Bond Price Volatility
“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy
Price Volatility

• Volatility measures how bond prices respond to interest rate changes.

• It is key to the risk management of interest-rate-sensitive securities.

• Assume level-coupon bonds throughout.
Price Volatility (concluded)

• What is the sensitivity of the percentage price change to changes in interest rates?

• Define price volatility by

\[ \frac{\partial P}{\partial y} \cdot \frac{1}{P} . \]
Price Volatility of Bonds

• The price volatility of a coupon bond is

\[
\left(\frac{C}{y}\right)^n - \left(\frac{C}{y^2}\right) \left((1 + y)^{n+1} - (1 + y)\right) - nF
\]

\[
\frac{\left(\frac{C}{y}\right)^n - \left(\frac{C}{y^2}\right) \left((1 + y)^{n+1} - (1 + y)\right) - nF}{\left(\frac{C}{y}\right) \left((1 + y)^{n+1} - (1 + y)\right) + F(1 + y)}.
\]

- \( F \) is the par value.
- \( C \) is the coupon payment per period.

• For bonds without embedded options,

\[
- \frac{\partial P}{\partial y} \frac{P}{P} > 0.
\]
Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.

- The weights are the cash flows’ PVs divided by the asset’s price.

- Formally,

\[
MD \equiv \frac{1}{P} \sum_{i=1}^{n} \frac{iC_i}{(1 + y)^i}.
\]

- The Macaulay duration, in periods, is equal to

\[
MD = -(1 + y) \frac{\partial P}{\partial y} \frac{1}{P}.
\] (7)
MD of Bonds

• The MD of a coupon bond is

\[
MD = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1 + y)^i} + \frac{nF}{(1 + y)^n} \right].
\] (8)

• It can be simplified to

\[
MD = \frac{c(1 + y) \left( (1 + y)^n - 1 \right) + ny(y - c)}{cy \left( (1 + y)^n - 1 \right) + y^2},
\]

where \( c \) is the period coupon rate.

• The MD of a zero-coupon bond equals its term to maturity \( n \).

• The MD of a coupon bond is less than its maturity.
Finesse

• Equations (7) on p. 76 and (8) on p. 77 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
  – That is, if the cash flow is independent of yields.

• To see this point, suppose the market yield declines.

• The MD will be lengthened.

• But for securities whose maturity actually decreases as a result, the MD (as originally defined) may actually decrease.
How Not To Think about MD

• The MD has its origin in measuring the length of time a bond investment is outstanding.

• The MD should be seen mainly as measuring *price volatility*.

• Many, if not most, duration-related terminology cannot be comprehended otherwise.
Conversion

• For the MD to be year-based, modify Eq. (8) on p. 77 to

\[
\frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} \frac{C}{(1 + \frac{y}{k})^i} + \frac{n}{k} \frac{F}{(1 + \frac{y}{k})^n} \right],
\]

where \( y \) is the annual yield and \( k \) is the compounding frequency per annum.

• Equation (7) on p. 76 also becomes

\[
MD = - \left( 1 + \frac{y}{k} \right) \frac{\partial P}{\partial y} \frac{1}{P}.
\]

• By definition, MD (in years) = \( \frac{MD \text{ (in periods)}}{k} \).
Modified Duration

• Modified duration is defined as

\[
\text{modified duration} \equiv -\frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}. \tag{9}
\]

• By Taylor expansion,

percent price change \approx -\text{modified duration} \times \text{yield change}.  

Example

• Consider a bond whose modified duration is 11.54 with a yield of 10%.

• If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$
Modified Duration of a Portfolio

- The modified duration of a portfolio equals

\[ \sum_i \omega_i D_i. \]

- \( D_i \) is the modified duration of the \( i \)th asset.
- \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

\[
P_\text{-} \frac{P_\text{-} - P_\text{+}}{P_0 (y_\text{+} - y_\text{-})}.
\]

- \( P_\text{-} \) is the price if the yield is decreased by \( \Delta y \).
- \( P_\text{+} \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.

- See plot on p. 85.
Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use
  \[
  \frac{P_0 - P_+}{P_0 \Delta y}.
  \]
  - More economical but less accurate.
The Practices

• Duration is usually expressed in percentage terms—call it $D\%$—for quick mental calculation.

• The percentage price change expressed in percentage terms is approximated by

$$-D\% \times \Delta r$$

when the yield increases instantaneously by $\Delta r\%$.

– Price will drop by 20% if $D\% = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.

• In fact, $D\%$ equals modified duration as originally defined (prove it!).
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as

  \[
  \text{modified duration} \times \text{price (\% of par)} = -\frac{\partial P}{\partial y}.
  \]

- The approximate dollar price change per $100 of par value is

  \[
  \text{price change} \approx -\text{dollar duration} \times \text{yield change}.
  \]
Convexity

- Convexity is defined as
  
  \[ \text{convexity (in periods)} \equiv \frac{\partial^2 P}{\partial y^2} \frac{1}{P}. \]

- The convexity of a coupon bond is positive (prove it!).

- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

- Hence, between two bonds with the same duration, the one with a higher convexity is more valuable.
Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

\[
\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}
\]

when there are \(k\) periods per annum.
Use of Convexity

• The approximation $\Delta P/P \approx -\text{duration} \times \text{yield change}$ works for small yield changes.

• To improve upon it for larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2$$

$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$ 

• Recall the figure on p. 90.
The Practices

• Convexity is usually expressed in percentage terms—call it $C\%$—for quick mental calculation.

• The percentage price change expressed in percentage terms is approximated by $-D\% \times \Delta r + C\% \times (\Delta r)^2/2$ when the yield increases instantaneously by $\Delta r\%$.
  
  – Price will drop by 17% if $D\% = 10$, $C\% = 1.5$, and $\Delta r = 2$ because
  
  $$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$ 

• In fact, $C\%$ equals convexity divided by 100 (prove it!).
Effective Convexity

- The effective convexity is defined as
  \[
  \frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2},
  \]
  - \(P_-\) is the price if the yield is decreased by \(\Delta y\).
  - \(P_+\) is the price if the yield is increased by \(\Delta y\).
  - \(P_0\) is the initial price, \(y\) is the initial yield.
  - \(\Delta y\) is small.

- Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

- Numerically, choosing the right \(\Delta y\) is a delicate matter.
Approximate $d^2 f(x)^2/dx^2$ at $x = 1$, Where $f(x) = x^2$

The difference of $( (1 + \Delta x)^2 + (1 - \Delta x)^2 - 2)/(\Delta x)^2$ and 2:
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest.
Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds forms the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.
Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots yields to maturity against maturity.
- A par yield curve is constructed from bonds trading near par.
Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.
- The PV of one dollar $i$ periods from now is $[1 + S(i)]^{-i}$.
- The one-period spot rate is called the short rate.
- Spot rate curve: Plot of spot rates against maturity.
Problems with the PV Formula

- In the bond price formula,
  \[ \sum_{i=1}^{n} \frac{C}{(1 + y)^i} + \frac{F}{(1 + y)^n}, \]
every cash flow is discounted at the same yield \( y \).

- Consider two riskless bonds with different yields to maturity because of their different cash flow streams:
  \[ \sum_{i=1}^{n_1} \frac{C}{(1 + y_1)^i} + \frac{F}{(1 + y_1)^{n_1}}, \]
  \[ \sum_{i=1}^{n_2} \frac{C}{(1 + y_2)^i} + \frac{F}{(1 + y_2)^{n_2}}. \]
Problems with the PV Formula (concluded)

• The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.

• But shouldn’t they be discounted at the same rate?
Spot Rate Discount Methodology

• A cash flow \( C_1, C_2, \ldots, C_n \) is equivalent to a package of zero-coupon bonds with the \( i \)th bond paying \( C_i \) dollars at time \( i \).

• So a level-coupon bond has the price

\[
P = \sum_{i=1}^{n} \frac{C}{(1 + S(i))^i} + \frac{F}{(1 + S(n))^n}. \tag{10}
\]

• This pricing method incorporates information from the term structure.

• Discount each cash flow at the corresponding spot rate.
Discount Factors

• In general, any riskless security having a cash flow \( C_1, C_2, \ldots, C_n \) should have a market price of

\[
P = \sum_{i=1}^{n} C_i d(i).
\]

  – Above, \( d(i) \equiv [1 + S(i)]^{-i}, i = 1, 2, \ldots, n \), are called discount factors.

  – \( d(i) \) is the PV of one dollar \( i \) periods from now.

• The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.
- Compute $S(2)$ from the two-period coupon bond price $P$ by solving

\[
P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.
\]
Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price $P$ of the $n$-period coupon bond and $S(1), S(2), \ldots, S(n-1)$.

- Then $S(n)$ can be computed from Eq. (10) on p. 105, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.$$

- The running time is $O(n)$ (see text).

- The procedure is called bootstrapping.
Some Problems

• Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).

• Some maturities might be missing from the data points (the incompleteness problem).

• Treasuries might not be of the same quality.

• Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.
  – Any economic justifications?
Yield Spread

• Consider a risky bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.

• Were this bond riskless, it would fetch

$$P^* = \sum_{t=1}^{n} \frac{C_t}{[1 + S(t)]^t}.$$ 

• Since riskiness must be compensated, $P < P^*$.

• Yield spread is the difference between the IRR of the risky bond and that of a riskless bond with comparable maturity.
Static Spread

- The static spread is the amount $s$ by which the spot rate curve has to shift in parallel to price the risky bond:

$$P = \sum_{t=1}^{n} \frac{C_t}{[1 + s + S(t)]^t}.$$ 

- Unlike the yield spread, the static spread incorporates information from the term structure.
Of Spot Rate Curve and Yield Curve

- $y_k$: yield to maturity for the $k$-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).

- If the yield curve is flat, the spot rate curve coincides with the yield curve.
Shapes

• The spot rate curve often has the same shape as the yield curve.
  – If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).

• But this is only a trend not a mathematical truth.\(^a\)

\(^a\)See a counterexample in the text.
Forward Rates

bullet The yield curve contains information regarding future interest rates currently “expected” by the market.

bullet Invest $1 for \(j\) periods to end up with \([1 + S(j)]^j\) dollars at time \(j\).
  - The maturity strategy.

bullet Invest $1 in bonds for \(i\) periods and at time \(i\) invest the proceeds in bonds for another \(j - i\) periods where \(j > i\).

bullet Will have \([1 + S(i)]^i[1 + S(i, j)]^{j-i}\) dollars at time \(j\).
  - \(S(i, j)\): \((j - i)\)-period spot rate \(i\) periods from now.
  - The rollover strategy.
Forward Rates (concluded)

• When \( S(i, j) \) equals

\[
f(i, j) \equiv \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1,
\]

we will end up with \( [1 + S(j)]^j \) dollars again.

• By definition, \( f(0, j) = S(j) \).

• \( f(i, j) \) is called the (implied) forward rates.
  – More precisely, the \( (j - i) \)-period forward rate \( i \) periods from now.
**Time Line**

$f(0, 1)$ | $f(1, 2)$ | $f(2, 3)$ | $f(3, 4)$

Time 0

$S(1)$  
$S(2)$  
$S(3)$  
$S(4)$
Forward Rates and Future Spot Rates

• We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  – This is the subject of the term structure theories.

• We merely looked for the future spot rate that, if realized, will equate two investment strategies.

• $f(i, i + 1)$ are instantaneous forward rates or one-period forward rates.
Spot Rates and Forward Rates

• When the spot rate curve is normal, the forward rate dominates the spot rates,

\[ f(i, j) > S(j) > \cdots > S(i). \]

• When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

\[ f(i, j) < S(j) < \cdots < S(i). \]
(a) forward rate curve
    spot rate curve
    yield curve

(b) yield curve
    spot rate curve
    forward rate curve
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of $1$ at time $n$ can be derived in two ways.
- Buy $n$-period zero-coupon bonds and receive
  \[ [1 + S(n)]^n. \]
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is
  \[ [1 + S(1)][1 + f(1, 2)] \cdots [1 + f(n - 1, n)]. \]
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curves

(concluded)

- Since they are identical,

\[
S(n) = \left\{ [1 + S(1)] [1 + f(1, 2)] \right\} \cdots \left\{ [1 + f(n - 1, n)] \right\}^{1/n} - 1. \tag{12}
\]

- Hence, the forward rates, specifically the one-period forward rates, determine the spot rate curve.

- Other equivalencies can be derived similarly, such as

\[
f(T, T + 1) = \frac{d(T)}{d(T + 1)} - 1.
\]
Locking in the Forward Rate $f(n, m)$

- Buy one $n$-period zero-coupon bond for $1/(1 + S(n))^n$.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ $m$-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow $1/(1 + S(n))^n$.
- At time $n$ there will be a cash inflow of $1$.
- At time $m$ there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.
- This implies the rate $f(n, m)$ between times $n$ and $m$. 
\[ \frac{(1 + S(m))^m}{(1 + S(n))^n} \]
Forward Contracts

• We generated the cash flow of a financial instrument called forward contract.

• Agreed upon today, it enables one to borrow money at time $n$ in the future and repay the loan at time $m > n$ with an interest rate equal to the forward rate $f(n, m)$.

• Can the spot rate curve be an arbitrary curve?\(^a\)

\(^a\)Contributed by Mr. Dai, Tian-Shyr (R86526008, D8852600) in 1998.
Spot and Forward Rates under Continuous Compounding

- The pricing formula:
  \[ P = \sum_{i=1}^{n} C e^{-iS(i)} + F e^{-nS(n)}. \]

- The market discount function:
  \[ d(n) = e^{-nS(n)}. \]

- The spot rate is an arithmetic average of forward rates,
  \[ S(n) = \frac{f(0, 1) + f(1, 2) + \cdots + f(n-1, n)}{n}. \]
Spot and Forward Rates under Continuous Compounding (concluded)

• The formula for the forward rate:

\[ f(i, j) = \frac{jS(j) - iS(i)}{j - i}. \]

• The one-period forward rate:

\[ f(j, j + 1) = -\ln \frac{d(j + 1)}{d(j)}. \]

•

\[ f(T) \equiv \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T \frac{\partial S}{\partial T}. \]

• \( f(T) > S(T) \) if and only if \( \partial S/\partial T > 0. \)
Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

\[ f(a, b) = E[S(a, b)]. \]  \hspace{1cm} (13)

- Does not imply that the forward rate is an accurate predictor for the future spot rate.

- Implies the maturity strategy and the rollover strategy produce the same result at the horizon on the average.
Unbiased Expectations Theory and Spot Rate Curve

- Implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - \( f(j, j+1) > S(j+1) \) if and only if \( S(j+1) > S(j) \) from Eq. (11) on p. 115.
  - So \( E[S(j, j+1)] > S(j+1) > \cdots > S(1) \) if and only if \( S(j+1) > \cdots > S(1) \).

- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.
More Implications

• The theory has been rejected by most empirical studies with the possible exception of the period prior to 1915.

• Since the term structure has been upward sloping about 80% of the time, the theory would imply that investors have expected interest rates to rise 80% of the time.

• Riskless bonds, regardless of their different maturities, are expected to earn the same return on the average.

• That would mean investors are indifferent to risk.
A “Bad” Expectations Theory

• The expected returns on all possible riskless bond strategies are equal for *all* holding periods.

• So

\[(1 + S(2))^2 = (1 + S(1)) E[1 + S(1, 2)] \quad (14)\]

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

• After rearrangement,

\[
\frac{1}{E[1 + S(1, 2)]} = \frac{1 + S(1)}{(1 + S(2))^2}.
\]
A “Bad” Expectations Theory (continued)

• Now consider two one-period strategies.
  – Strategy one buys a two-period bond and sells it after one period.
  – The expected return is
    \[ E\left[ (1 + S(1, 2))^{-1} \right] (1 + S(2))^2. \]
  – Strategy two buys a one-period bond with a return of \( 1 + S(1) \).

• The theory says the returns are equal:
  \[
  \frac{1 + S(1)}{(1 + S(2))^2} = E \left[ \frac{1}{1 + S(1, 2)} \right].
  \]
A “Bad” Expectations Theory (concluded)

• Combine this with Eq. (14) on p. 130 to obtain

\[ E \left[ \frac{1}{1 + S(1, 2)} \right] = \frac{1}{E[1 + S(1, 2)]}. \]

• But this is impossible save for a certain economy.
  – Jensen’s inequality states that \( E[g(X)] > g(E[X]) \)
    for any nondegenerate random variable \( X \) and
    strictly convex function \( g \) (i.e., \( g''(x) > 0 \)).
  – Use \( g(x) \equiv (1 + x)^{-1} \) to prove our point.
Local Expectations Theory

- The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:

\[
E \left[ \frac{(1 + S(1,n))^{-(n-1)}}{(1 + S(n))^{-n}} \right] = 1 + S(1) \quad \text{for all } n > 1.
\]

- This theory is the basis of many interest rate models.
Duration Revisited

• To handle more general types of spot rate curve changes, define a vector \([c_1, c_2, \ldots, c_n]\) that characterizes the perceived type of change.
  
  – Parallel shift: \([1, 1, \ldots, 1]\).
  
  – Twist: \([1, 1, \ldots, 1, -1, \ldots, -1]\).
  
  – ...

• Let \(P(y) \equiv \sum_i C_i/(1 + S(i) + yc_i)^i\) be the price associated with the cash flow \(C_1, C_2, \ldots\).

• Define duration as
  \[
  \left. -\frac{\partial P(y)/P(0)}{\partial y} \right|_{y=0}.
  \]