Bond Price Volatility

"Well, Beethoven, what is this?"
— Attributed to Prince Anton Esterházy

Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest-rate-sensitive securities.
- Assume level-coupon bonds throughout.

Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?
- Define price volatility by

$$-\frac{\frac{\partial P}{\partial y}}{P}.$$

Price Volatility of Bonds

- The price volatility of a coupon bond is
$$-\frac{(C/y)\,n-\left(C/y^2\right)\left(((1+y)^{n+1}-(1+y)\right)-nF}{(C/y)\,((1+y)^{n+1}-(1+y))+F(1+y)}.$$
  - F is the par value.
  - C is the coupon payment per period.
- For bonds without embedded options,
$$-\frac{\frac{\partial P}{\partial y}}{P}>0.$$

Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' PVs divided by the asset's price.
- Formally,

$$\text{MD} \equiv \frac{1}{P} \sum_{i=1}^{n} \frac{iC_i}{(1+y)^i}.$$

- The Macaulay duration, in periods, is equal to

$$\text{MD} = -(1+y)\,\frac{\partial P}{\partial y}\,\frac{1}{P}. \tag{7}$$

MD of Bonds

- The MD of a coupon bond is

$$\text{MD} = \frac{1}{P}\left[\sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n}\right]. \tag{8}$$

- It can be simplified to

$$\text{MD} = \frac{c(1+y)\,[\,(1+y)^n - 1\,] + ny(y-c)}{cy\,[\,(1+y)^n - 1\,] + y^2},$$

where c is the period coupon rate.

- The MD of a zero-coupon bond equals its term to maturity n.

- The MD of a coupon bond is less than its maturity.

Finesse

- Equations (7) on p. 76 and (8) on p. 77 hold only if the coupon C, the par value F, and the maturity n are all independent of the yield y.
  - That is, if the cash flow is independent of yields.
- To see this point, suppose the market yield declines.
- The MD will be lengthened.
- But for securities whose maturity actually decreases as a result, the MD (as originally defined) may actually decrease.

How Not To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- The MD should be seen mainly as measuring *price volatility*.
- Many, if not most, duration-related terminology cannot be comprehended otherwise.

Conversion

- For the MD to be year-based, modify Eq. (8) on p. 77 to

$$\frac{1}{P}\left[\sum_{i=1}^{n}\frac{i}{k}\,\frac{C}{\left(1+\frac{y}{k}\right)^{i}}+\frac{n}{k}\,\frac{F}{\left(1+\frac{y}{k}\right)^{n}}\right],$$

  where y is the *annual* yield and k is the compounding frequency per annum.

- Equation (7) on p. 76 also becomes

$$\text{MD}=-\left(1+\frac{y}{k}\right)\frac{\partial P}{\partial y}\,\frac{1}{P}.$$

- By definition, MD (in years) $=\dfrac{\text{MD (in periods)}}{k}$.

Modified Duration

- Modified duration is defined as

$$\text{modified duration} \equiv -\frac{\partial P}{\partial y}\frac{1}{P} = \frac{\text{MD}}{(1+y)}. \tag{9}$$

- By Taylor expansion,

$$\text{percent price change} \approx -\text{modified duration} \times \text{yield change}.$$

Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

Modified Duration of a Portfolio

- The modified duration of a portfolio equals

$$\sum_i \omega_i D_i.$$

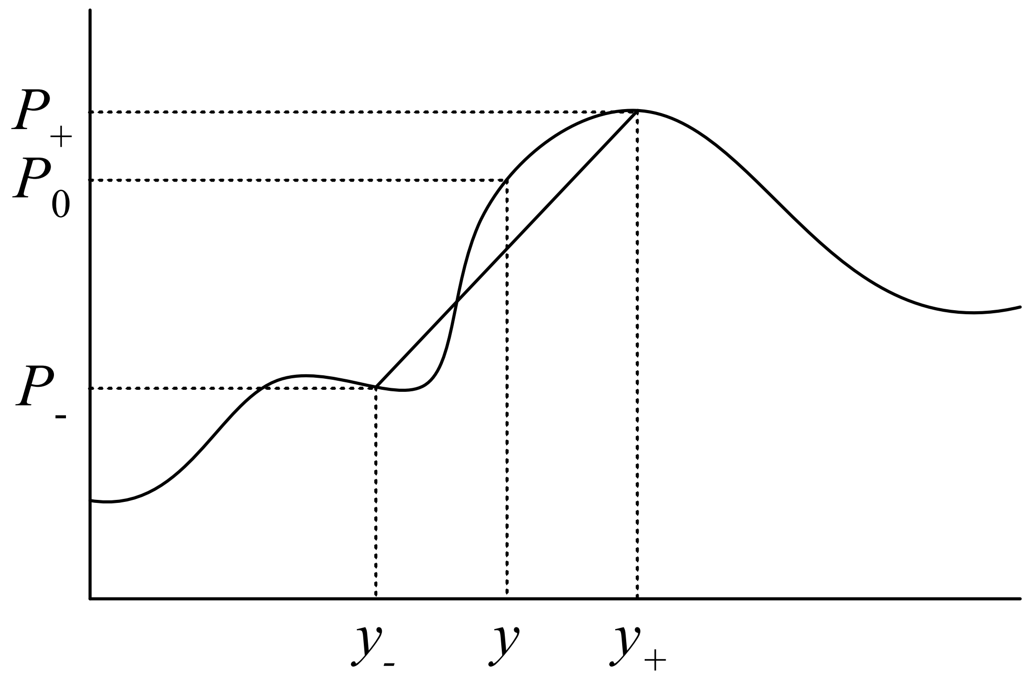
  - D_i is the modified duration of the ith asset.
  - ω_i is the market value of that asset expressed as a percentage of the market value of the portfolio.

Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

$$\frac{P_- - P_+}{P_0(y_+ - y_-)}.$$

  - P_- is the price if the yield is decreased by Δy.
  - P_+ is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.
- See plot on p. 85.



P_+
P_0
P_-
y_-
y
y_+

Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use

$$\frac{P_0 - P_+}{P_0\,\Delta y}.$$

  - More economical but less accurate.

The Practices

- Duration is usually expressed in percentage terms—call it $D_{\%}$—for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by

$$-D_{\%} \times \Delta r$$

when the yield increases instantaneously by $\Delta r\%$.

  – Price will drop by 20% if $D_{\%} = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.

- In fact, $D_{\%}$ equals modified duration as originally defined (prove it!).

Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as

$$\text{modified duration} \times \text{price} \ (\% \text{ of par}) = -\frac{\partial P}{\partial y}.$$

- The approximate dollar price change per \$100 of par value is

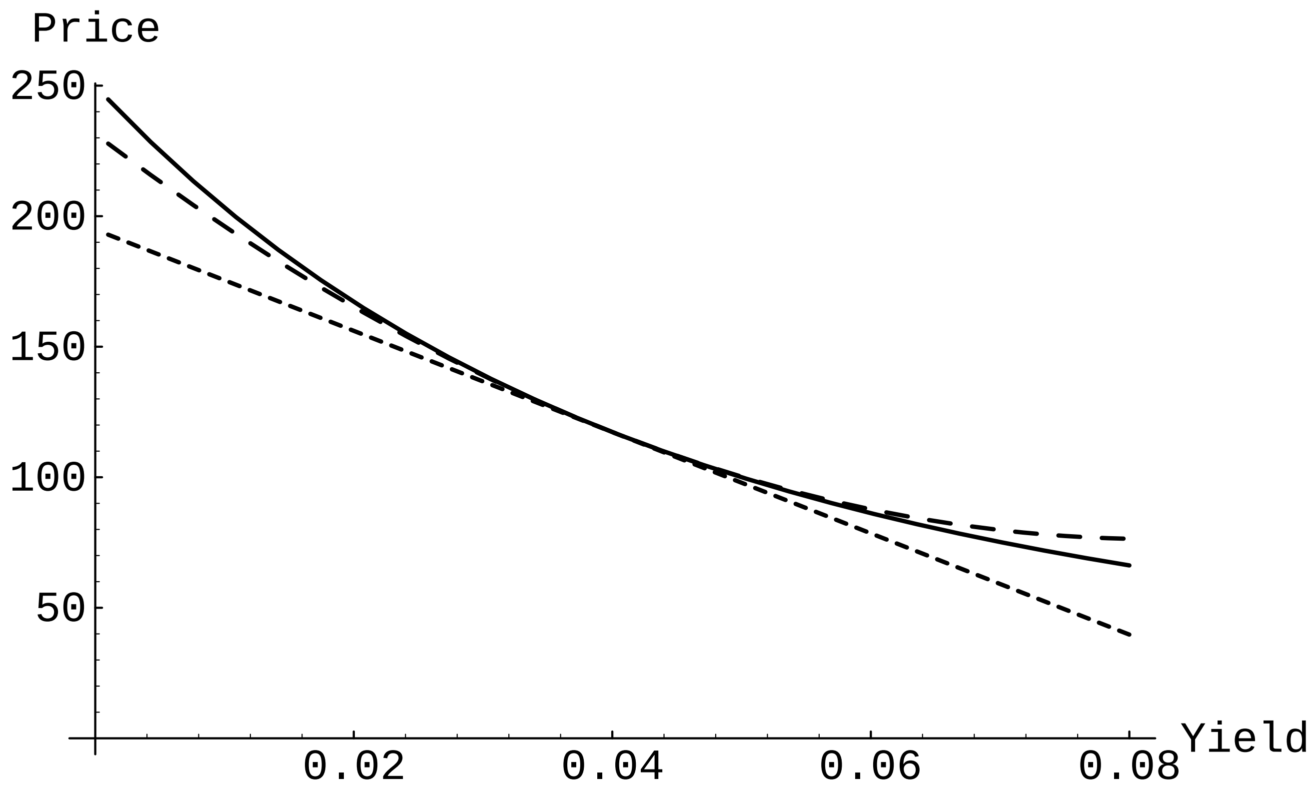
$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$

Convexity

- Convexity is defined as

$$\text{convexity (in periods)} \equiv \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.$$

- The convexity of a coupon bond is positive (prove it!).
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).
- Hence, between two bonds with the same duration, the one with a higher convexity is more valuable.


Price
250
200
150
100
50
0.02
0.04
0.06
0.08
Yield

Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

$$\textsf{convexity (in years)} = \frac{\textsf{convexity (in periods)}}{k^2}$$

when there are k periods per annum.

Use of Convexity

- The approximation $\Delta P/P \approx -$ duration $\times$ yield change works for small yield changes.
- To improve upon it for larger yield changes, use

$$\begin{aligned} \frac{\Delta P}{P} &\approx \frac{\partial P}{\partial y}\frac{1}{P}\,\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\frac{1}{P}\,(\Delta y)^2 \\ &= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2. \end{aligned}$$

- Recall the figure on p. 90.

The Practices

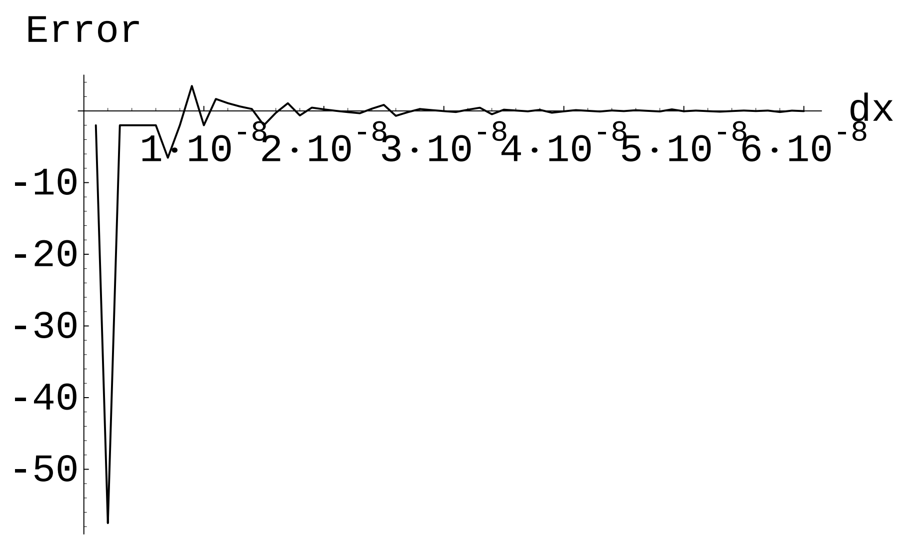
- Convexity is usually expressed in percentage terms—call it $C_{\%}$—for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by $-D_{\%} \times \Delta r + C_{\%} \times (\Delta r)^2/2$ when the yield increases instantaneously by $\Delta r\%$.
  - Price will drop by 17% if $D_{\%} = 10$, $C_{\%} = 1.5$, and $\Delta r = 2$ because
$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$
- In fact, $C_{\%}$ equals convexity divided by 100 (prove it!).

Effective Convexity

- The effective convexity is defined as
$$\frac{P_+ + P_- - 2P_0}{P_0\,(0.5\times(y_+ - y_-))^2},$$
  - P_- is the price if the yield is decreased by Δy.
  - P_+ is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.
- Effective convexity is most relevant when a bond's cash flow is interest rate sensitive.
- Numerically, choosing the right Δy is a delicate matter.

Approximate $d^2 f(x)^2/dx^2$ at $x = 1$, Where $f(x) = x^2$

The difference of $((1+\Delta x)^2 + (1-\Delta x)^2 - 2)/(\Delta x)^2$ and 2:



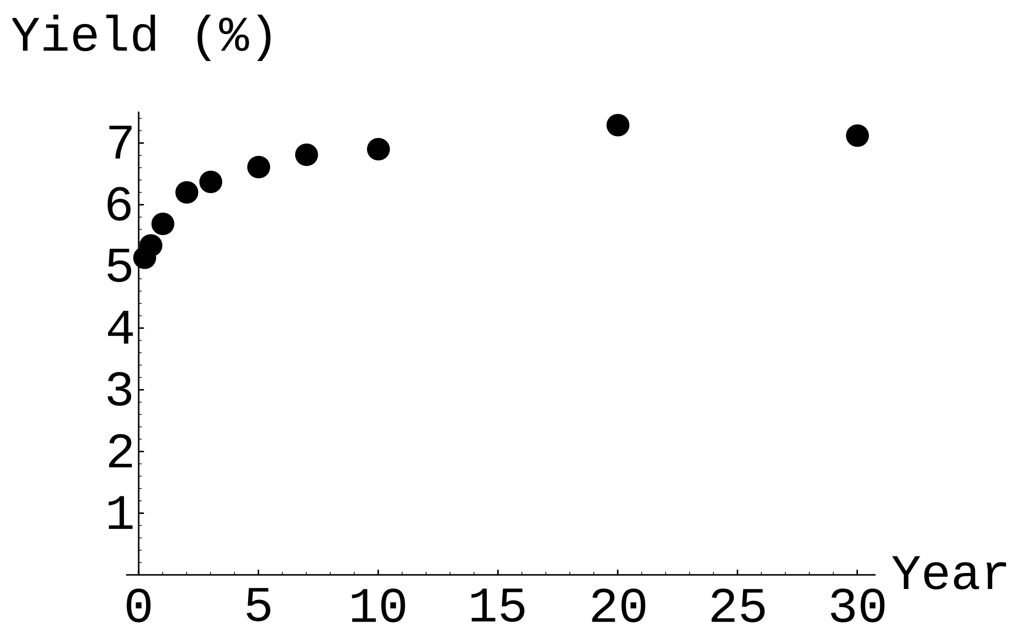
Term Structure of Interest Rates

Why is it that the interest of money is lower,
when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don't lend it at interest.
Rather, give [it] to someone
from whom you won't get it back.
— Thomas Gospel 95

Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds forms the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.


Yield (%)
7
6
5
4
3
2
1
0
5
10
15
20
25
30
Year

Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots yields to maturity against maturity.
- A par yield curve is constructed from bonds trading near par.

Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

Spot Rates

- The i-period spot rate $S(i)$ is the yield to maturity of an i-period zero-coupon bond.
- The PV of one dollar i periods from now is

$$[\,1 + S(i)\,]^{-i}.$$

- The one-period spot rate is called the short rate.
- Spot rate curve: Plot of spot rates against maturity.

Problems with the PV Formula

- In the bond price formula,

$$\sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n},$$

every cash flow is discounted at the same yield y.

- Consider two riskless bonds with different yields to maturity because of their different cash flow streams:

$$\sum_{i=1}^{n_1} \frac{C}{(1+y_1)^i} + \frac{F}{(1+y_1)^{n_1}},$$

$$\sum_{i=1}^{n_2} \frac{C}{(1+y_2)^i} + \frac{F}{(1+y_2)^{n_2}}.$$

Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.
- But shouldn't they be discounted at the same rate?

Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the ith bond paying C_i dollars at time i.
- So a level-coupon bond has the price

$$P = \sum_{i=1}^{n} \frac{C}{[\,1 + S(i)\,]^i} + \frac{F}{[\,1 + S(n)\,]^n}. \tag{10}$$

- This pricing method incorporates information from the term structure.
- Discount each cash flow at the corresponding spot rate.

Discount Factors

- In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of
$$P = \sum_{i=1}^{n} C_i d(i).$$
  - Above, $d(i) \equiv [\,1 + S(i)\,]^{-i}$, $i = 1, 2, \ldots, n$, are called discount factors.
  - $d(i)$ is the PV of one dollar i periods from now.
- The discount factors are often interpolated to form a continuous function called the discount function.

Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.
- Compute $S(2)$ from the two-period coupon bond price P by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[\,1 + S(2)\,]^2}.$$

Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price P of the n-period coupon bond and $S(1), S(2), \dots, S(n-1)$.
- Then $S(n)$ can be computed from Eq. (10) on p. 105, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[\,1 + S(i)\,]^i} + \frac{F}{[\,1 + S(n)\,]^n}.$$

- The running time is $O(n)$ (see text).
- The procedure is called bootstrapping.

Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.
  - Any economic justifications?

Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for P.
- Were this bond riskless, it would fetch

$$P^* = \sum_{t=1}^{n} \frac{C_t}{[\,1 + S(t)\,]^t}.$$

- Since riskiness must be compensated, $P < P^*$.
- Yield spread is the difference between the IRR of the risky bond and that of a riskless bond with comparable maturity.

Static Spread

- The static spread is the amount s by which the spot rate curve has to shift in parallel to price the risky bond:

$$P = \sum_{t=1}^{n} \frac{C_t}{[\,1 + s + S(t)\,]^t}.$$

- Unlike the yield spread, the static spread incorporates information from the term structure.

Of Spot Rate Curve and Yield Curve

- y_k: yield to maturity for the k-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But this is only a trend not a mathematical truth.[a]

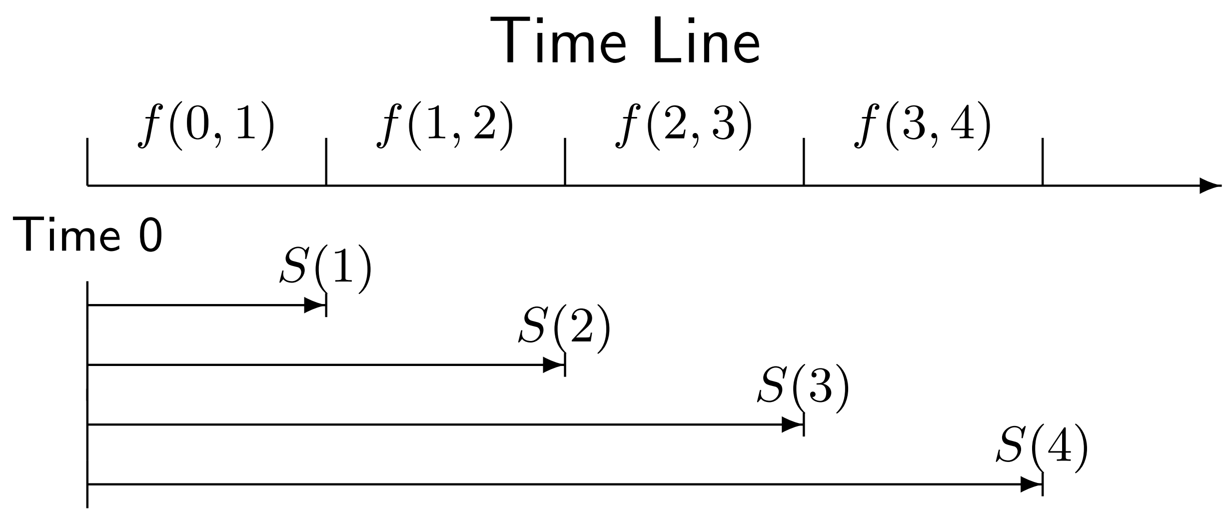
[a] See a counterexample in the text.

Forward Rates

- The yield curve contains information regarding future interest rates currently "expected" by the market.
- Invest \$1 for j periods to end up with $[\,1 + S(j)\,]^j$ dollars at time j.
  - The maturity strategy.
- Invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j - i$ periods where $j > i$.
- Will have $[\,1 + S(i)\,]^i[\,1 + S(i,j)\,]^{j-i}$ dollars at time j.
  - $S(i,j)$: $(j - i)$-period spot rate i periods from now.
  - The rollover strategy.

Forward Rates (concluded)

- When $S(i,j)$ equals
$$f(i,j) \equiv \left[\frac{(1+S(j))^j}{(1+S(i))^i} \right]^{1/(j-i)} - 1, \tag{11}$$
we will end up with $[\,1+S(j)\,]^j$ dollars again.
- By definition, $f(0,j) = S(j)$.
- $f(i,j)$ is called the (implied) forward rates.
  - More precisely, the $(j-i)$-period forward rate i periods from now.



Time Line
$f(0,1)$
$f(1,2)$
$f(2,3)$
$f(3,4)$
Time 0
$S(1)$
$S(2)$
$S(3)$
$S(4)$

Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate two investment strategies.
- $f(i, i+1)$ are instantaneous forward rates or one-period forward rates.

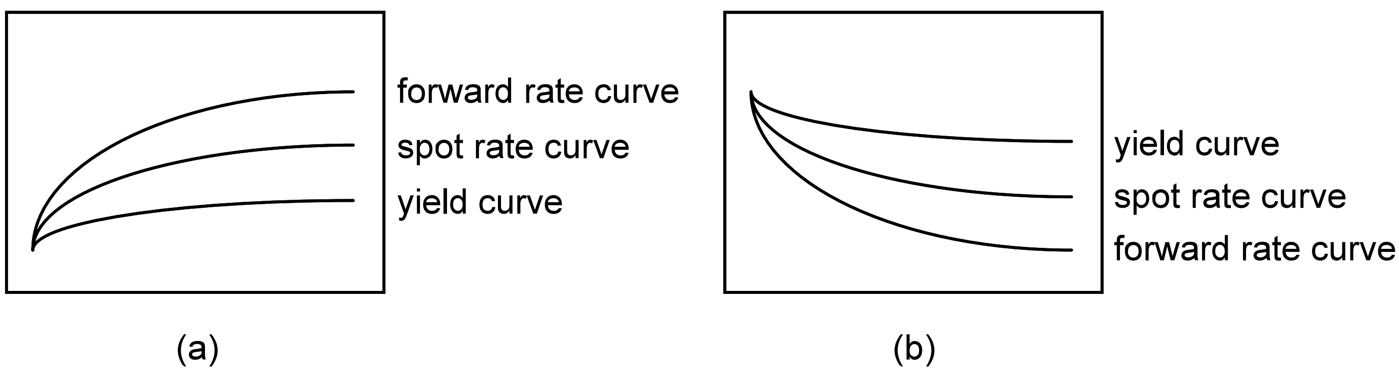
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i, j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i, j) < S(j) < \cdots < S(i).$$


forward rate curve
spot rate curve
yield curve
(a)
yield curve
spot rate curve
forward rate curve
(b)

Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of \$1 at time n can be derived in two ways.
- Buy n-period zero-coupon bonds and receive

$$[\,1 + S(n)\,]^n.$$

- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is

$$[\,1 + S(1)\,][\,1 + f(1,2)\,]\cdots[\,1 + f(n-1,n)\,].$$

Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curves (concluded)

- Since they are identical,

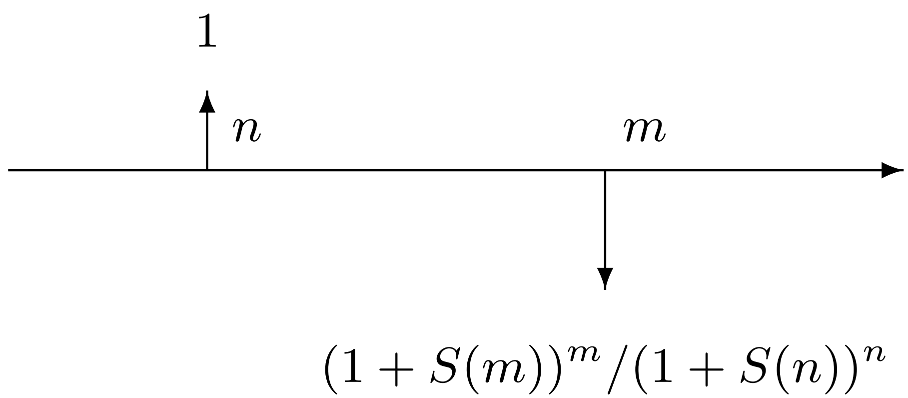
$$\begin{aligned} S(n) = \{ [\,1 + S(1)\,][\,1 + f(1,2)\,] \\ \cdots [\,1 + f(n-1,n)\,] \}^{1/n} - 1. \end{aligned} \tag{12}$$

- Hence, the forward rates, specifically the one-period forward rates, determine the spot rate curve.
- Other equivalencies can be derived similarly, such as

$$f(T, T+1) = \frac{d(T)}{d(T+1)} - 1.$$

Locking in the Forward Rate $f(n, m)$

- Buy one n-period zero-coupon bond for $1/(1 + S(n))^n$.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ m-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow $1/(1 + S(n))^n$.
- At time n there will be a cash inflow of \$1.
- At time m there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.
- This implies the rate $f(n, m)$ between times n and m.


1
n
m
(1 + S(m))^m/(1 + S(n))^n

Forward Contracts

- We generated the cash flow of a financial instrument called forward contract.
- Agreed upon today, it enables one to borrow money at time n in the future and repay the loan at time $m > n$ with an interest rate equal to the forward rate

$$f(n, m).$$

- Can the spot rate curve be an arbitrary curve?[a]

[a]Contributed by Mr. Dai, Tian-Shyr (R86526008, D8852600) in 1998.

Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} C e^{-iS(i)} + F e^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

Spot and Forward Rates under Continuous Compounding (concluded)

- The formula for the forward rate:

$$f(i,j) = \frac{jS(j) - iS(i)}{j - i}.$$

- The one-period forward rate:

$$f(j,j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

- $$f(T) \equiv \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T\,\frac{\partial S}{\partial T}.$$

- $f(T) > S(T)$ if and only if $\partial S / \partial T > 0$.

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a,b) = E[\,S(a,b)\,]. \tag{13}$$

- Does not imply that the forward rate is an accurate predictor for the future spot rate.
- Implies the maturity strategy and the rollover strategy produce the same result at the horizon on the average.

Unbiased Expectations Theory and Spot Rate Curve

- Implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from Eq. (11) on p. 115.
  - So $E[\, S(j, j+1)\,] > S(j+1) > \cdots > S(1)$ if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

More Implications

- The theory has been rejected by most empirical studies with the possible exception of the period prior to 1915.
- Since the term structure has been upward sloping about 80% of the time, the theory would imply that investors have expected interest rates to rise 80% of the time.
- Riskless bonds, regardless of their different maturities, are expected to earn the same return on the average.
- That would mean investors are indifferent to risk.

A "Bad" Expectations Theory

- The expected returns on all possible riskless bond strategies are equal for *all* holding periods.
- So

$$(1 + S(2))^2 = (1 + S(1))\, E[\, 1 + S(1, 2)\,] \tag{14}$$

  because of the equivalency between buying a two-period bond and rolling over one-period bonds.
- After rearrangement,

$$\frac{1}{E[\, 1 + S(1, 2)\,]} = \frac{1 + S(1)}{(1 + S(2))^2}.$$

A "Bad" Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond and sells it after one period.
  - The expected return is $E[\,(1+S(1,2))^{-1}\,]\,(1+S(2))^2$.
  - Strategy two buys a one-period bond with a return of $1+S(1)$.
- The theory says the returns are equal:

$$\frac{1+S(1)}{(1+S(2))^2}=E\left[\frac{1}{1+S(1,2)}\right].$$

A "Bad" Expectations Theory (concluded)

- Combine this with Eq. (14) on p. 130 to obtain

$$E\left[\frac{1}{1+S(1,2)}\right]=\frac{1}{E[\,1+S(1,2)\,]}.$$

- But this is impossible save for a certain economy.
  - Jensen's inequality states that $E[\,g(X)\,]>g(E[\,X\,])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x)>0$).
  - Use $g(x)\equiv(1+x)^{-1}$ to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over *a single period* equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}} = 1+S(1) \quad \text{for } \textit{all } n>1.$$

- This theory is the basis of many interest rate models.

Duration Revisited

- To handle more general types of spot rate curve changes, define a vector $[\, c_1, c_2, \ldots, c_n \,]$ that characterizes the perceived type of change.
  - Parallel shift: $[\, 1, 1, \ldots, 1 \,]$.
  - Twist: $[\, 1, 1, \ldots, 1, -1, \ldots, -1 \,]$.
  - $\cdots$
- Let $P(y) \equiv \sum_i C_i / (1 + S(i) + y c_i)^i$ be the price associated with the cash flow $C_1, C_2, \ldots$.
- Define duration as

$$-\frac{\partial P(y)/P(0)}{\partial y}\Bigg|_{y=0}.$$