Binomial Distribution

- Denote the binomial distribution with parameters n and p by

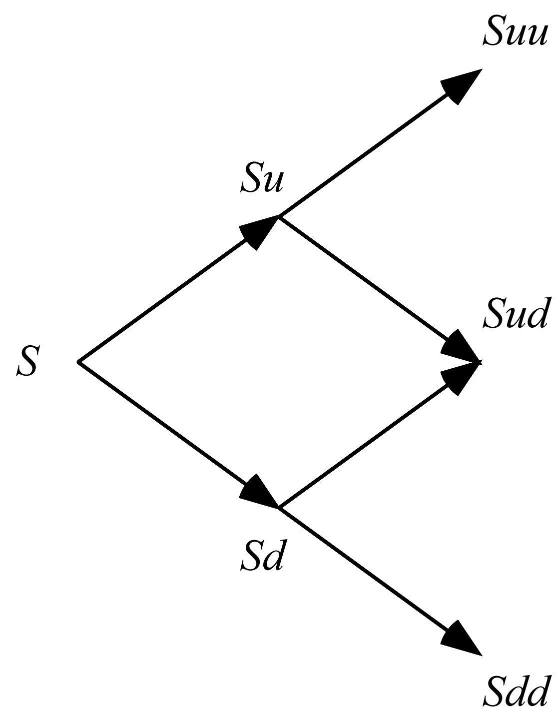
$$b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

  - $n! = n \times (n-1) \cdots 2 \times 1$ with the convention $0! = 1$.

- Suppose you toss a coin n times with p being the probability of getting heads.
- Then $b(j; n, p)$ is the probability of getting j heads.

Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on three possible prices at time two: Suu, Sud, and Sdd.
  - There are 4 paths.
  - But the tree combines.
- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.


Suu
Su
Sud
S
Sd
Sdd

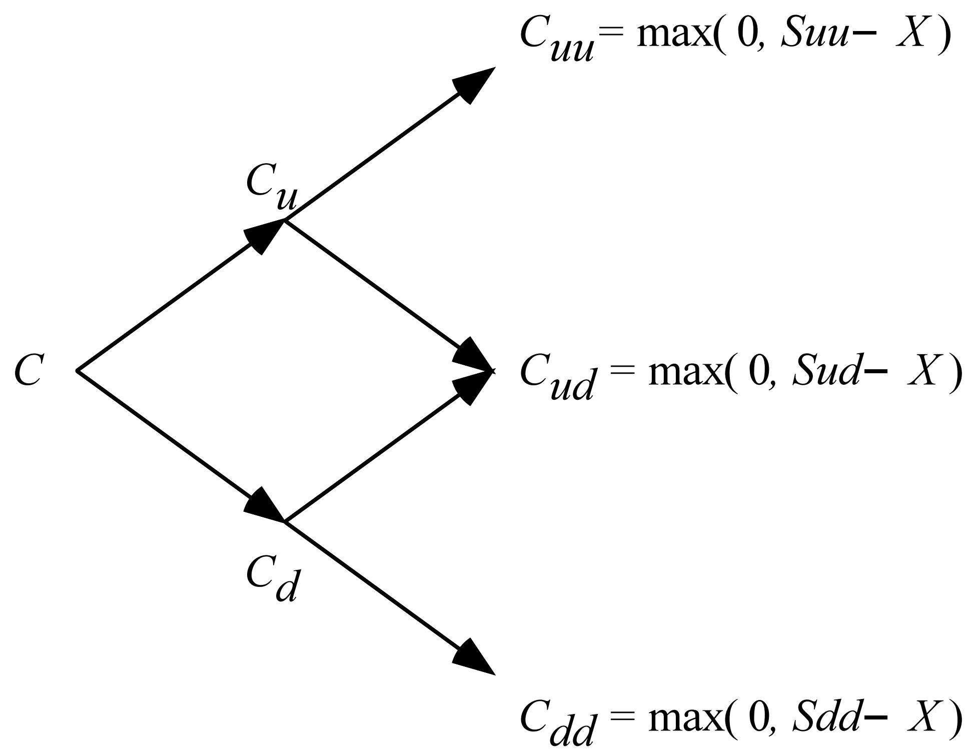
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let C_{uu} be the call's value at time two if the stock price is Suu.
- Thus,

$$C_{uu} = \max(0, Suu - X).$$

- C_{ud} and C_{dd} can be calculated analogously,

$$\begin{aligned} C_{ud} &= \max(0, Sud - X), \\ C_{dd} &= \max(0, Sdd - X). \end{aligned}$$


$C_{uu} = \max(0, Suu - X)$
C_u
C
$C_{ud} = \max(0, Sud - X)$
C_d
$C_{dd} = \max(0, Sdd - X)$

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time one can be obtained by applying the same logic:

$$
\begin{aligned}
C_u &= \frac{pC_{uu} + (1-p)\,C_{ud}}{R}, \qquad (21) \\
C_d &= \frac{pC_{ud} + (1-p)\,C_{dd}}{R}.
\end{aligned}
$$

- Deltas can be derived from Eq. (19) on p. 200.
- For example, the delta at C_u is

$$
\frac{C_{uu} - C_{ud}}{Suu - Sud}.
$$

Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- An equivalent portfolio of h shares of stock and $\$B$ riskless bonds can be set up for the call that costs C_u (C_d, resp.) if the stock price goes to Su (Sd, resp.).
- The values of h and B can be derived from Eqs. (19)–(20) on p. 200.
- That is, compute

$$\frac{pC_u + (1-p)\,C_d}{R}$$

as the price.

Early Exercise

- Since the call will not be exercised at time one even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.
- Therefore,

$$\begin{aligned} hS + B &= \frac{pC_u + (1-p)\,C_d}{R} \geq \frac{[\,pu + (1-p)\,d\,]\,S - X}{R} \\ &= S - \frac{X}{R} > S - X. \end{aligned}$$

  – The call again will not be exercised at present.

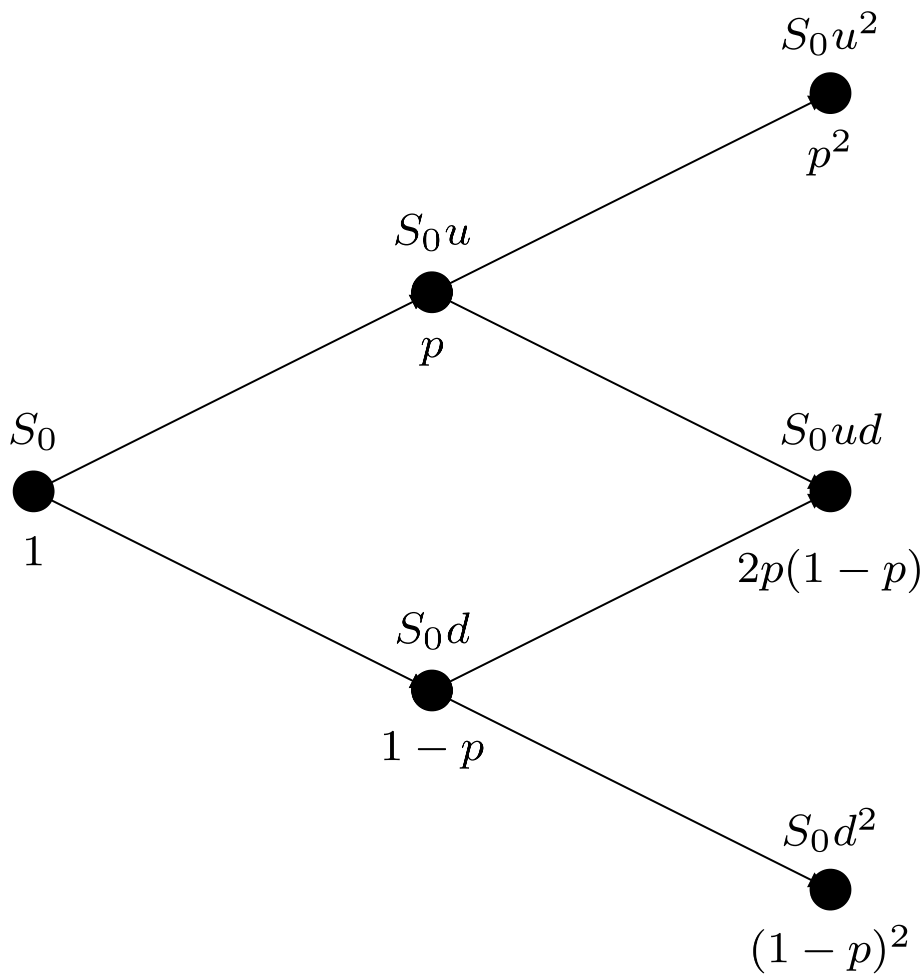
- So

$$C = hS + B = \frac{pC_u + (1-p)\,C_d}{R}.$$

Backward Induction of Zermelo (1871–1953)

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happened at C_u and C_d, too, as demonstrated in Eq. (21) on p. 211.
- This recursive procedure is called backward induction.
- Now, C equals

$$\begin{aligned} & [\, p^2 C_{uu} + 2p(1-p)\, C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\ = \; & [\, p^2 \max\left(0, Su^2 - X\right) + 2p(1-p) \max\left(0, Sud - X\right) \\ & + (1-p)^2 \max\left(0, Sd^2 - X\right)]/R^2. \end{aligned}$$


S_0u^2
p^2
S_0u
p
S_0
1
S_0ud
$2p(1-p)$
S_0d
$1-p$
S_0d^2
$(1-p)^2$

Backward Induction (concluded)

- In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, S u^j d^{n-j} - X\right)}{R^n}.$$

  - The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- The value of a European put is

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, X - S u^j d^{n-j}\right)}{R^n}.$$

Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-\hat{r}n} E^{\pi}[\,\mathcal{D}\,].$$

  - E^{π} means the expectation is taken under the risk-neutral probability.
- The "equivalence" between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.

Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- The maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.
  - Changes in value are due entirely to capital gains.

The Binomial Option Pricing Formula

- The stock prices at time n are

$$Su^n, Su^{n-1}d, \ldots, Sd^n.$$

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer such that

$$Su^a d^{n-a} \geq X,$$

or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

The Binomial Option Pricing Formula (concluded)

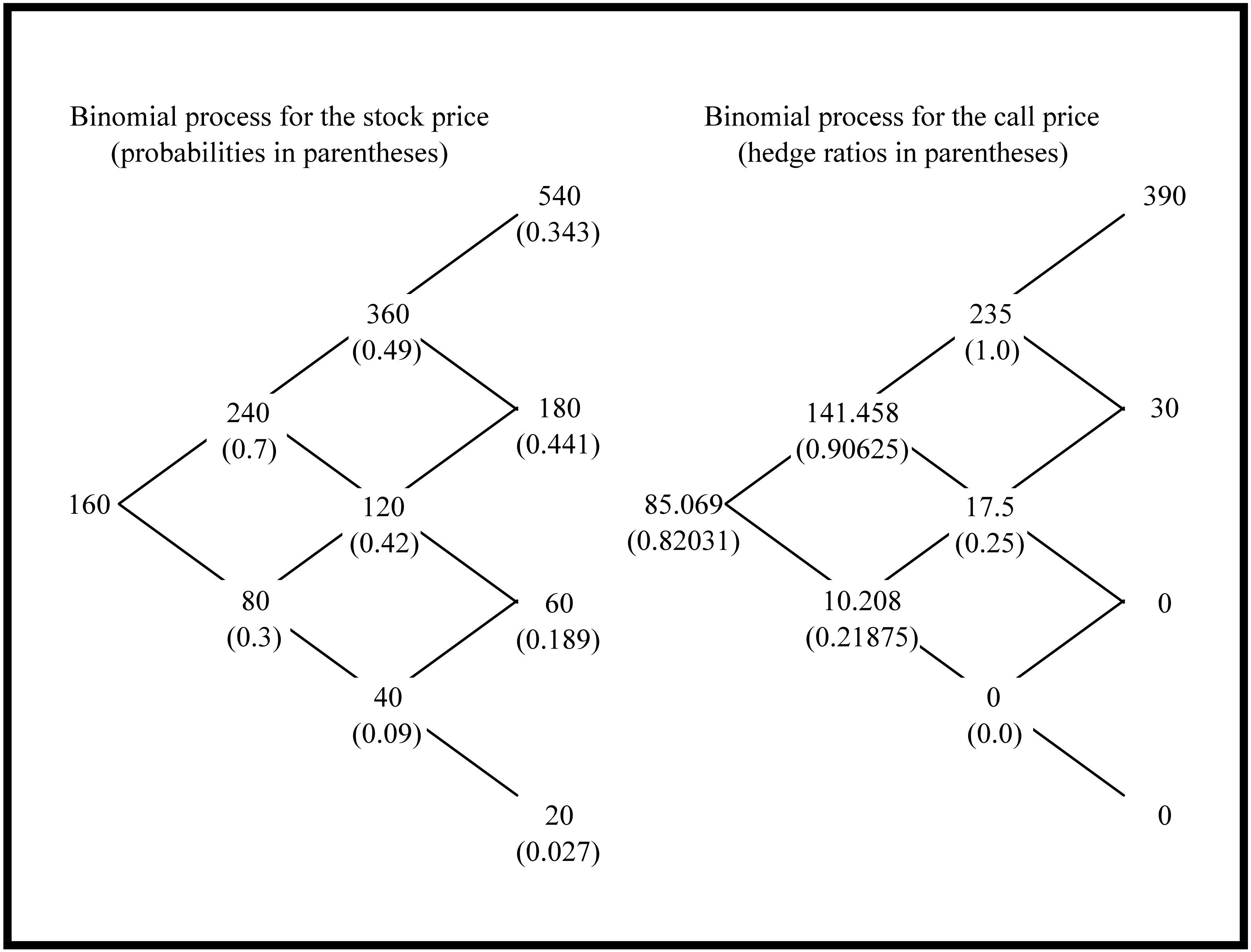
- Hence,

$$
\begin{aligned}
& C \\
= \;& \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(S u^j d^{n-j} - X \right)}{R^n} \qquad (22) \\
= \;& S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j [\,(1-p)\, d\,]^{n-j}}{R^n} \\
& - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\
= \;& S \sum_{j=a}^{n} b\,(j; n, pu/R) - X e^{-\hat{r} n} \sum_{j=a}^{n} b(j; n, p).
\end{aligned}
$$

Numerical Examples

- A non-dividend-paying stock is selling for $160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is $85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$


Binomial process for the stock price
(probabilities in parentheses)
540
(0.343)
360
(0.49)
240
(0.7)
180
(0.441)
160
120
(0.42)
80
(0.3)
60
(0.189)
40
(0.09)
20
(0.027)
Binomial process for the call price
(hedge ratios in parentheses)
390
235
(1.0)
141.458
(0.90625)
30
85.069
(0.82031)
17.5
(0.25)
10.208
(0.21875)
0
0
(0.0)
0

Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90 and invest \$85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars},$$

  is the arbitrage profit as we will see.

Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy
$$0.90625 - 0.82031 = 0.08594$$
more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to

$$76.04232 \times 1.2 - 78.75 = 12.5$$

dollars.

Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to \$180.
- The call we wrote finishes in the money.
- For a loss of $180 - 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

Numerical Examples (concluded)

Time 3 (the case of declining price):

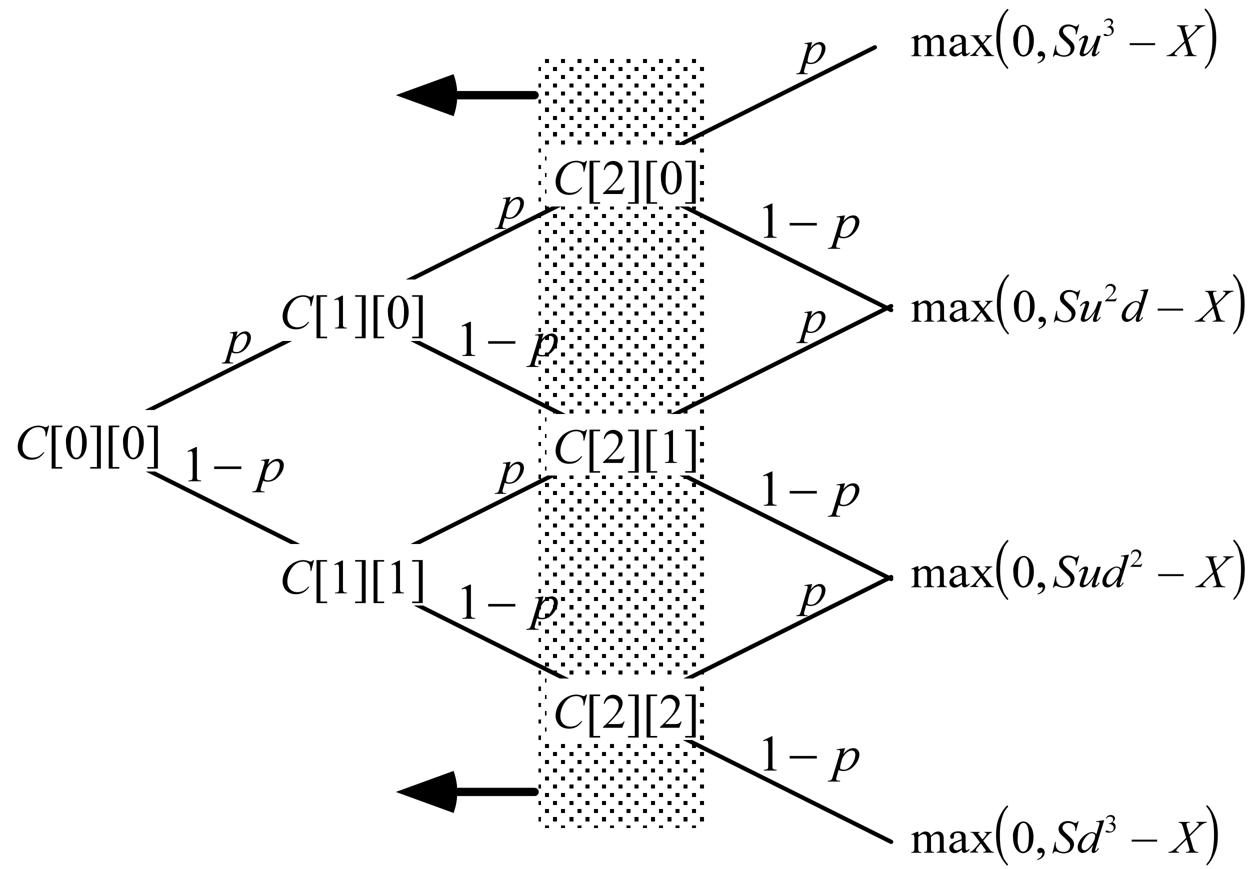
- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

$$0.25 \times 60 = 15$$

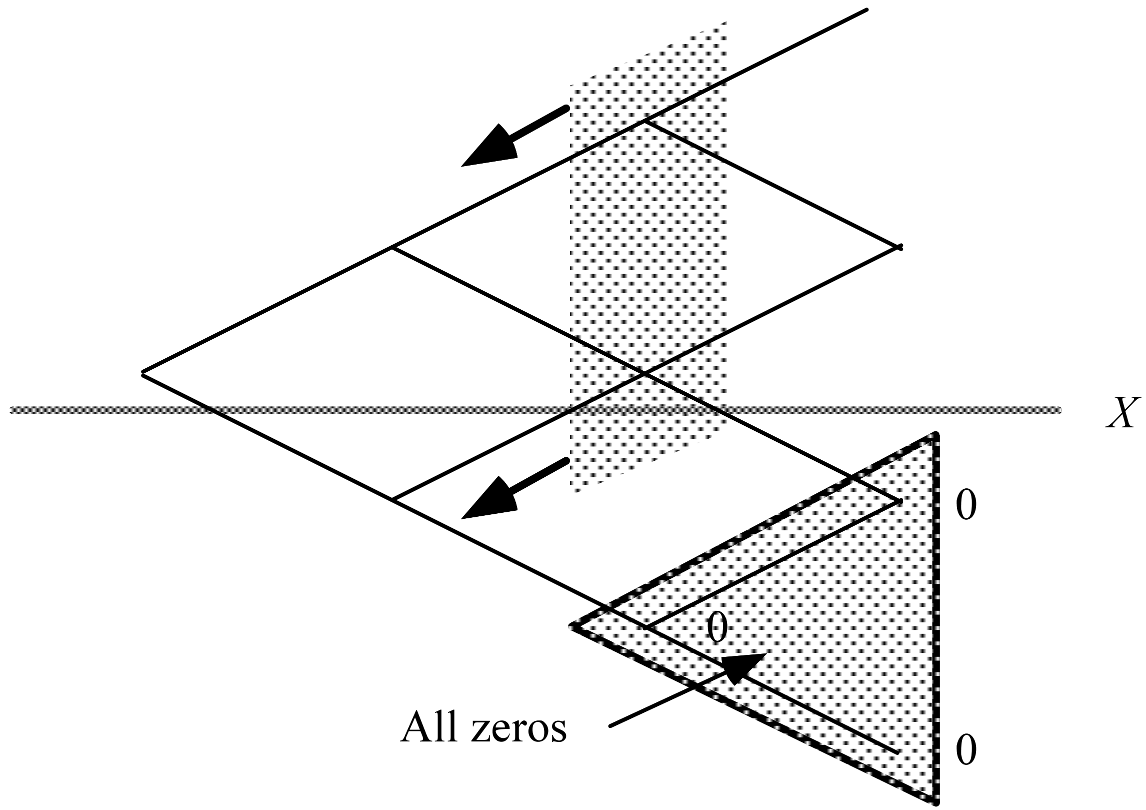
  dollars.
- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.

Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$.
- The memory requirement is $O(n^2)$.
  - Can be further reduced to $O(n)$ by reusing space
- To price European puts, simply replace the payoff.


C[0][0]
C[1][0]
C[1][1]
C[2][0]
C[2][1]
C[2][2]
p
1 − p
max(0, Su³ − X)
max(0, Su²d − X)
max(0, Sud² − X)
max(0, Sd³ − X)

Further Improvement for Calls



Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j; n, p) = \frac{p(n-j+1)}{(1-p)\, j}\, b(j-1; n, p).$$

Optimal Algorithm (continued)

- The following program computes $b(j; n, p)$ in $b[\,j\,]$:

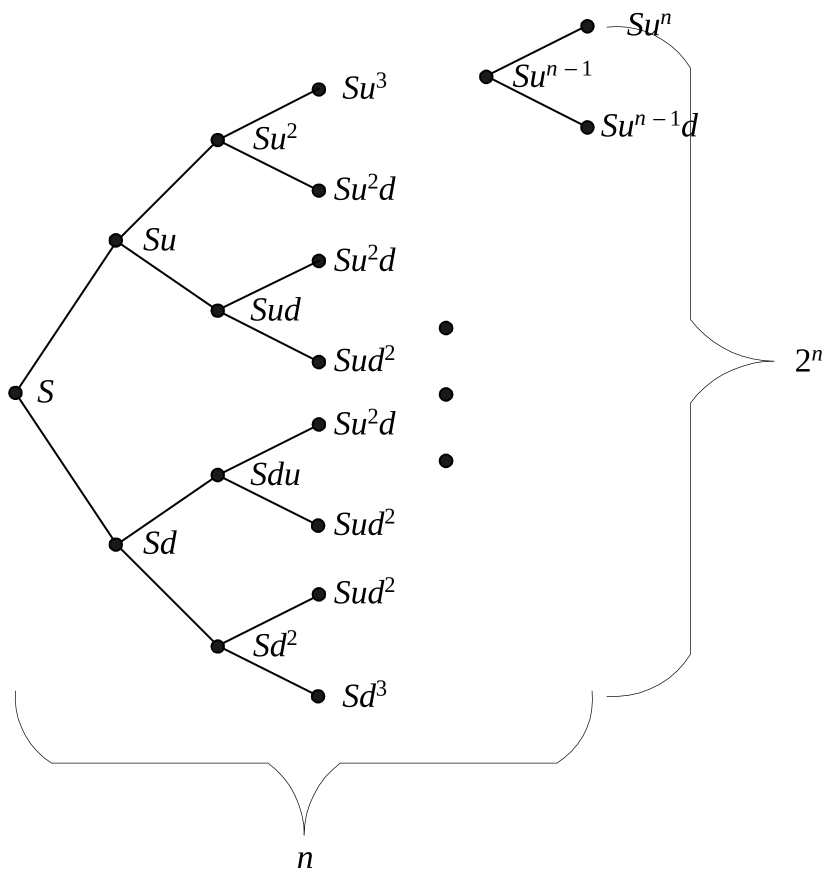
$b[\,a\,] := \binom{n}{a}\, p^a (1-p)^{n-a}$;
for $j = a+1, a+2, \ldots, n$ **do**
  $b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j)$;
end for

- It runs in $O(n)$ steps.

Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (22) on p. 220 is trivial to compute.
- We only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique *cannot* be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

On the Bushy Tree



Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As n increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.
- Any proper calibration of the model parameters makes the BOPM converge to the continuous-time model.
- We now skim through the proof.

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as n goes to infinity.
- First, $\hat{r} = r\tau/n$.
  - The period gross return $R = e^{\hat{r}}$.

Toward the Black-Scholes Formula (continued)

- Use

$$\widehat{\mu} \equiv \frac{1}{n}\, E\left[\ln \frac{S_\tau}{S}\right] \quad \text{and} \quad \widehat{\sigma}^2 \equiv \frac{1}{n}\, \text{Var}\left[\ln \frac{S_\tau}{S}\right]$$

to denote, resp., the expected value and variance of the continuously compounded rate of return per period.

- Under the BOPM, it is not hard to show that

$$\begin{aligned} \widehat{\mu} &= q \ln(u/d) + \ln d, \\ \widehat{\sigma}^2 &= q(1-q) \ln^2(u/d). \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- Assume the stock's true continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
  - Call σ the stock's (annualized) volatility.
- The BOPM converges to the distribution only if

$$
\begin{aligned}
n\widehat{\mu} &= n(q\ln(u/d) + \ln d) \to \mu\tau, \\
n\widehat{\sigma}^2 &= nq(1-q)\ln^2(u/d) \to \sigma^2\tau.
\end{aligned}
$$

- Impose $ud = 1$ to make nodes at the same horizontal level of the tree have identical price (review p. 231).
  - Other choices are possible (see text).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\,\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}\,. \quad (23)$$

- With Eqs. (23),

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right]\sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $u > R > d$ may not hold under Eqs. (23) on p. 240.
  - If this happens, the risk-neutral probability may lie outside $[0, 1]$.
- The problem disappears when n satisfies

$$e^{\sigma\sqrt{\tau/n}} > e^{r\tau/n},$$

or when $n > r^2\tau/\sigma^2$ (check it).

  - So it goes away if n is large enough.
  - Other solutions will be presented later.

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$?
- The central limit theorem says $\ln(S_\tau/S)$ converges to the normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$.
- So $\ln S_\tau$ approaches the normal distribution with mean $\mu\tau + \ln S$ and variance $\sigma^2\tau$.
- S_τ has a lognormal distribution in the limit.

Toward the Black-Scholes Formula (continued)

Lemma 7 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r - \sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability $p \equiv (e^{r\tau/n} - d)/(u - d)$.
- Let $n \to \infty$.

Toward the Black-Scholes Formula (continued)

- By Lemma 7 (p. 243) and Eq. (17) on p. 147, the expected stock price at expiration in a risk-neutral economy is $Se^{r\tau}$.
- The stock's expected annual rate of return[a] is thus the riskless rate r.

---

[a] In the sense of $(1/\tau) \ln E[\, S_\tau/S\,]$ (arithmetic average rate of return) not $(1/\tau) E[\,\ln(S_\tau/S)\,]$ (geometric average rate of return).

Toward the Black-Scholes Formula (concluded)

Theorem 8 (The Black-Scholes Formula)

$$
\begin{aligned}
C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\
P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x),
\end{aligned}
$$

where

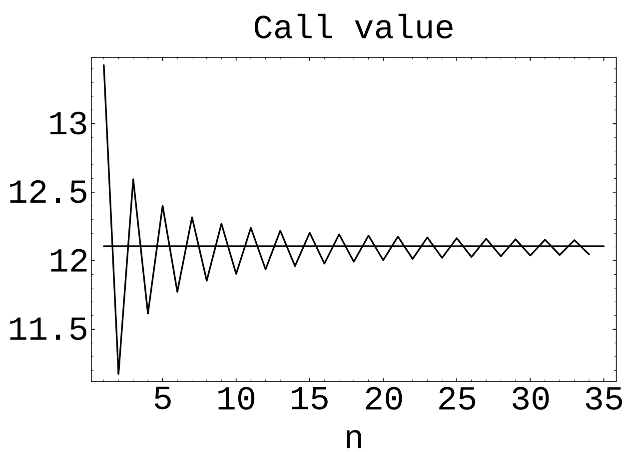
$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

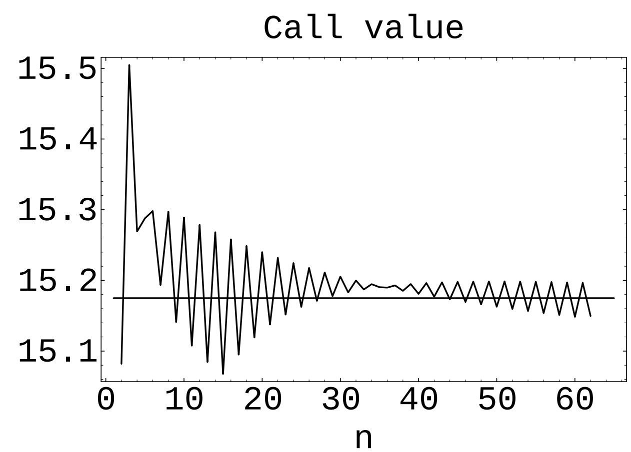
BOPM and Black-Scholes Model

- The Black-Scholes formula needs five parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take six inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$u = e^{\sigma\sqrt{\tau/n}},\ d = e^{-\sigma\sqrt{\tau/n}},\ \hat{r} = r\tau/n.$$

- The binomial tree algorithms converge reasonably fast.
- Oscillations can be dealt with by the judicious choices of u and d (see text).





$S = 100$, $X = 100$ (left), and $X = 95$ (right).

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?
- This volatility is called the implied volatility.
- Implied volatility is often preferred to historical volatility in practice.[a]

[a] It is like driving a car with your eyes on the rearview mirror?

Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.

Problems; the Smile (concluded)

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.[a]
  - σ measures the volatility of stock price one year from now (regardless of what happens in between).
- How to incorporate these two different ways of day count into the Black-Scholes formula and binomial tree algorithms?

[a]Fama (1965); French (1980); French and Roll (1986).

Trading Days and Calendar Days (concluded)

- Suppose a year has 260 trading days.
- A quick and dirty way is to replace σ with[a]

$$\sigma \sqrt{\frac{365}{260} \, \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}\,.$$

- How about binomial tree algorithms?

[a] French (1984).

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs

$$\max(0, X - Su^j d^{n-j})$$

  and applies backward induction.
- At each intermediate node, it checks for early exercise by comparing the payoff if exercised with the continuation value.

Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- The only exception is early exercise is considered for only those nodes when early exercise is permitted.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the prevailing stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\,u$, $(Su - D)\,d$, $(Sd - D)\,u$, $(Sd - D)\,d$.
  - The binomial tree no longer combines.

$$
\begin{array}{ccccc}
 & & & & (Su-D)\,u \\
 & & & \nearrow & \\
 & & Su-D & & \\
 & \nearrow & & \searrow & \\
 & & & & (Su-D)\,d \\
S & & & & \\
 & & & & (Sd-D)\,u \\
 & \searrow & & \nearrow & \\
 & & Sd-D & & \\
 & & & \searrow & \\
 & & & & (Sd-D)\,d
\end{array}
$$

An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends (Roll, 1977).
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - σ equal to the volatility of the process followed by the risky component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all *future* dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

A General Approach[a]

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases.
- Other approaches include adjusting σ and approximating the known dividend with a dividend yield.

---

[a]Dai and Lyuu (2004).

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.

Continuous Dividend Yields (continued)

- The Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$ (Merton, 1973):

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{24}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{24'}$$

where

$$x \equiv \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (24) and (24') remain valid as long as the dividend yield is predictable.
- Replace q with the average annualized dividend yield.

Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace u with $ue^{-q\Delta t}$ and d with $de^{-q\Delta t}$, where $\Delta t \equiv \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- Other than the changes, binomial tree algorithms stay the same.
  - In particular, p should use the *original* u and d.[a]

[a] Contributed by Ms. Chuan-Ju Wang (R95922018) on May 2, 2007.

Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \tag{25}$$

where $\Delta t \equiv \tau/n$.

  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- The u and d remain unchanged.

- Other than the change in Eq. (25), binomial tree algorithms stay the same.