concluded)

- This method is not recommended because of its high variance.
- A much better approach is to use common random numbers to lower the variance:

$$e^{-r\tau}\, E\left[\,\frac{P(S+\epsilon)-P(S-\epsilon)}{2\epsilon}\,\right].$$

- Here, the *same* random numbers are used for $P(S+\epsilon)$ and $P(S-\epsilon)$.
- This holds for gamma and cross gammas (for multivariate derivatives).

## Gamma

- The finite-difference formula for gamma is

$$e^{-r\tau}\, E\left[\, \frac{P(S+\epsilon) - 2 \times P(S) + P(S-\epsilon)}{\epsilon^2} \,\right].$$

- For a correlation option with multiple underlying assets, the finite-difference formula for the cross gammas $\partial^2 P(S_1, S_2, \dots)/(\partial S_1 \partial S_2)$ is:

$$e^{-r\tau}\, E\left[\, \frac{P(S_1+\epsilon_1, S_2+\epsilon_2) - P(S_1-\epsilon_1, S_2+\epsilon_2)}{4\epsilon_1\epsilon_2} \right.$$

$$\left. \frac{-P(S_1+\epsilon_1, S_2-\epsilon_2) + P(S_1-\epsilon_1, S_2-\epsilon_2)}{} \,\right].$$

## Gamma (concluded)

- Choosing an $\epsilon$ of the right magnitude can be challenging.
  - If $\epsilon$ is too large, inaccurate Greeks result.
  - If $\epsilon$ is too small, unstable Greeks result.
- This phenomenon is sometimes called the curse of differentiation.
- Need formulas for Greeks which are integrals (thus avoiding finite differences and resimulation).[a]

[a] Lyuu and Teng (2008).

# Biases in Pricing Continuously Monitored Options with Monte Carlo

- We are asked to price a continuously monitored up-and-out call with barrier $H$.
- The Monte Carlo method samples the stock price at $n$ discrete time points $t_1, t_2, \ldots, t_n$.
- A sample path $S(t_0), S(t_1), \ldots, S(t_n)$ is produced.
  - Here, $t_0 = 0$ is the current time, and $t_n = T$ is the expiration time of the option.

## Biases in Pricing Continuously Monitored Options with Monte Carlo (continued)

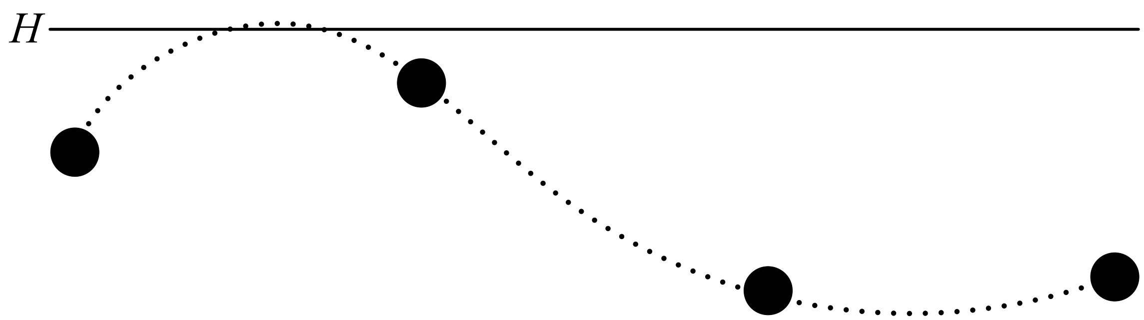
- If all of the sampled prices are below the barrier, this sample path pays $\max(S(t_n) - X, 0)$.
- Repeating these steps and averaging the payoffs yield a Monte Carlo estimate.

```
C := 0; hit := 0;
for i = 1, 2, 3, ..., m do
    P := S;
    for j = 1, 2, 3, ..., n do
        P := P \times e^{(r - \sigma^2/2)(T/n) + \sigma \sqrt{(T/n)}\, \xi};
        if P \geq H then
            hit := 1;
            break;
        end if
    end for
    if hit = 0 then
        C := C + \max(P - X, 0);
    end if
end for
return C e^{-rT} / m;
```

## Biases in Pricing Continuously Monitored Options with Monte Carlo (continued)

- This estimate is biased.
  - Suppose none of the sampled prices on a sample path equals or exceeds the barrier $H$.
  - It remains possible for the continuous sample path that passes through them to hit the barrier between sampled time points (see plot on next page).


$H$

## Biases in Pricing Continuously Monitored Options with Monte Carlo (concluded)

- The bias can certainly be lowered by increasing the number of observations along the sample path.
- However, even daily sampling may not suffice.
- The computational cost also rises as a result.

## Brownian Bridge Approach to Pricing Barrier Options

- We desire an unbiased estimate efficiently.
- So the above-mentioned payoff should be multiplied by the probability $p$ that a continuous sample path does not hit the barrier conditional on the sampled prices.
- This methodology is called the Brownian bridge approach.
- Formally, we have

$$p \equiv \text{Prob}[\, S(t) < H, 0 \leq t \leq T \mid S(t_0), S(t_1), \ldots, S(t_n) \,].$$

## Brownian Bridge Approach to Pricing Barrier Options (continued)

- As a barrier is hit over a time interval if and only if the maximum stock price over that period is at least $H$,

$$p = \text{Prob}\left[ \max_{0 \le t \le T} S(t) < H \mid S(t_0), S(t_1), \ldots, S(t_n) \right].$$

- Luckily, the conditional distribution of the maximum over a time interval given the beginning and ending stock prices is known.

## Brownian Bridge Approach to Pricing Barrier Options (continued)

**Lemma 19** *Assume* $S$ *follows* $dS/S = \mu\, dt + \sigma\, dW$ *and define*

$$\zeta(x) \equiv \exp\left[ -\frac{2 \ln(x/S(t)) \ln(x/S(t+\Delta t))}{\sigma^2 \Delta t} \right].$$

*(1) If* $H > \max(S(t), S(t+\Delta t))$, *then*

$$\text{Prob}\left[ \max_{t \leq u \leq t+\Delta t} S(u) < H \,\middle|\, S(t), S(t+\Delta t) \right] = 1 - \zeta(H).$$

*(2) If* $h < \min(S(t), S(t+\Delta t))$, *then*

$$\text{Prob}\left[ \min_{t \leq u \leq t+\Delta t} S(u) > h \,\middle|\, S(t), S(t+\Delta t) \right] = 1 - \zeta(h).$$

## Brownian Bridge Approach to Pricing Barrier Options (continued)

- Lemma 19 gives the probability that the barrier is not hit in a time interval, given the starting and ending stock prices.
- For our up-and-out call, choose $n = 1$.
- As a result,

$$p = \begin{cases} 1 - \exp\left[ -\frac{2\ln(H/S(0))\ln(H/S(T))}{\sigma^2 T} \right], & \text{if } H > \max(S(0), S(T)), \\ 0, & \text{otherwise.} \end{cases}$$

# Brownian Bridge Approach to Pricing Barrier Options (continued)

$C := 0;$

**for** $i = 1, 2, 3, \ldots, m$ **do**

    $P := S \times e^{(r-q-\sigma^2/2)\, T + \sigma \sqrt{T}\; \xi(\,)};$

    **if** $(S < H \text{ and } P < H)$ or $(S > H \text{ and } P > H)$ **then**

        $C := C + \max(P - X, 0) \times \left\{ 1 - \exp\left[ -\frac{2 \ln(H/S) \times \ln(H/P)}{\sigma^2 T} \right] \right\};$

    **end if**

**end for**

`return` $Ce^{-rT}/m;$

# Brownian Bridge Approach to Pricing Barrier Options (concluded)

- The idea can be generalized.
- For example, we can handle more complex barrier options.
- Consider an up-and-out call with barrier $H_i$ for the time interval $(t_i, t_{i+1}]$, $0 \leq i < n$.
- This option thus contains $n$ barriers.
- It is a simple matter of multiplying the probabilities for the $n$ time intervals properly to obtain the desired probability adjustment term.