Principles of Financial Computing

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Class Information

- Yuh-Dauh Lyuu. *Financial Engineering & Computation: Principles, Mathematics, Algorithms*. Cambridge University Press. 2002.
- Official Web page is

  `www.csie.ntu.edu.tw/~lyuu/finance1.html`
- Check

  `www.csie.ntu.edu.tw/~lyuu/capitals.html`

  for some of the software.

Class Information (concluded)

- Please ask many questions in class.
  - The best way for me to remember you in a large class.[a]
- Teaching assistants will be announced later.

[a] "[A] science concentrator [...] said that in his eighth semester of [Harvard] college, there was not a single science professor who could identify him by name." (*New York Times*, September 3, 2003.)

Useful Journals

- *Applied Mathematical Finance.*
- *Finance and Stochastics.*
- *Financial Analysts Journal.*
- *Journal of Banking & Finance.*
- *Journal of Computational Finance.*
- *Journal of Derivatives.*
- *Journal of Economic Dynamics & Control.*
- *Journal of Finance.*
- *Journal of Financial Economics.*

Useful Journals (continued)

- *Journal of Fixed Income.*
- *Journal of Futures Markets.*
- *Journal of Financial and Quantitative Analysis.*
- *Journal of Portfolio Management.*
- *Journal of Real Estate Finance and Economics.*
- *Management Science.*
- *Mathematical Finance.*

Useful Journals (concluded)

- *Quantitative Finance.*
- *Review of Financial Studies.*
- *Review of Derivatives Research.*
- *Risk Magazine.*
- *Stochastics and Stochastics Reports.*

Introduction

[An] investment bank could be
more collegial than a college.
— Emanuel Derman,
My Life as a Quant (2004)

The two most dangerous words in Wall Street
vocabulary are "financial engineering."
— Wilbur Ross (2007)

What This Course Is About

- Financial theories in pricing.
- Mathematical backgrounds.
- Derivative securities.
- Pricing models.
- Efficient algorithms in pricing financial instruments.
- Research problems.
- Finding your thesis directions.

What This Course Is *Not* About

- How to program.
- Basic mathematics in calculus, probability, and algebra.
- Details of the financial markets.
- How to be rich.
- How the markets will perform tomorrow.

Analysis of Algorithms

It is unworthy of excellent men
to lose hours like slaves
in the labor of computation.
— Gottfried Wilhelm Leibniz (1646–1716)

Computability and Algorithms

- Algorithms are precise procedures that can be turned into computer programs.
- Uncomputable problems.
  - Does this program have infinite loops?
  - Is this program bug free?
- Computable problems.
  - Intractable problems.
  - Tractable problems.

Complexity

- Start with a set of basic operations which will be assumed to take one unit of time.
- The total number of these operations is the total work done by an algorithm (its computational complexity).
- The space complexity is the amount of memory space used by an algorithm.
- Concentrate on the abstract complexity of an algorithm instead of its detailed implementation.
- Complexity is a good guide to an algorithm's *actual* running time.

Asymptotics

- Consider the search algorithm on p. 15.
- The worst-case complexity is n comparisons (why?).
- There are operations besides comparison.
- We care only about the asymptotic growth rate not the exact number of operations.
  - So the complexity of maintaining the loop is subsumed by the complexity of the body of the loop.
- The complexity is hence $O(n)$.

Algorithm for Searching an Element

```
for k = 1, 2, 3, ..., n do
    if x = A_k then
        return k;
    end if
end for
return not-found;
```

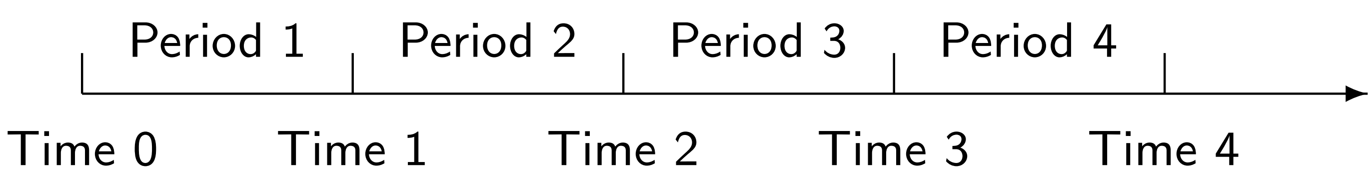
Common Complexities

- Let n stand for the "size" of the problem.
  - Number of elements, number of cash flows, etc.
- Linear time if the complexity is $O(n)$.
- Quadratic time if the complexity is $O(n^2)$.
- Cubic time if the complexity is $O(n^3)$.
- Exponential time if the complexity is $2^{O(n)}$.
- Superpolynomial if the complexity is less than exponential but higher than polynomials, say $2^{O(\sqrt{n})}$.

Basic Financial Mathematics

In the fifteenth century
mathematics was mainly concerned with
questions of commercial arithmetic and
the problems of the architect.
— Joseph Alois Schumpeter (1883–1950)

The Time Line



Time Value of Money

$$\begin{aligned} \text{FV} &= \text{PV}(1+r)^n, \\ \text{PV} &= \text{FV} \times (1+r)^{-n}. \end{aligned}$$

- FV (future value).
- PV (present value).
- r: interest rate.

Periodic Compounding

- Suppose the interest is compounded m times per annum, then

$$1 \to \left(1 + \frac{r}{m}\right) \to \left(1 + \frac{r}{m}\right)^2 \to \left(1 + \frac{r}{m}\right)^3 \to \cdots$$

- Hence

$$\text{FV} = \text{PV}\left(1 + \frac{r}{m}\right)^{nm}. \tag{1}$$

Common Compounding Methods

- Annual compounding: $m = 1$.
- Semiannual compounding: $m = 2$.
- Quarterly compounding: $m = 4$.
- Monthly compounding: $m = 12$.
- Weekly compounding: $m = 52$.
- Daily compounding: $m = 365$.

Easy Translations

- An interest rate of r compounded m times a year is "equivalent to" an interest rate of r/m per $1/m$ year.
- If a loan asks for a return of 1% per month, the annual interest rate will be 12% *with monthly compounding*.

Example

- Annual interest rate is 10% compounded twice per annum.
- Each dollar will grow to be

$$[\,1 + (0.1/2)\,]^2 = 1.1025$$

one year from now.

- The rate is equivalent to an interest rate of 10.25% compounded once *per annum*.

Continuous Compounding

- Let $m \to \infty$ so that
$$\left(1+\frac{r}{m}\right)^m \to e^r$$
in Eq. (1) on p. 21.
- Then
$$\text{FV} = \text{PV} \times e^{rn},$$
where $e = 2.71828\ldots$.

Continuous Compounding (concluded)

- Continuous compounding is easier to work with.
- Suppose the annual interest rate is r_1 for n_1 years and r_2 for the following n_2 years.
- Then the FV of one dollar will be

$$e^{r_1 n_1 + r_2 n_2}.$$

Efficient Algorithms for PV and FV

- The PV of the cash flow $C_1, C_2, \ldots, C_n$ at times $1, 2, \ldots, n$ is

$$\frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_n}{(1+y)^n}.$$

- This formula and its variations are the engine behind most of financial calculations.[a]
  - What is y?
  - What are C_i?
  - What is n?

[a] "Asset pricing theory all stems from one simple concept [...]: price equals expected discounted payoff" (see Cochrane (2005)).

Algorithm for Evaluating PV

```
x := 0;
d := 1 + y;
for i = n, n − 1, . . . , 1 do
    x := (x + C_i)/d;
end for
return x;
```

Horner's Rule: The Idea Behind p. 28

- This idea is

$$\left(\cdots \left(\left(\frac{C_n}{1+y} + C_{n-1} \right) \frac{1}{1+y} + C_{n-2} \right) \frac{1}{1+y} + \cdots \right) \frac{1}{1+y}.$$

  - Due to Horner (1786–1837) in 1819.

- The algorithm takes $O(n)$ time.

- It is the most efficient possible in terms of the absolute number of arithmetic operations.[a]

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[a] Borodin and Munro (1975).

Conversion between Compounding Methods

- Suppose r_1 is the annual rate with continuous compounding.
- Suppose r_2 is the equivalent rate compounded m times per annum.
- How are they related?

Conversion between Compounding Methods (concluded)

- Both interest rates must produce the same amount of money after one year.
- That is,

$$\left(1+\frac{r_2}{m}\right)^m = e^{r_1}.$$

- Therefore,

$$\begin{aligned} r_1 &= m \ln\left(1+\frac{r_2}{m}\right), \\ r_2 &= m\left(e^{r_1/m}-1\right). \end{aligned}$$

Annuities

- An annuity pays out the same C dollars at the end of each year for n years.
- With a rate of r, the FV at the end of the nth year is

$$\sum_{i=0}^{n-1} C(1+r)^i = C\,\frac{(1+r)^n - 1}{r}. \tag{2}$$

General Annuities

- If m payments of C dollars each are received per year (the general annuity), then Eq. (2) becomes

$$C \, \frac{\left(1+\frac{r}{m}\right)^{nm}-1}{\frac{r}{m}}.$$

- The PV of a general annuity is

$$\sum_{i=1}^{nm} C \left(1+\frac{r}{m}\right)^{-i} = C \, \frac{1-\left(1+\frac{r}{m}\right)^{-nm}}{\frac{r}{m}}. \tag{3}$$

Amortization

- It is a method of repaying a loan through regular payments of interest *and* principal.
- The size of the loan (the original balance) is reduced by the principal part of each payment.
- The interest part of each payment pays the interest incurred on the remaining principal balance.
- As the principal gets paid down over the term of the loan, the interest part of the payment diminishes.

Example: Home Mortgage

- By paying down the principal consistently, the risk to the lender is lowered.
- When the borrower sells the house, the remaining principal is due the lender.
- Consider the equal-payment case, i.e., fixed-rate, level-payment, fully amortized mortgages.
  - They are called traditional mortgages in the U.S.

A Numerical Example

- Consider a 15-year, \$250,000 loan at 8.0% interest rate.
- Solving Eq. (3) on p. 33 with $\text{PV} = 250000$, $n = 15$, $m = 12$, and $r = 0.08$ gives a monthly payment of $C = 2389.13$.
- The amortization schedule is shown on p. 37.
- In every month (1) the principal and interest parts add up to \$2,389.13, (2) the remaining principal is reduced by the amount indicated under the Principal heading, and (3) the interest is computed by multiplying the remaining balance of the previous month by 0.08/12.

Month	Payment	Interest	Principal	Remaining principal
				250,000.000
1	2,389.13	1,666.667	722.464	249,277.536
2	2,389.13	1,661.850	727.280	248,550.256
3	2,389.13	1,657.002	732.129	247,818.128
		...		
178	2,389.13	47.153	2,341.980	4,730.899
179	2,389.13	31.539	2,357.591	2,373.308
180	2,389.13	15.822	2,373.308	0.000
Total	430,043.438	180,043.438	250,000.000	

Method 1 of Calculating the Remaining Principal

- Go down the amortization schedule until you reach the particular month you are interested in.
  - A month's principal payment equals the monthly payment subtracted by the previous month's remaining principal times the monthly interest rate.
  - A month's remaining principal equals the previous month's remaining principal subtracted by the principal payment calculated above.

Method 1 of Calculating the Remaining Principal (concluded)

- This method is relatively slow but is universal in its applicability.
- It can, for example, accommodate prepayment and variable interest rates.

Method 2 of Calculating the Remaining Principal

- Right after the kth payment, the remaining principal is the PV of the future $nm - k$ cash flows,

$$\sum_{i=1}^{nm-k} C \left(1 + \frac{r}{m}\right)^{-i} = C \, \frac{1 - \left(1 + \frac{r}{m}\right)^{-nm+k}}{\frac{r}{m}}.$$

- This method is faster but more limited in applications.

Yields

- The term yield denotes the return of investment.
- Two widely used yields are the bond equivalent yield (BEY) and the mortgage equivalent yield (MEY).
- Recall Eq. (1) on p. 21: $\text{FV} = \text{PV}\left(1 + \frac{r}{m}\right)^{nm}$.
- BEY corresponds to the r above that equates PV with FV when $m = 2$.
- MEY corresponds to the r above that equates PV with FV when $m = 12$.

Internal Rate of Return (IRR)

- It is the interest rate which equates an investment's PV with its price P,

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_n}{(1+y)^n}.$$

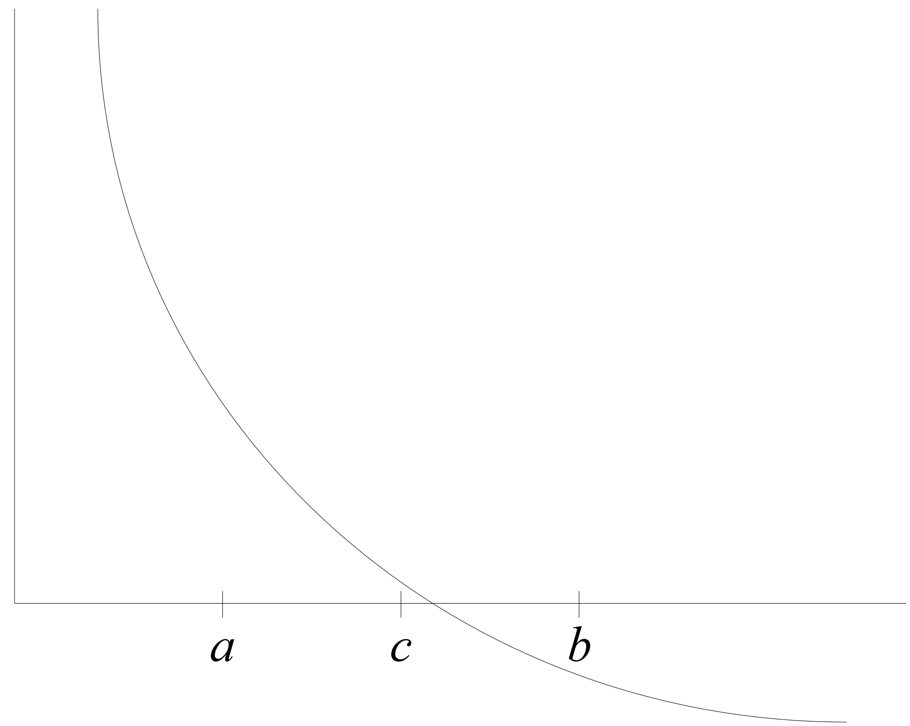
- The above formula is the foundation upon which pricing methodologies are built.

Numerical Methods for Yields

- Solve $f(y) = 0$ for $y \geq -1$, where
$$f(y) \equiv \sum_{t=1}^{n} \frac{C_t}{(1+y)^t} - P.$$
  - P is the market price.
- The function $f(y)$ is monotonic in y if $C_t > 0$ for all t.
- A unique solution exists for a monotonic $f(y)$.

The Bisection Method

- Start with a and b where $a < b$ and $f(a)\,f(b) < 0$.
- Then $f(\xi)$ must be zero for some $\xi \in [\,a, b\,]$.
- If we evaluate f at the midpoint $c \equiv (a + b)/2$, either (1) $f(c) = 0$, (2) $f(a)\,f(c) < 0$, or (3) $f(c)\,f(b) < 0$.
- In the first case we are done, in the second case we continue the process with the new bracket $[\,a, c\,]$, and in the third case we continue with $[\,c, b\,]$.
- The bracket is halved in the latter two cases.
- After n steps, we will have confined ξ within a bracket of length $(b - a)/2^n$.



a
c
b

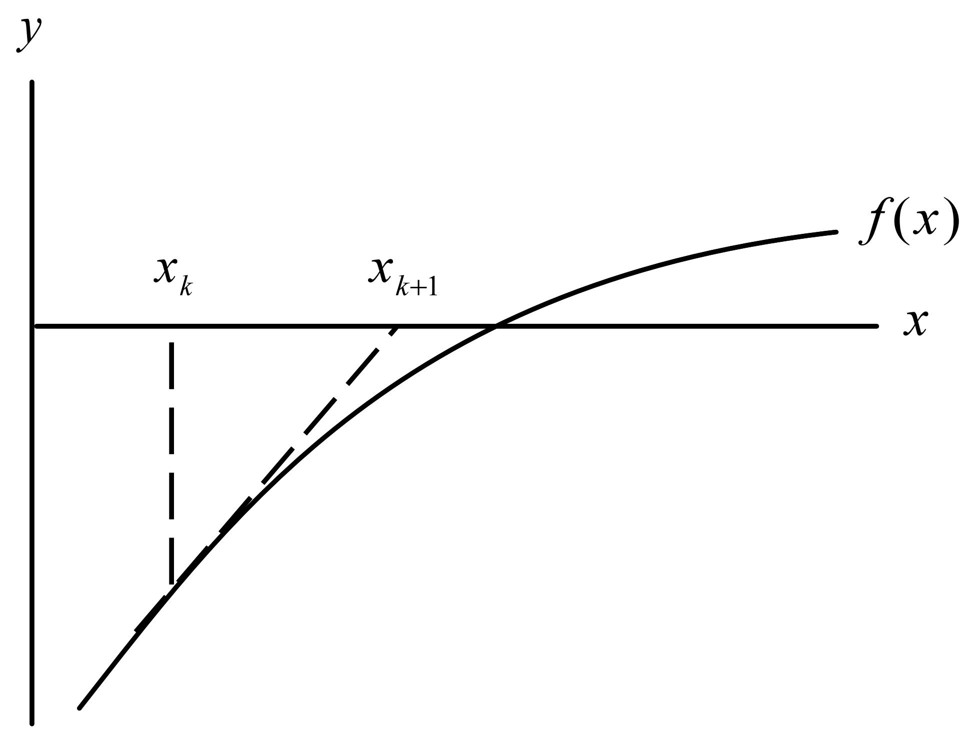
The Newton-Raphson Method

- Converges faster than the bisection method.
- Start with a first approximation x_0 to a root of $f(x) = 0$.
- Then

$$x_{k+1} \equiv x_k - \frac{f(x_k)}{f'(x_k)}.$$

- When computing yields,

$$f'(x) = -\sum_{t=1}^{n} \frac{tC_t}{(1+x)^{t+1}}.$$


y
f(x)
x_k
x_k+1
x

The Secant Method

- A variant of the Newton-Raphson method.
- Replace differentiation with difference.
- Start with two approximations x_0 and x_1.
- Then compute the $(k+1)$st approximation with

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

The Secant Method (concluded)

- Its convergence rate, 1.618.
- This is slightly worse than the Newton-Raphson method's 2.
- But the secant method does not need to evaluate $f'(x_k)$ needed by the Newton-Raphson method.
- This saves about 50% in computation efforts per iteration.
- The convergence rate of the bisection method is 1.

Solving Systems of Nonlinear Equations

- It is not easy to extend the bisection method to higher dimensions.
- But the Newton-Raphson method can be extended to higher dimensions.
- Let (x_k, y_k) be the kth approximation to the solution of the two simultaneous equations,

$$\begin{aligned} f(x,y) &= 0, \\ g(x,y) &= 0. \end{aligned}$$

Solving Systems of Nonlinear Equations (concluded)

- The $(k+1)$st approximation (x_{k+1}, y_{k+1}) satisfies the following linear equations,

$$\begin{bmatrix} \frac{\partial f(x_k, y_k)}{\partial x} & \frac{\partial f(x_k, y_k)}{\partial y} \\ \frac{\partial g(x_k, y_k)}{\partial x} & \frac{\partial g(x_k, y_k)}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{bmatrix} = - \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix},$$

  where unknowns $\Delta x_{k+1} \equiv x_{k+1} - x_k$ and $\Delta y_{k+1} \equiv y_{k+1} - y_k$.

- The above has a unique solution for $(\Delta x_{k+1}, \Delta y_{k+1})$ when the 2×2 matrix is invertible.

- Set $(x_{k+1}, y_{k+1}) = (x_k + \Delta x_{k+1}, y_k + \Delta y_{k+1})$.

Zero-Coupon Bonds (Pure Discount Bonds)

- The price of a zero-coupon bond that pays F dollars in n periods is

$$F/(1+r)^n,$$

where r is the interest rate per period.

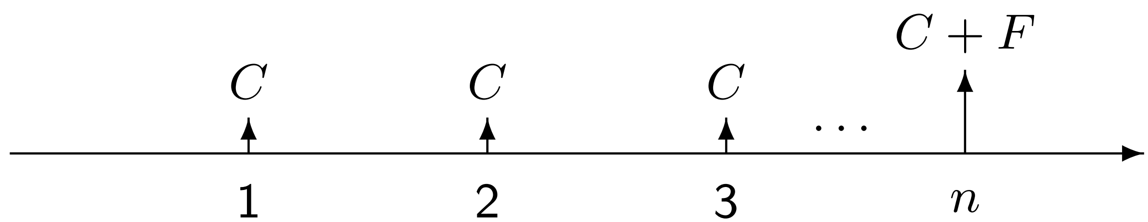
- Can meet future obligations without reinvestment risk.

Example

- The interest rate is 8% compounded semiannually.
- A zero-coupon bond that pays the par value 20 years from now will be priced at $1/(1.04)^{40}$, or 20.83%, of its par value.
- It will be quoted as 20.83.
- If the bond matures in 10 years instead of 20, its price would be 45.64.

Level-Coupon Bonds

- Coupon rate.
- Par value, paid at maturity.
- F denotes the par value, and C denotes the coupon.
- Cash flow:



- Coupon bonds can be thought of as a matching package of zero-coupon bonds, at least theoretically.

Pricing Formula

$$
\begin{aligned}
P &= \sum_{i=1}^{n} \frac{C}{\left(1+\frac{r}{m}\right)^{i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}} \\
&= C\, \frac{1-\left(1+\frac{r}{m}\right)^{-n}}{\frac{r}{m}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}}. \qquad (4)
\end{aligned}
$$

- n: number of cash flows.
- m: number of payments per year.
- r: annual rate compounded m times per annum.
- $C = Fc/m$ when c is the annual coupon rate.
- Price P can be computed in $O(1)$ time.

Yields to Maturity

- It is the r that satisfies Eq. (4) on p. 55 with P being the bond price.
- For a 15% BEY, a 10-year bond with a coupon rate of 10% paid semiannually sells for

$$\begin{aligned} & 5 \times \frac{1 - [\,1 + (0.15/2)\,]^{-2\times 10}}{0.15/2} + \frac{100}{[\,1 + (0.15/2)\,]^{2\times 10}} \\ = \;& 74.5138 \end{aligned}$$

percent of par.

Price Behavior (1)

- Bond prices fall when interest rates rise, and vice versa.
- "Only 24 percent answered the question correctly."[a]

[a] CNN, December 21, 2001.

Price Behavior (2)

- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate is above the market interest rate;
  - at par (at its par value) when its coupon rate is equal to the market interest rate;
  - at a discount (below its par value) when its coupon rate is below the market interest rate.

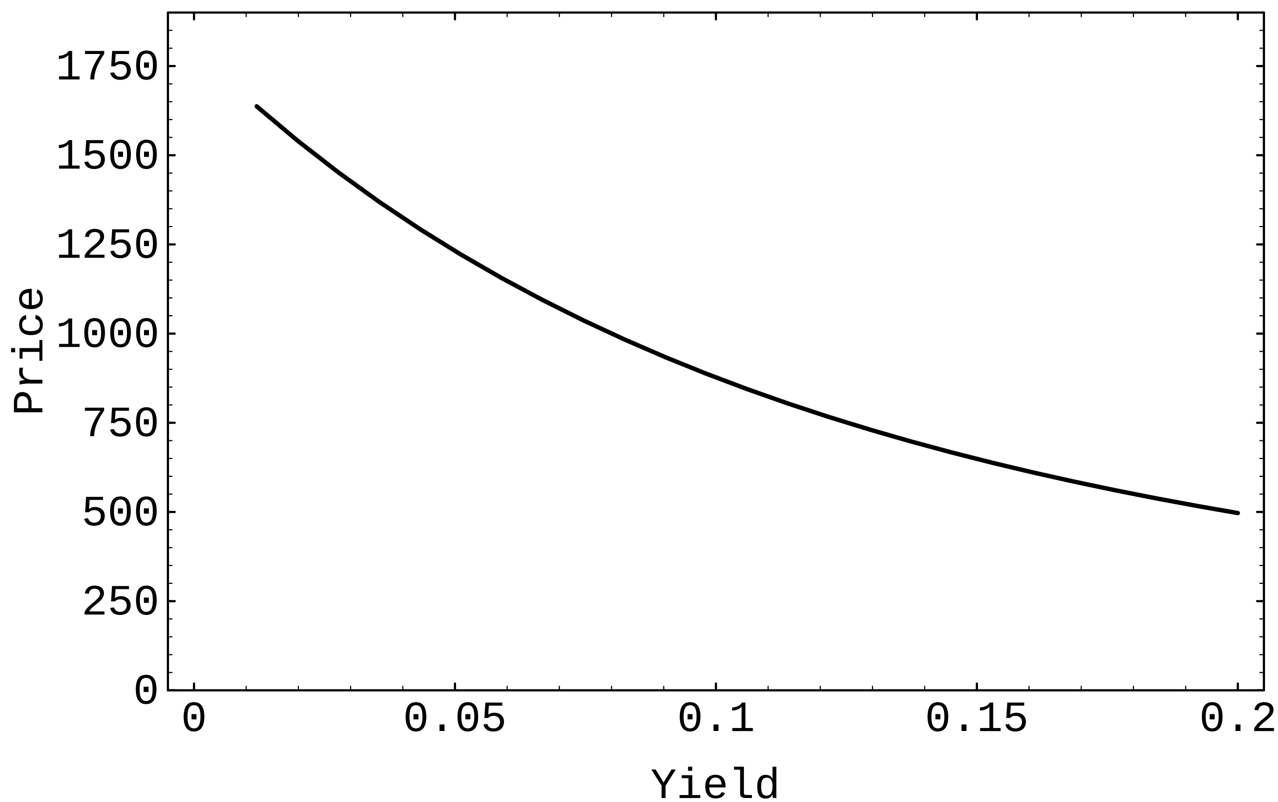
9% Coupon Bond

Yield (%)	Price (% of par)
7.5	113.37
8.0	108.65
8.5	104.19
9.0	100.00
9.5	96.04
10.0	92.31
10.5	88.79

Terminology

- Bonds selling at par are called par bonds.
- Bonds selling at a premium are called premium bonds.
- Bonds selling at a discount are called discount bonds.

Price Behavior (3): Convexity



Day Count Conventions: Actual/Actual

- The first "actual" refers to the actual number of days in a month.
- The second refers to the actual number of days in a coupon period.
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ to date $D_2 \equiv (y_2, m_2, d_2)$ is

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1).$$

- Complications: 31, Feb 28, and Feb 29.

Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

$$\omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}. \tag{5}$$

- The price is now calculated by

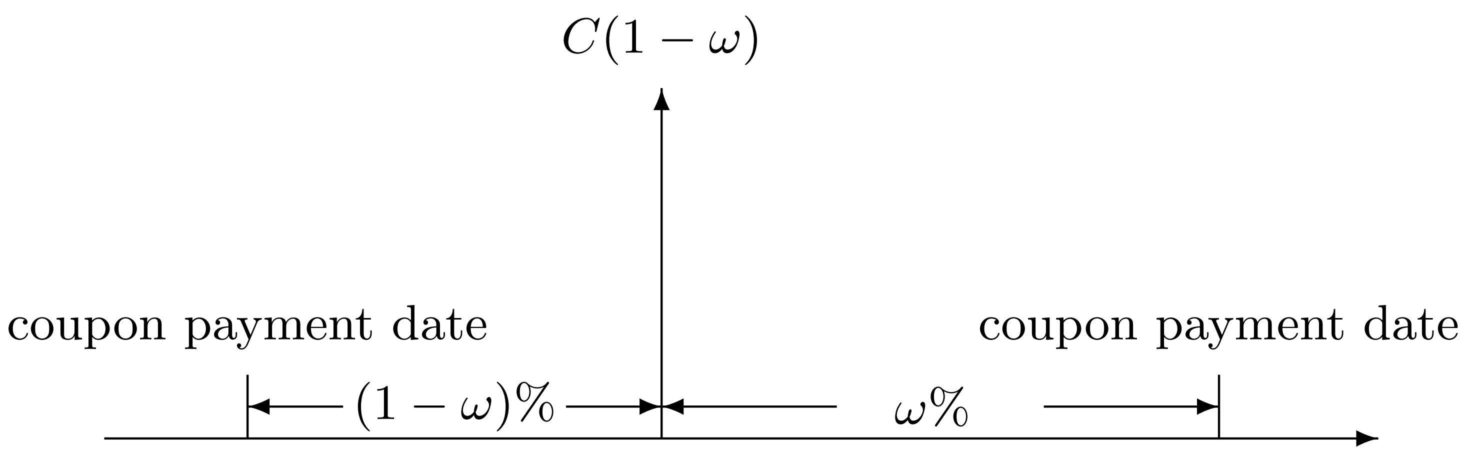
$$\text{PV} = \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}. \tag{6}$$

Accrued Interest

- The buyer pays the quoted price plus the accrued interest — the invoice price:

$$C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega).$$

- The yield to maturity is the r satisfying Eq. (6) when P is the invoice price.
- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.


C(1 − ω)
coupon payment date
coupon payment date
(1 − ω)%
ω%

Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.

Example ("30/360") (concluded)

- The accrued interest is $(10/2) \times \frac{180-60}{180} = 3.3333$ per \$100 of par value.
- The yield to maturity is 3%.
- This can be verified by Eq. (6) on p. 64 with
  - $\omega = 60/180$,
  - $m = 2$,
  - $C = 5$,
  - PV$= 111.2891 + 3.3333$,
  - $r = 0.03$.

Price Behavior (2) Revisited

- Before: A bond selling at par if the yield to maturity equals the coupon rate.
- But it assumed that the settlement date is on a coupon payment date.
- Now suppose the settlement date for a bond selling at par (i.e., the *quoted price* is equal to the par value) falls between two coupon payment dates.
- Then its yield to maturity is less than the coupon rate.
  - The short reason: Exponential growth is replaced by linear growth, hence "overpaying" the coupon.