## Sample Term Structure

- We shall construct interest rate trees consistent with the sample term structure in the following table.
- Assume the short rate volatility is such that $v \equiv r_{\mathrm{h}}/r_{\ell} = 1.5$, independent of time.

| Period | 1 | 2 | 3 |
|---|---|---|---|
| Spot rate (%) | 4 | 4.2 | 4.3 |
| One-period forward rate (%) | 4 | 4.4 | 4.5 |
| Discount factor | 0.96154 | 0.92101 | 0.88135 |

## An Approximate Calibration Scheme

- Start with the implied one-period forward rates and then equate the expected short rate with the forward rate (see Exercise 5.6.6 in text).
- For the first period, the forward rate is today's one-period spot rate.
- In general, let $f_j$ denote the forward rate in period $j$.
- This forward rate can be derived from the market discount function via $f_j = (d(j)/d(j+1)) - 1$ (see Exercise 5.6.3 in text).

## An Approximate Calibration Scheme (continued)

- Since the $i$th short rate, $1 \leq i \leq j$, occurs with probability $2^{-(j-1)} \binom{j-1}{i-1}$, this means

$$\sum_{i=1}^{j} 2^{-(j-1)} \binom{j-1}{i-1} r_j v_j^{i-1} = f_j .$$

- Thus

$$r_j = \left( \frac{2}{1+v_j} \right)^{j-1} f_j . \tag{78}$$

- The binomial interest rate tree is trivial to set up.

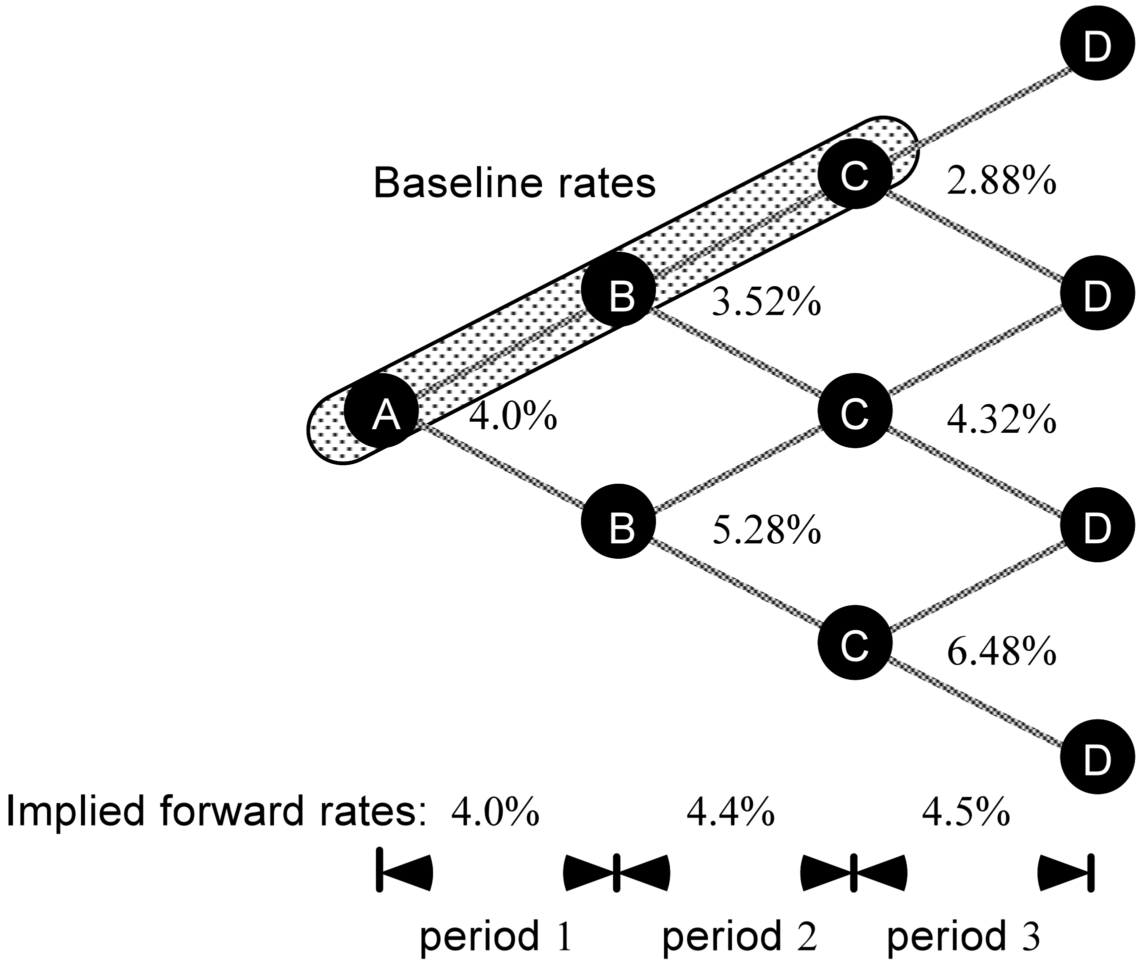
## An Approximate Calibration Scheme (concluded)

- The ensuing tree for the sample term structure appears in figure next page.
- For example, the price of the zero-coupon bond paying \$1 at the end of the third period is

$$\frac{1}{4} \times \frac{1}{1.04} \times \left( \frac{1}{1.0352} \times \left( \frac{1}{1.0288} + \frac{1}{1.0432} \right) + \frac{1}{1.0528} \times \left( \frac{1}{1.0432} + \frac{1}{1.0648} \right) \right)$$

or 0.88155, which exceeds discount factor 0.88135.

- The tree is thus not calibrated.
- Indeed, this bias is inherent (see text).


D
C
2.88%
Baseline rates
B
3.52%
D
C
A
4.0%
4.32%
B
5.28%
D
C
6.48%
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

## Issues in Calibration

- The model prices generated by the binomial interest rate tree should match the observed market prices.
- Perhaps the most crucial aspect of model building.
- Treat the backward induction for the model price of the $m$-period zero-coupon bond as computing some function of the unknown baseline rate $r_m$ called $f(r_m)$.
- A root-finding method is applied to solve $f(r_m) = P$ for $r_m$ given the zero's price $P$ and $r_1, r_2, \ldots, r_{m-1}$.
- This procedure is carried out for $m = 1, 2, \ldots, n$.
- It runs in cubic time, hopelessly slow.

## Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in quadratic time by the use of forward induction (Jamshidian, 1991).
- The scheme records how much \$1 at a node contributes to the model price.
- This number is called the state price.
  - It is the price of a state contingent claim that pays \$1 at that particular node (state) and 0 elsewhere.
- The column of state prices will be established by moving *forward* from time 1 to time $n$.

## Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at time $j$ and there are $j + 1$ nodes.
  - The baseline rate for period $j$ is $r \equiv r_j$.
  - The multiplicative ratio be $v \equiv v_j$.
  - $P_1, P_2, \ldots, P_j$ are the state prices a period prior, corresponding to rates $r, rv, \ldots, rv^{j-1}$.
- By definition, $\sum_{i=1}^{j} P_i$ is the price of the $(j - 1)$-period zero-coupon bond.

## Binomial Interest Rate Tree Calibration (continued)

- One dollar at time $j$ has a known market value of $1/[\,1 + S(j)\,]^j$, where $S(j)$ is the $j$-period spot rate.
- Alternatively, this dollar has a present value of

$$g(r) \equiv \frac{P_1}{(1+r)} + \frac{P_2}{(1+rv)} + \frac{P_3}{(1+rv^2)} + \cdots + \frac{P_j}{(1+rv^{j-1})}.$$

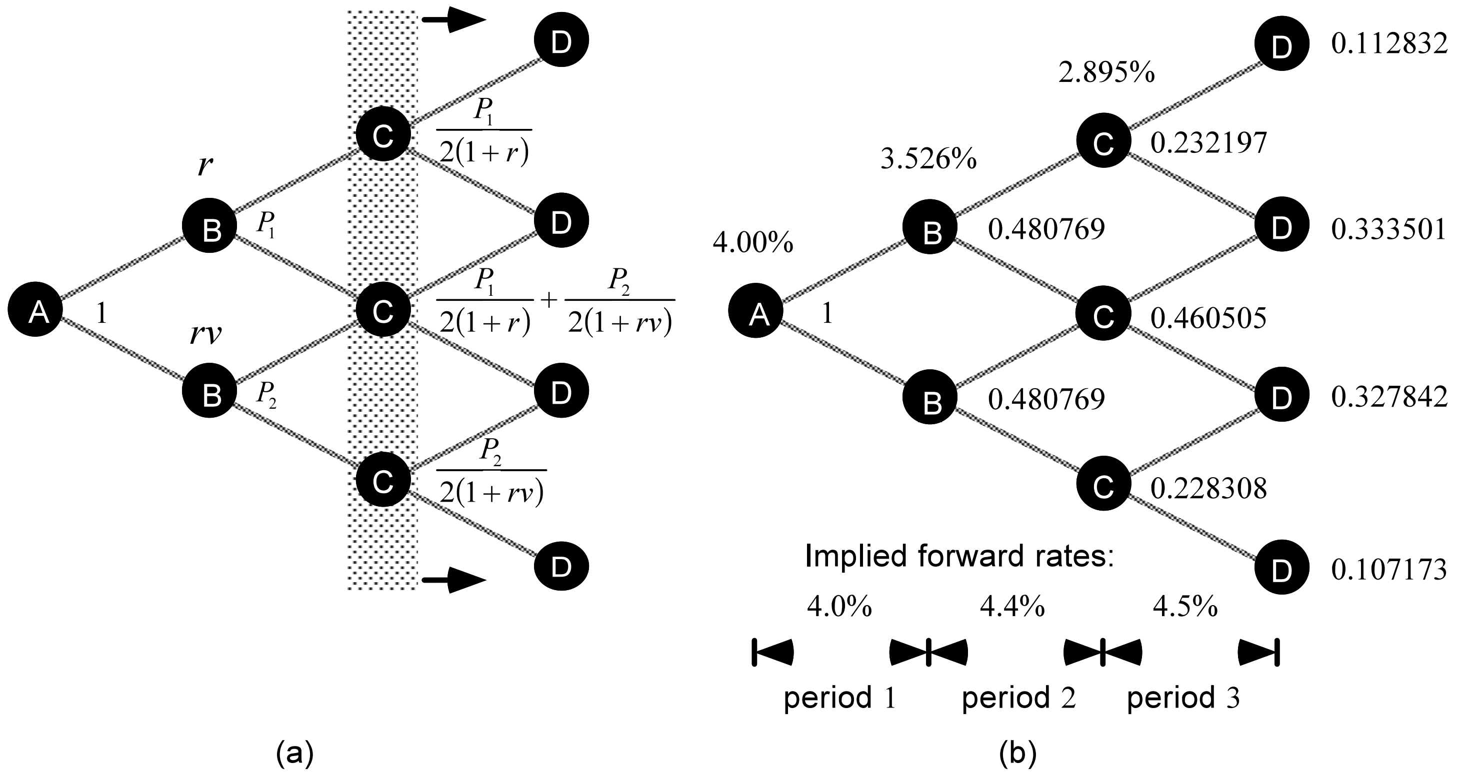
- So we solve

$$g(r) = \frac{1}{[\,1 + S(j)\,]^j} \tag{79}$$

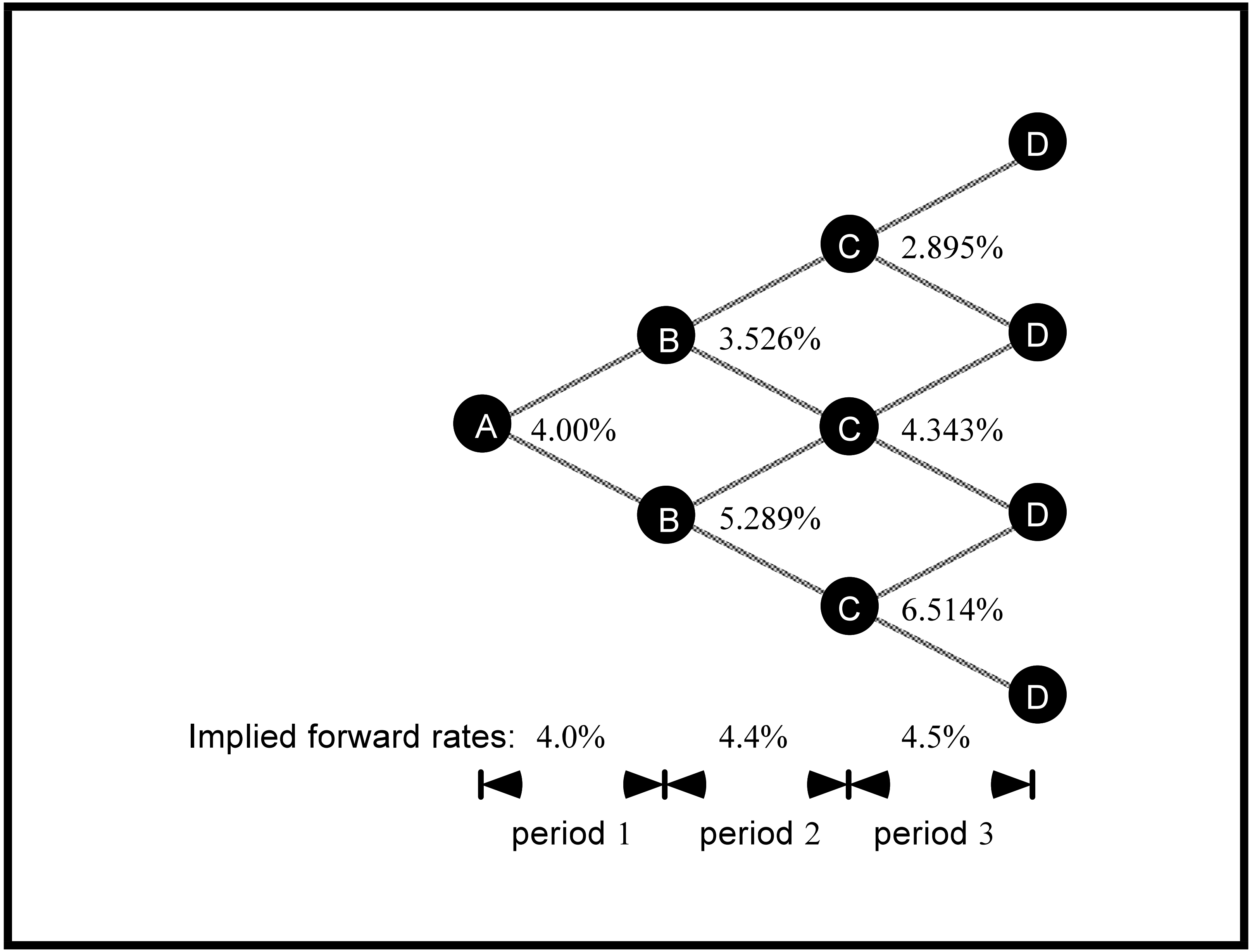
for $r$.

## Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive solution for $r$ is guaranteed.
- The state prices at time $j$ can now be calculated (see figure (a) next page).
- We call a tree with these state prices a binomial state price tree (see figure (b) next page).
- The calibrated tree is depicted on p. 761.



A
B
C
D
$r$
$rv$
1
$P_1$
$P_2$
$\frac{P_1}{2(1+r)}$
$\frac{P_1}{2(1+r)} + \frac{P_2}{2(1+rv)}$
$\frac{P_2}{2(1+rv)}$
(a)
4.00%
3.526%
2.895%
0.480769
0.480769
0.232197
0.460505
0.228308
0.112832
0.333501
0.327842
0.107173
Implied forward rates:
4.0%
4.4%
4.5%
period 1
period 2
period 3
(b)


D
C
2.895%
B
3.526%
D
A
4.00%
C
4.343%
B
5.289%
D
C
6.514%
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

## Binomial Interest Rate Tree Calibration (concluded)

- The Newton-Raphson method can be used to solve for the $r$ in Eq. (79) on p. 758 as $g'(r)$ is easy to evaluate.
- The monotonicity and the convexity of $g(r)$ also facilitate root finding.
- The total running time is $O(\mathcal{C}n^2)$, where $\mathcal{C}$ is the maximum number of times the root-finding routine iterates, each consuming $O(n)$ work.
- With a good initial guess, the Newton-Raphson method converges in only a few steps[a]

[a] Lyuu (1999).

# A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.
- The baseline rate for the second period, $r_2$, satisfies

$$\frac{0.480769}{1+r_2} + \frac{0.480769}{1+1.5 \times r_2} = 0.92101.$$

- The result is $r_2 = 3.526\%$.
- This is used to derive the next column of state prices shown in figure (b) on p. 760 as 0.232197, 0.460505, and 0.228308.
- Their sum gives the correct market discount factor 0.92101.

## A Numerical Example (concluded)

- The baseline rate for the third period, $r_3$, satisfies

$$\frac{0.232197}{1+r_3} + \frac{0.460505}{1+1.5\times r_3} + \frac{0.228308}{1+(1.5)^2\times r_3} = 0.88135.$$

- The result is $r_3 = 2.895\%$.
- Now, redo the calculation on p. 753 using the new rates:

$$\frac{1}{4}\times\frac{1}{1.04}\times[\frac{1}{1.03526}\times(\frac{1}{1.02895}+\frac{1}{1.04343})+\frac{1}{1.05289}\times(\frac{1}{1.04343}+\frac{1}{1.06514})],$$

  which equals 0.88135, an exact match.

- The tree on p. 761 prices without bias the benchmark securities.

- The term structure dynamics is shown on p. 765.

$$
\begin{array}{ccccc}
 & & & & \begin{bmatrix} 0.971865 \end{bmatrix} \\
 & & & \nearrow & \\
 & & \begin{bmatrix} 0.965941 \\ 0.932250 \end{bmatrix} & & \\
 & \nearrow & & \searrow & \\
\begin{bmatrix} 0.96154 \\ 0.92101 \\ 0.88135 \end{bmatrix} & & & & \begin{bmatrix} 0.958378 \end{bmatrix} \\
 & \searrow & & \nearrow & \\
 & & \begin{bmatrix} 0.949767 \\ 0.900959 \end{bmatrix} & & \\
 & & & \searrow & \\
 & & & & \begin{bmatrix} 0.938844 \end{bmatrix}
\end{array}
$$

## Spread of Nonbenchmark Bonds

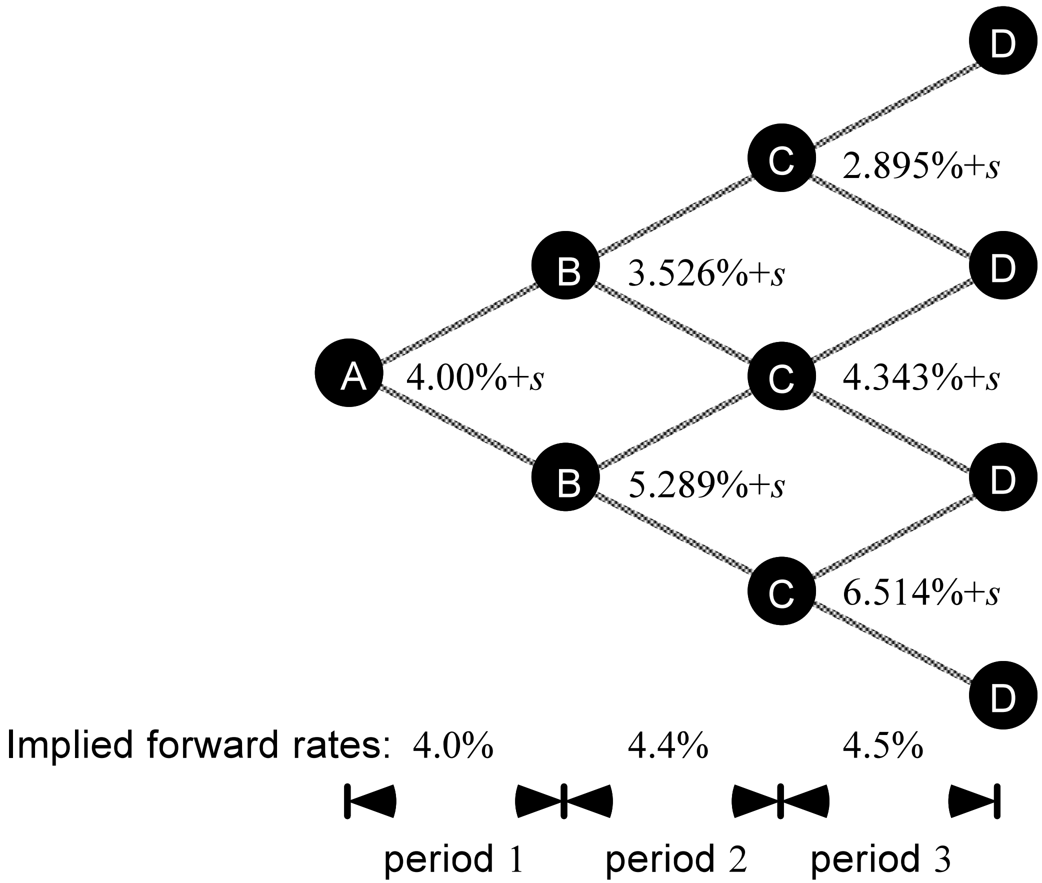
- Model prices calculated by the calibrated tree as a rule do not match market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- We look for the spread that, when added uniformly over the short rates in the tree, makes the model price equal the market price.
- We will apply the spread concept to option-free bonds here.

## Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 768.
- Consider a security with cash flow $C_i$ at time $i$ for $i = 1, 2, 3$.
- Its model price is $p(s)$, which is equal to

$$\frac{1}{1.04+s} \times \left[ C_1 + \frac{1}{2} \times \frac{1}{1.03526+s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.02895+s} + \frac{C_3}{1.04343+s} \right) \right) + \frac{1}{2} \times \frac{1}{1.05289+s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.04343+s} + \frac{C_3}{1.06514+s} \right) \right) \right].$$

- Given a market price of $P$, the spread is the $s$ that solves $P = p(s)$.


D
C
2.895%+s
B
3.526%+s
D
A
4.00%+s
C
4.343%+s
B
5.289%+s
D
C
6.514%+s
D
Implied forward rates: 4.0%
4.4%
4.5%
period 1
period 2
period 3

## Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of $s$.
- We will employ the Newton-Raphson root-finding method to solve $p(s) - P = 0$ for $s$.
- But a quick look at the equation above reveals that evaluating $p'(s)$ directly is infeasible.
- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.

## Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate $r$.
- In the process of computing the model price $p(s)$, a price $p_{\mathrm{A}}(s)$ is computed at A.
- Prices computed at A's two successor nodes B and C are discounted by $r + s$ to obtain $p_{\mathrm{A}}(s)$ as follows,

$$p_{\mathrm{A}}(s) = c + \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1 + r + s)},$$

where $c$ denotes the cash flow at A.

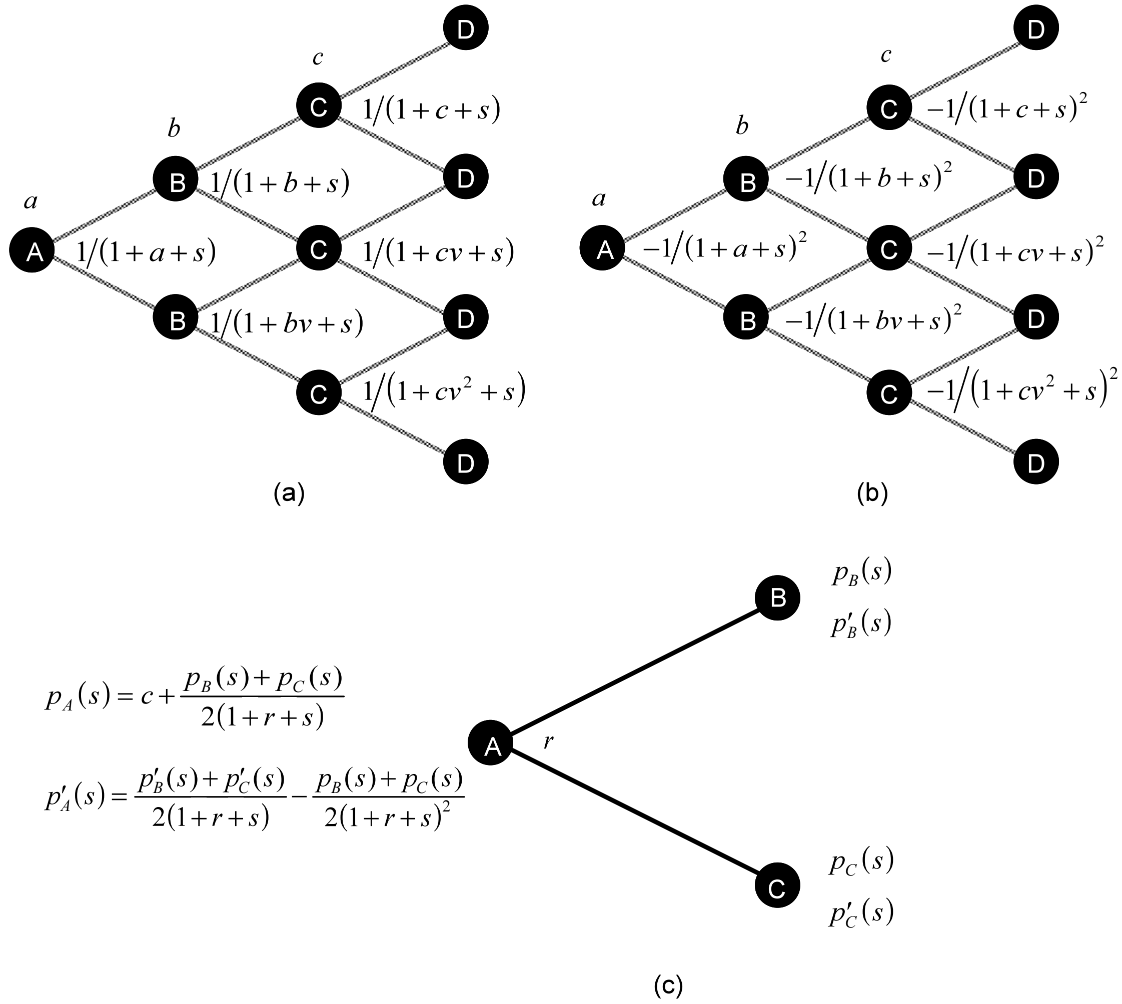
## Spread of Nonbenchmark Bonds (continued)

- To compute $p'_{\mathrm{A}}(s)$ as well, node A calculates

$$p'_{\mathrm{A}}(s) = \frac{p'_{\mathrm{B}}(s) + p'_{\mathrm{C}}(s)}{2(1+r+s)} - \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1+r+s)^2}. \tag{80}$$

- This is easy if $p'_{\mathrm{B}}(s)$ and $p'_{\mathrm{C}}(s)$ are also computed at nodes B and C.
- Apply the above procedure inductively to yield $p(s)$ and $p'(s)$ at the root (p. 772).
- This is called the differential tree method.[a]

[a]Lyuu (1999).



a
b
c
A
B
B
C
C
C
D
D
D
D
$1/(1+a+s)$
$1/(1+b+s)$
$1/(1+bv+s)$
$1/(1+c+s)$
$1/(1+cv+s)$
$1/(1+cv^2+s)$
(a)
$-1/(1+a+s)^2$
$-1/(1+b+s)^2$
$-1/(1+bv+s)^2$
$-1/(1+c+s)^2$
$-1/(1+cv+s)^2$
$-1/(1+cv^2+s)^2$
(b)
$p_A(s) = c + \frac{p_B(s)+p_C(s)}{2(1+r+s)}$
$p'_A(s) = \frac{p'_B(s)+p'_C(s)}{2(1+r+s)} - \frac{p_B(s)+p_C(s)}{2(1+r+s)^2}$
r
$p_B(s)$
$p'_B(s)$
$p_C(s)$
$p'_C(s)$
(c)

## Spread of Nonbenchmark Bonds (continued)

- Let $\mathcal{C}$ represent the number of times the tree is traversed, which takes $O(n^2)$ time.
- The total running time is $O(\mathcal{C}n^2)$.
- In practice $\mathcal{C}$ is a small constant.
- The memory requirement is $O(n)$.

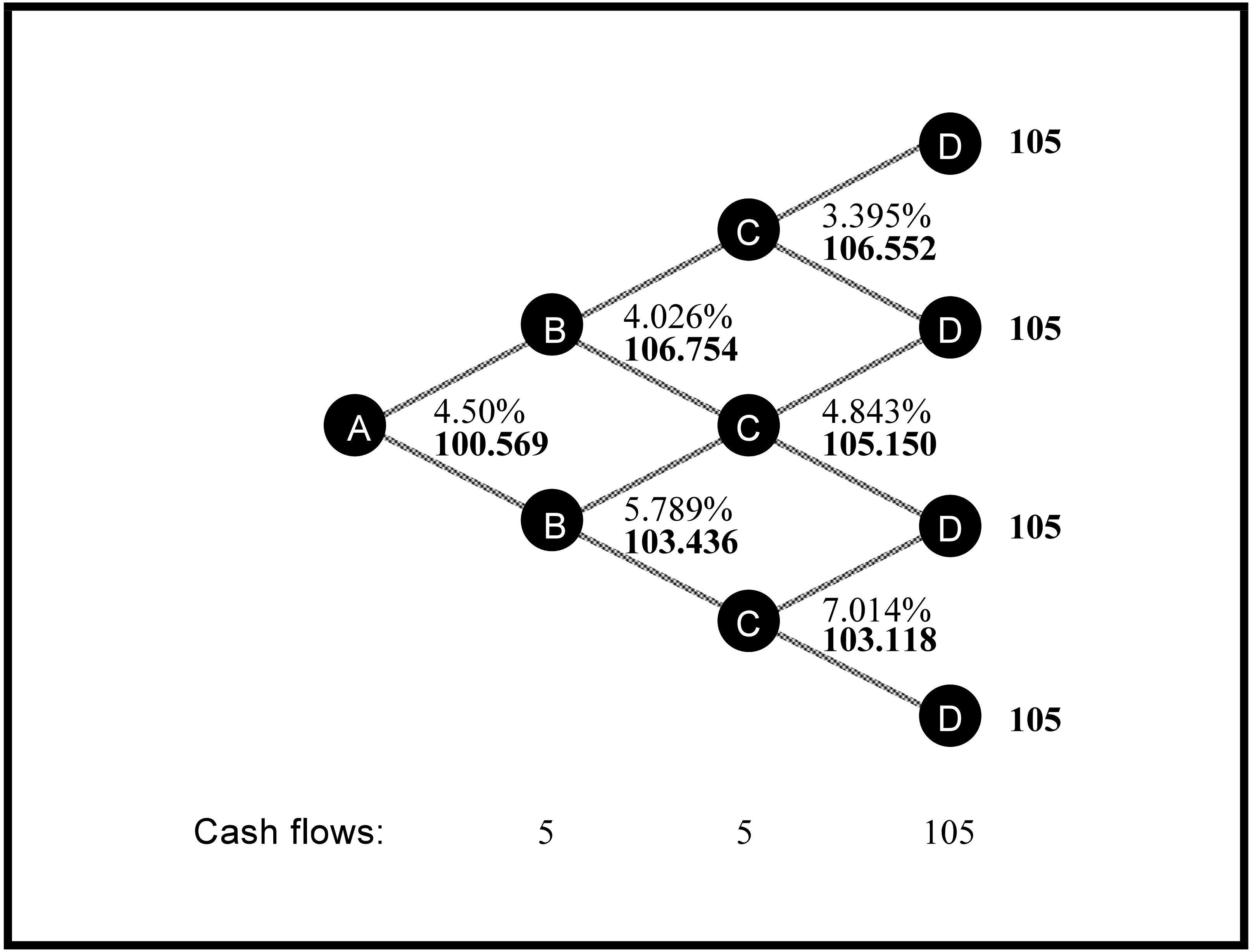
# Spread of Nonbenchmark Bonds (continued)

| Number of partitions $n$ | Running time (s) | Number of iterations | Number of partitions | Running time (s) | Number of iterations |
|---|---|---|---|---|---|
| 500 | 7.850 | 5 | 10500 | 3503.410 | 5 |
| 1500 | 71.650 | 5 | 11500 | 4169.570 | 5 |
| 2500 | 198.770 | 5 | 12500 | 4912.680 | 5 |
| 3500 | 387.460 | 5 | 13500 | 5714.440 | 5 |
| 4500 | 641.400 | 5 | 14500 | 6589.360 | 5 |
| 5500 | 951.800 | 5 | 15500 | 7548.760 | 5 |
| 6500 | 1327.900 | 5 | 16500 | 8502.950 | 5 |
| 7500 | 1761.110 | 5 | 17500 | 9523.900 | 5 |
| 8500 | 2269.750 | 5 | 18500 | 10617.370 | 5 |
| 9500 | 2834.170 | 5 | .......... | ......... | .......... |

75MHz Sun SPARCstation 20.

## Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread can be shown to be 50 basis points over the tree (p. 776).
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread and static spread of the nonbenchmark bond over an otherwise identical benchmark bond.


D
105
C
3.395%
106.552
B
4.026%
106.754
D
105
A
4.50%
100.569
C
4.843%
105.150
B
5.789%
103.436
D
105
C
7.014%
103.118
D
105
Cash flows:
5
5
105

# More Applications of the Differential Tree: Calibrating Black-Derman-Toy (in seconds)

| Number of years | Running time | Number of years | Running time | Number of years | Running time |
|---|---|---|---|---|---|
| 3000 | 398.880 | 39000 | 8562.640 | 75000 | 26182.080 |
| 6000 | 1697.680 | 42000 | 9579.780 | 78000 | 28138.140 |
| 9000 | 2539.040 | 45000 | 10785.850 | 81000 | 30230.260 |
| 12000 | 2803.890 | 48000 | 11905.290 | 84000 | 32317.050 |
| 15000 | 3149.330 | 51000 | 13199.470 | 87000 | 34487.320 |
| 18000 | 3549.100 | 54000 | 14411.790 | 90000 | 36795.430 |
| 21000 | 3990.050 | 57000 | 15932.370 | 120000 | 63767.690 |
| 24000 | 4470.320 | 60000 | 17360.670 | 150000 | 98339.710 |
| 27000 | 5211.830 | 63000 | 19037.910 | 180000 | 140484.180 |
| 30000 | 5944.330 | 66000 | 20751.100 | 210000 | 190557.420 |
| 33000 | 6639.480 | 69000 | 22435.050 | 240000 | 249138.210 |
| 36000 | 7611.630 | 72000 | 24292.740 | 270000 | 313480.390 |

75MHz Sun SPARCstation 20, one period per year.

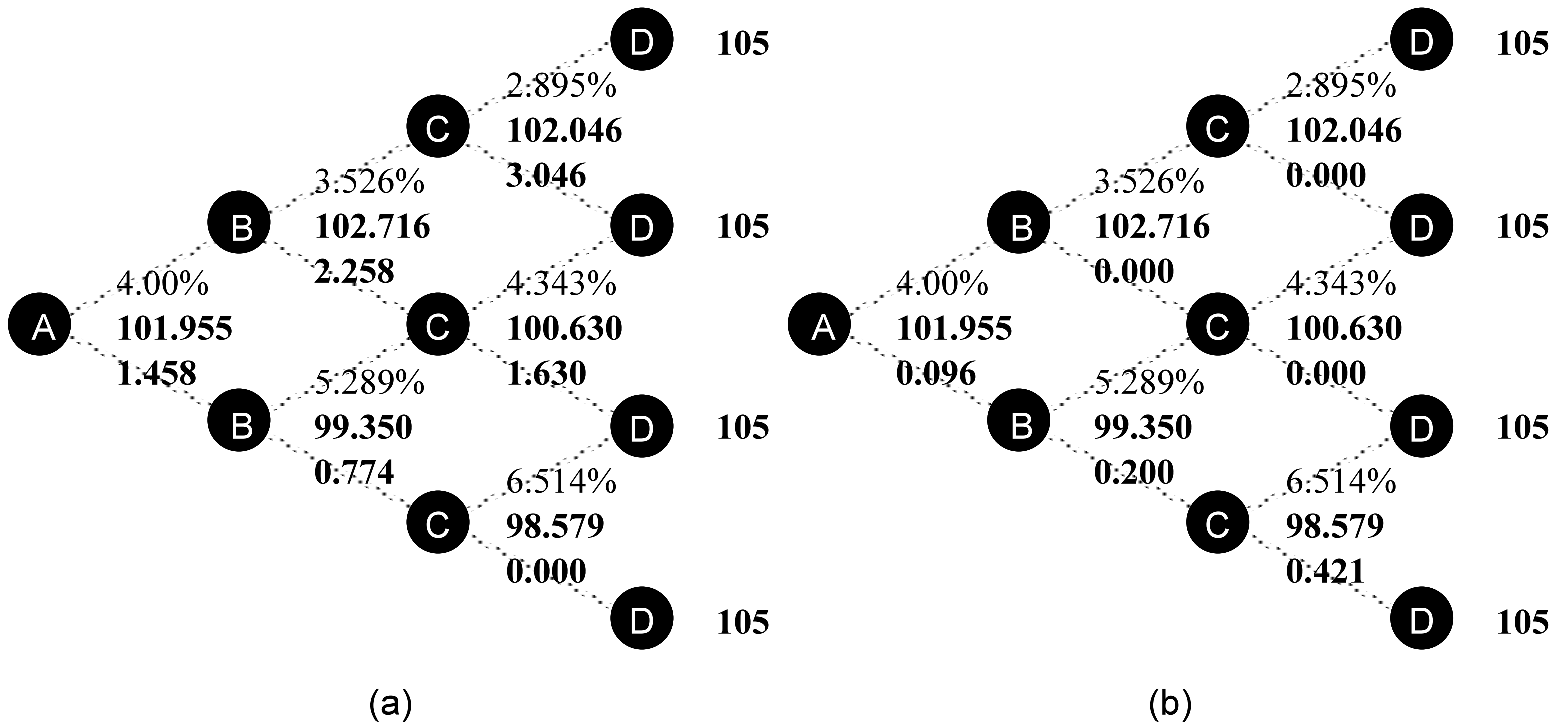
# More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)

| American call | | | American put | | |
|---|---|---|---|---|---|
| Number of partitions | Running time | Number of iterations | Number of partitions | Running time | Number of iterations |
| 100 | 0.008210 | 2 | 100 | 0.013845 | 3 |
| 200 | 0.033310 | 2 | 200 | 0.036335 | 3 |
| 300 | 0.072940 | 2 | 300 | 0.120455 | 3 |
| 400 | 0.129180 | 2 | 400 | 0.214100 | 3 |
| 500 | 0.201850 | 2 | 500 | 0.333950 | 3 |
| 600 | 0.290480 | 2 | 600 | 0.323260 | 2 |
| 700 | 0.394090 | 2 | 700 | 0.435720 | 2 |
| 800 | 0.522040 | 2 | 800 | 0.569605 | 2 |

Intel 166MHz Pentium, running on Microsoft Windows 95.

# Fixed-Income Options

- Consider a two-year 99 European call on the three-year, 5% Treasury.
- Assume the Treasury pays annual interest.
- From p. 780 the three-year Treasury's price minus the \$5 interest could be \$102.046, \$100.630, or \$98.579 two years from now.
- Since these prices do not include the accrued interest, we should compare the strike price against them.
- The call is therefore in the money in the first two scenarios, with values of \$3.046 and \$1.630, and out of the money in the third scenario.



A
4.00%
101.955
1.458
B
3.526%
102.716
2.258
B
5.289%
99.350
0.774
C
2.895%
102.046
3.046
C
4.343%
100.630
1.630
C
6.514%
98.579
0.000
D
105
D
105
D
105
D
105
(a)
A
4.00%
101.955
0.096
B
3.526%
102.716
0.000
B
5.289%
99.350
0.200
C
2.895%
102.046
0.000
C
4.343%
100.630
0.000
C
6.514%
98.579
0.421
D
105
D
105
D
105
D
105
(b)

## Fixed-Income Options (continued)

- The option value is calculated to be \$1.458 on p. 780(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only if the Treasury is worth \$98.579 without the accrued interest.
- The option value is computed to be \$0.096 on p. 780(b).

# Fixed-Income Options (concluded)

- The present value of the strike price is $\mathrm{PV}(X) = 99 \times 0.92101 = 91.18$.
- The Treasury is worth $B = 101.955$.
- The present value of the interest payments during the life of the options is

$$\mathrm{PV}(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275.$$

- The call and the put are worth $C = 1.458$ and $P = 0.096$, respectively.
- Hence the put-call parity is preserved:

$$C = P + B - \mathrm{PV}(I) - \mathrm{PV}(X).$$

# Delta or Hedge Ratio

- How much does the option price change in response to changes in the price of the underlying bond?
- This relation is called delta (or hedge ratio) defined as

$$\frac{O_{\mathrm{h}} - O_{\ell}}{P_{\mathrm{h}} - P_{\ell}}.$$

- In the above $P_{\mathrm{h}}$ and $P_{\ell}$ denote the bond prices if the short rate moves up and down, respectively.
- Similarly, $O_{\mathrm{h}}$ and $O_{\ell}$ denote the option values if the short rate moves up and down, respectively.

## Delta or Hedge Ratio (concluded)

- Since delta measures the sensitivity of the option value to changes in the underlying bond price, it shows how to hedge one with the other.
- Take the call and put on p. 780 as examples.
- Their deltas are

$$\begin{aligned} \frac{0.774 - 2.258}{99.350 - 102.716} &= 0.441, \\ \frac{0.200 - 0.000}{99.350 - 102.716} &= -0.059, \end{aligned}$$

respectively.

## Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an $n$-period zero-coupon bond.
- First find its yield to maturity $y_{\mathrm{h}}$ ($y_\ell$, respectively) at the end of the initial period if the rate rises (declines, respectively).
- The yield volatility for our model is defined as $(1/2)\ln(y_{\mathrm{h}}/y_\ell)$.

## Volatility Term Structures (continued)

- For example, based on the tree on p. 761, the two-year zero's yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.
- Its yield volatility is therefore

$$\frac{1}{2}\,\ln\left(\frac{0.05289}{0.03526}\right) = 20.273\%.$$

## Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the rate rises, the price of the zero one year from now will be

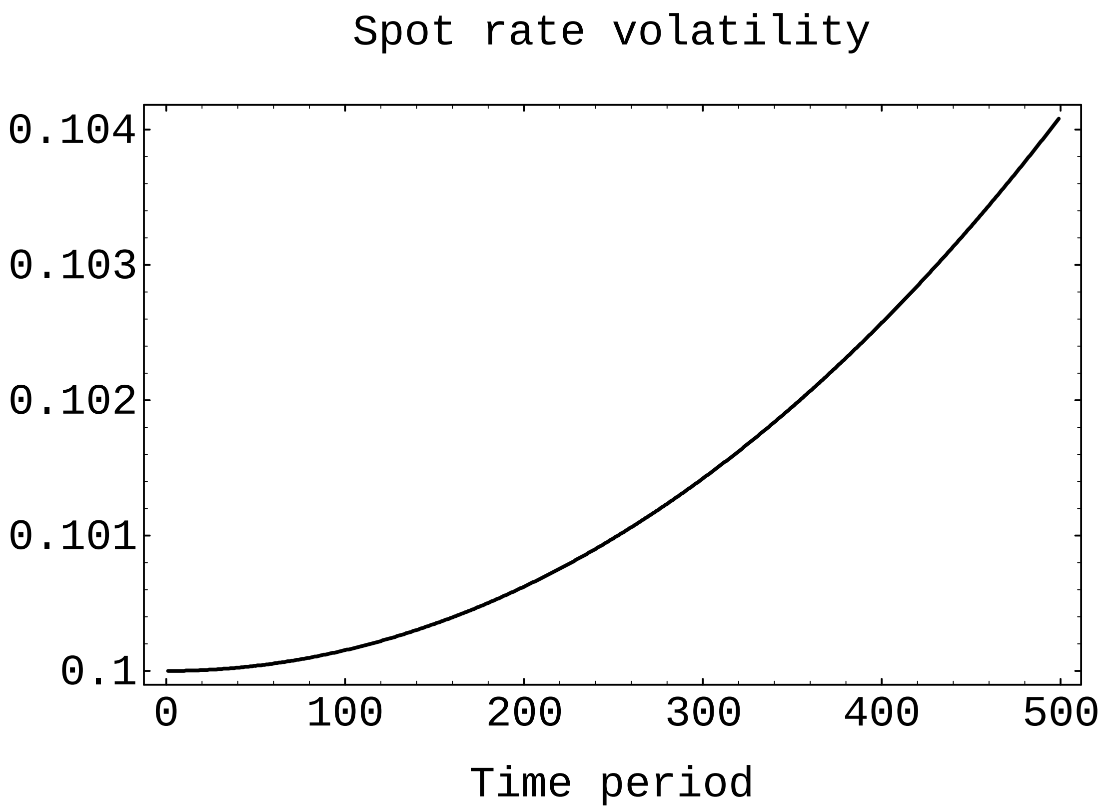
$$\frac{1}{2} \times \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.$$

- Thus its yield is $\sqrt{\frac{1}{0.90096}} - 1 = 0.053531$.
- If the rate declines, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.$$

## Volatility Term Structures (continued)

- Thus its yield is $\sqrt{\frac{1}{0.93225}} - 1 = 0.0357$.
- The yield volatility is hence
$$\frac{1}{2}\ln\left(\frac{0.053531}{0.0357}\right) = 20.256\%,$$
slightly less than the one-year yield volatility.
- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.




Short rate volatility given flat %10 volatility term structure.

# Volatility Term Structures (continued)

- We started with $v_i$ and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The $v_i$—hence the short rate volatilities via Eq. (76) on p. 741—and the $r_i$ are then simultaneously determined.
- The result is the Black-Derman-Toy model.

## Volatility Term Structures (concluded)

- Suppose the user supplies the volatility term structure which results in $(v_1, v_2, v_3, \ldots)$ for the tree.
- The volatility term structure one period from now will be determined by $(v_2, v_3, v_4, \ldots)$ not $(v_1, v_2, v_3, \ldots)$.
- The volatility term structure supplied by the user is hence not maintained through time.
- This issue will be addressed by other types of (complex) models.

*Foundations of Term Structure Modeling*

[Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader.

— Roger Lowenstein, *When Genius Failed* (2000)

[The] fixed-income traders I knew
seemed smarter than the equity trader $[\cdots]$
there's no competitive edge to
being smart in the equities business[.]
— Emanuel Derman,
*My Life as a Quant* (2004)

## Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time $t$, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

## Standard Notations

The following notation will be used throughout.

$t$**:** a point in time.

$r(t)$**:** the one-period riskless rate prevailing at time $t$ for repayment one period later (the instantaneous spot rate, or short rate, at time $t$).

$P(t, T)$**:** the present value at time $t$ of one dollar at time $T$.

## Standard Notations (continued)

$r(t, T)$**:** the $(T - t)$-period interest rate prevailing at time $t$ stated on a per-period basis and compounded once per period—in other words, the $(T - t)$-period spot rate at time $t$.

$F(t, T, M)$**:** the forward price at time $t$ of a forward contract that delivers at time $T$ a zero-coupon bond maturing at time $M \geq T$.

## Standard Notations (concluded)

$f(t, T, L)$**:** the $L$-period forward rate at time $T$ implied at time $t$ stated on a per-period basis and compounded once per period.

$f(t, T)$**:** the one-period or instantaneous forward rate at time $T$ as seen at time $t$ stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$.

# Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t,T) = \begin{cases} (1 + r(t,T))^{-(T-t)}, & \text{in discrete time,} \\ e^{-r(t,T)(T-t)}, & \text{in continuous time.} \end{cases}$$

- $r(t,T)$ as a function of $T$ defines the spot rate curve at time $t$.

- By definition,

$$f(t,t) = \begin{cases} r(t,t+1), & \text{in discrete time,} \\ r(t,t), & \text{in continuous time.} \end{cases}$$

## Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \tag{81}$$

  - The forward price equals the future value at time $T$ of the underlying asset (see text for proof).

- Equation (81) holds whether the model is discrete-time or continuous-time.

# Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t,T,L) = \left(\frac{1}{F(t,T,T+L)}\right)^{1/L} - 1 = \left(\frac{P(t,T)}{P(t,T+L)}\right)^{1/L} - 1 \tag{82}$$

  in discrete time.

  - $f(t,T,L) = \frac{1}{L}\left(\frac{P(t,T)}{P(t,T+L)} - 1\right)$ is the analog to Eq. (82) under simple compounding.

## Fundamental Relations (continued)

- In continuous time,

$$f(t,T,L) = -\frac{\ln F(t,T,T+L)}{L} = \frac{\ln(P(t,T)/P(t,T+L))}{L} \tag{83}$$

  by Eq. (81) on p. 800.

- Furthermore,

$$\begin{aligned} f(t,T,\Delta t) &= \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t,T)}{\partial T} \\ &= -\frac{\partial P(t,T)/\partial T}{P(t,T)}. \end{aligned}$$

## Fundamental Relations (continued)

- So

$$f(t,T) \equiv \lim_{\Delta t \to 0} f(t,T,\Delta t) = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \leq T. \tag{84}$$

- Because Eq. (84) is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}, \tag{85}$$

the spot rate curve is

$$r(t,T) = \frac{1}{T-t} \int_t^T f(t,s)\,ds.$$

# Fundamental Relations (concluded)

- The discrete analog to Eq. (85) is

$$P(t,T) = \frac{1}{(1+r(t))(1+f(t,t+1))\cdots(1+f(t,T-1))}.$$

- The short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}.$$

## Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t + 1 < T$,
$$\frac{E_t[\,P(t+1,T)\,]}{P(t,T)} = 1 + r(t). \tag{86}$$
  - Relation (86) in fact follows from the risk-neutral valuation principle.[a]

[a] Theorem 14 on p. 429.

## Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability $\pi$.
- Rewrite Eq. (86) as

$$\frac{E_t^{\pi}[\,P(t+1,T)\,]}{1+r(t)} = P(t,T).$$

  - It says the current spot rate curve equals the expected spot rate curve one period from now discounted by the short rate.

## Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

$$
\begin{aligned}
& P(t,T) \\
= \quad & E_t^{\pi}\left[\,\frac{P(t+1,T)}{1+r(t)}\,\right] \\
= \quad & E_t^{\pi}\left[\,\frac{E_{t+1}^{\pi}[\,P(t+2,T)\,]}{(1+r(t))(1+r(t+1))}\,\right] = \cdots \\
= \quad & E_t^{\pi}\left[\,\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))}\,\right]. \qquad (87)
\end{aligned}
$$

## Risk-Neutral Pricing (concluded)

- Equation (86) on p. 805 can also be expressed as

$$E_t[\,P(t+1,T)\,] = F(t,t+1,T).$$

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.

# Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

$$P(t, T) = E_t \left[ e^{-\int_t^T r(s)\, ds} \right], \quad t < T. \tag{88}$$

- Note that $e^{\int_t^T r(s)\, ds}$ is the bank account process, which denotes the rolled-over money market account.
- When the local expectations theory holds, riskless arbitrage opportunities are impossible.

# Interest Rate Swaps

- Consider an interest rate swap made at time $t$ with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is $c$ per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant $\Delta t$ for all $i$, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

## Interest Rate Swaps (continued)

- The amount to be paid out at time $t_{i+1}$ is $(f_i - c)\,\Delta t$ for the floating-rate payer.
- Simple rates are adopted here.
- Hence $f_i$ satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

## Interest Rate Swaps (continued)

- The value of the swap at time $t$ is thus

$$
\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi}\left[ e^{-\int_t^{t_i} r(s)\,ds} (f_{i-1} - c)\,\Delta t \right] \\
= \;& \sum_{i=1}^{n} E_t^{\pi}\left[ e^{-\int_t^{t_i} r(s)\,ds} \left( \frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\
= \;& \sum_{i=1}^{n} [\, P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \,] \\
= \;& P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}
$$

## Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.

## Swap Rate

- The swap rate, which gives the swap zero value, equals

$$S_n(t) \equiv \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i)\, \Delta t}. \tag{89}$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t, t_0) = 1$.

# The Binomial Model

- The analytical framework can be nicely illustrated with the binomial model.
- Suppose the bond price $P$ can move with probability $q$ to $Pu$ and probability $1 - q$ to $Pd$, where $u > d$:

$$
\begin{tikzcd}
 & Pd \\
P \arrow[ur, "1-q"] \arrow[dr, "q"'] & \\
 & Pu
\end{tikzcd}
$$

## The Binomial Model (continued)

- Over the period, the bond's expected rate of return is

$$\widehat{\mu} \equiv \frac{qPu + (1-q)\,Pd}{P} - 1 = qu + (1-q)\,d - 1. \tag{90}$$

- The variance of that return rate is

$$\widehat{\sigma}^2 \equiv q(1-q)(u-d)^2. \tag{91}$$

- The bond whose maturity is only one period away will move from a price of $1/(1+r)$ to its par value \$1.
- This is the money market account modeled by the short rate.

## The Binomial Model (continued)

- The market price of risk is defined as $\lambda \equiv (\widehat{\mu} - r)/\widehat{\sigma}$.
- As in the continuous-time case, it can be shown that $\lambda$ is independent of the maturity of the bond (see text).

# The Binomial Model (concluded)

- Now change the probability from $q$ to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1+r)-d}{u-d}, \tag{92}$$

  which is independent of bond maturity and $q$.
  - Recall the BOPM.
- The bond's expected rate of return becomes

$$\frac{pPu + (1-p)\,Pd}{P} - 1 = pu + (1-p)\,d - 1 = r.$$

- The local expectations theory hence holds under the new probability measure $p$.

# Numerical Examples

- Assume this spot rate curve:

| Year | 1 | 2 |
|---|---|---|
| Spot rate | 4% | 5% |

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

$$\begin{tikzcd}[row sep=small] & 8\% \\ 4\% \arrow[ur] \arrow[dr] & \\ & 2\% \end{tikzcd}$$

## Numerical Examples (continued)

- No real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$100/1.04 = 96.154, 100/(1.05)^2 = 90.703.$$

- They follow the binomial processes on p. 821.

## Numerical Examples (continued)

$$
\begin{tikzcd}[column sep=small]
 & 92.593 \ (= 100/1.08) \\
90.703 \arrow[ur] \arrow[dr] & \\
 & 98.039 \ (= 100/1.02)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
 & 100 \\
96.154 \arrow[ur] \arrow[dr] & \\
 & 100
\end{tikzcd}
$$

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

## Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

where $p$ denotes the risk-neutral probability of an up move in rates.

## Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

## Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,

$$
\begin{tikzcd}[row sep=small]
 & 0.000 \\
C \arrow[ur] \arrow[dr] & \\
 & 3.039
\end{tikzcd}
$$

- To solve for the option value $C$, we replicate the call by a portfolio of $x$ one-year and $y$ two-year zeros.

## Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$\begin{aligned} x \times 100 + y \times 92.593 &= 0.000, \\ x \times 100 + y \times 98.039 &= 3.039. \end{aligned}$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

# Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

## Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

  the same as before.
- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

## Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where $r$ is the one-year rate at maturity:

$$F \begin{array}{l} \nearrow \; 92 \; (= 100 - 8) \\ \searrow \; 98 \; (= 100 - 2) \end{array}$$

- As the futures price $F$ is the expected future payoff (see text), $F = (1 - p) \times 92 + p \times 98 = 93.914$.

- On the other hand, the forward price for a one-year forward contract on a one-year zero-coupon bond equals $90.703/96.154 = 94.331\%$.

- The forward price exceeds the futures price.

## Numerical Examples: Mortgage-Backed Securities

- Consider a 5%-coupon, two-year mortgage-backed security without amortization, prepayments, and default risk.
- Its cash flow and price process are illustrated on p. 830.
- Its fair price is

$$M = \frac{(1-p) \times 102.222 + p \times 107.941}{1.04} = 100.045.$$

- Identical results could have been obtained via arbitrage considerations.

$$
\begin{tikzcd}[row sep=small]
 & & 105 \\
 & 5 \arrow[ur] \arrow[dr] & \\
 & & 105 \\
0 \arrow[uur] \arrow[ddr] & & \\
 & & 105 \\
 & 5 \arrow[ur] \arrow[dr] & \\
 & & 105
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 102.222\ (= 5 + (105/1.08)) \\
M \arrow[ur] \arrow[dr] & \\
 & 107.941\ (= 5 + (105/1.02))
\end{tikzcd}
$$

The left diagram depicts the cash flow; the right diagram illustrates the price process.

## Numerical Examples: MBSs (continued)

- Suppose that the security can be prepaid at par.
- It will be prepaid only when its price is higher than par.
- Prepayment will hence occur only in the "down" state when the security is worth 102.941 (excluding coupon).
- The price therefore follows the process,

$$
\begin{array}{ccc}
 & \nearrow & 102.222 \\
M & & \\
 & \searrow & 105
\end{array}
$$

- The security is worth

$$M = \frac{(1-p) \times 102.222 + p \times 105}{1.04} = 99.142.$$

## Numerical Examples: MBSs (continued)

- The cash flow of the principal-only (PO) strip comes from the mortgage's principal cash flow.
- The cash flow of the interest-only (IO) strip comes from the interest cash flow (p. 833(a)).
- Their prices hence follow the processes on p. 833(b).
- The fair prices are

$$\begin{aligned} \text{PO} &= \frac{(1-p)\times 92.593 + p\times 100}{1.04} = 91.304, \\ \text{IO} &= \frac{(1-p)\times 9.630 + p\times 5}{1.04} = 7.839. \end{aligned}$$

$$
\begin{array}{ccccc}
 & & \text{PO:} & & 100 \\
 & & & \nearrow & \\
 & & 0 & & \\
 & \nearrow & & \searrow & \\
 & & & & 100 \\
0 & & & & \\
 & & & & 0 \\
 & \searrow & & \nearrow & \\
 & & 100 & & \\
 & & & \searrow & \\
 & & & & 0
\end{array}
\qquad
\begin{array}{ccccc}
 & & \text{IO:} & & 5 \\
 & & & \nearrow & \\
 & & 5 & & \\
 & \nearrow & & \searrow & \\
 & & & & 5 \\
0 & & & & \\
 & & & & 0 \\
 & \searrow & & \nearrow & \\
 & & 5 & & \\
 & & & \searrow & \\
 & & & & 0
\end{array}
$$

(a)

$$
\begin{array}{ccc}
 & & 92.593 \\
 & \nearrow & \\
\text{PO} & & \\
 & \searrow & \\
 & & 100
\end{array}
\qquad
\begin{array}{ccc}
 & & 9.630 \\
 & \nearrow & \\
\text{IO} & & \\
 & \searrow & \\
 & & 5
\end{array}
$$

(b)

The price $9.630$ is derived from $5 + (5/1.08)$.

## Numerical Examples: MBSs (continued)

- Suppose the mortgage is split into half floater and half inverse floater.
- Let the floater (FLT) receive the one-year rate.
- Then the inverse floater (INV) must have a coupon rate of

$$(10\% - \text{one-year rate})$$

  to make the overall coupon rate 5%.
- Their cash flows as percentages of par and values are shown on p. 835.

$$
\begin{array}{ccccc}
 & & \text{FLT:} & & 108 \\
 & & & \nearrow & \\
 & & 4 & & \\
 & \nearrow & & \searrow & \\
 & & & & 108 \\
0 & & & & \\
 & & & & 0 \\
 & \searrow & & \nearrow & \\
 & & 104 & & \\
 & & & \searrow & \\
 & & & & 0
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & \text{INV:} & & 102 \\
 & & & \nearrow & \\
 & & 6 & & \\
 & \nearrow & & \searrow & \\
 & & & & 102 \\
0 & & & & \\
 & & & & 0 \\
 & \searrow & & \nearrow & \\
 & & 106 & & \\
 & & & \searrow & \\
 & & & & 0
\end{array}
$$

(a)

$$
\begin{array}{ccc}
 & & 104 \\
 & \nearrow & \\
\text{FLT} & & \\
 & \searrow & \\
 & & 104
\end{array}
\qquad\qquad\qquad
\begin{array}{ccc}
 & & 100.444 \\
 & \nearrow & \\
\text{INV} & & \\
 & \searrow & \\
 & & 106
\end{array}
$$

(b)

## Numerical Examples: MBSs (concluded)

- On p. 835, the floater's price in the up node, 104, is derived from $4 + (108/1.08)$.
- The inverse floater's price 100.444 is derived from $6 + (102/1.08)$.
- The current prices are

$$\begin{aligned} \text{FLT} &= \frac{1}{2} \times \frac{104}{1.04} = 50, \\ \text{INV} &= \frac{1}{2} \times \frac{(1-p) \times 100.444 + p \times 106}{1.04} = 49.142. \end{aligned}$$

*Equilibrium Term Structure Models*

8. What's your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*

## Introduction

- This chapter surveys equilibrium models.
- Since the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t},$$

  the discount function $P(t,T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

## The Vasicek Model[a]

- The short rate follows

$$dr = \beta(\mu - r)\, dt + \sigma\, dW.$$

- The short rate is pulled to the long-term mean level $\mu$ at rate $\beta$.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\, dW$.
- Since the process is an Ornstein-Uhlenbeck process,

$$E[\, r(T) \,|\, r(t) = r\,] = \mu + (r - \mu)\, e^{-\beta(T-t)}$$

from Eq. (52) on p. 485.

[a] Vasicek (1977).

# The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{93}$$

where

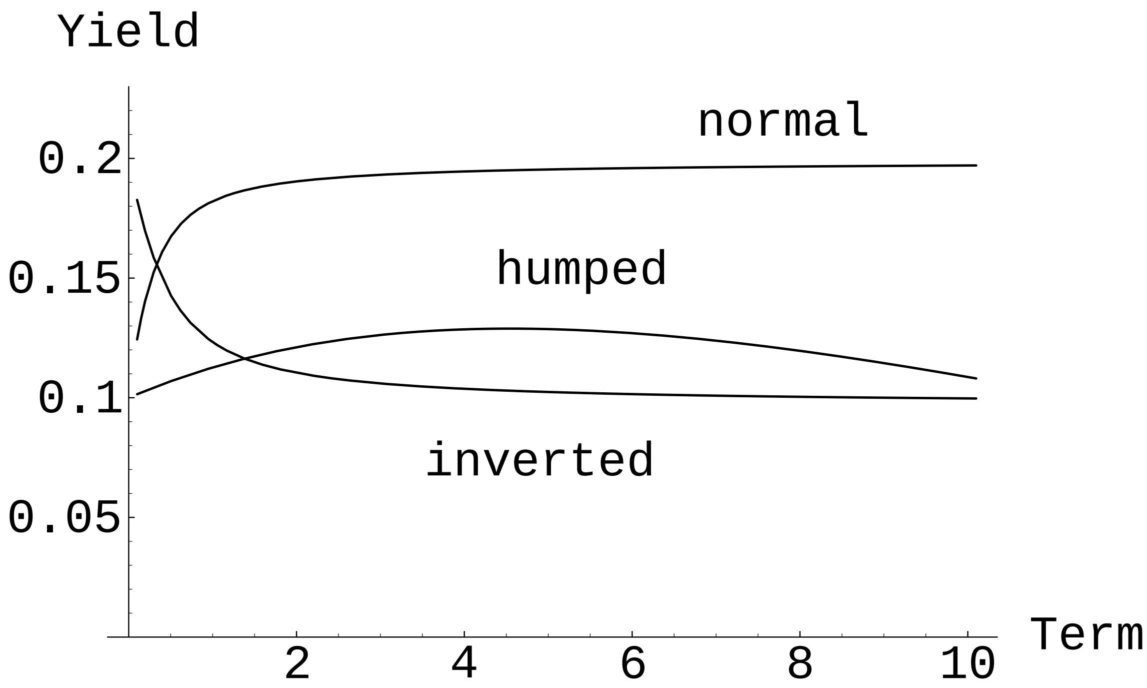
$$A(t,T) = \begin{cases} \exp\left[ \frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\ \exp\left[ \frac{\sigma^2 (T-t)^3}{6} \right] & \text{if } \beta = 0. \end{cases}$$

and

$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T-t & \text{if } \beta = 0. \end{cases}$$

# The Vasicek Model (concluded)

- If $\beta = 0$, then $P$ goes to infinity as $T \to \infty$.
- Sensibly, $P$ goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, $P$ may exceed one for a finite $T$.
- The spot rate volatility structure is the curve $(\partial r(t, T)/\partial r)\, \sigma = \sigma B(t, T)/(T - t)$.
- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, $\beta$, controls the shape of the curve.
- Iindeed, higher $\beta$ leads to greater attenuation of volatility with maturity.


Yield
normal
0.2
humped
0.15
0.1
inverted
0.05
Term
2
4
6
8
10

## The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price $X$ expiring at time $T$ on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t, s)\, N(x) - X P(t, T)\, N(x - \sigma_v).$$

[a] Jamshidian (1989).

## The Vasicek Model: Options on Zeros (concluded)

- Above

$$
\begin{aligned}
x &\equiv \frac{1}{\sigma_v} \ln \left( \frac{P(t,s)}{P(t,T)\,X} \right) + \frac{\sigma_v}{2}, \\
\sigma_v &\equiv v(t,T)\,B(T,s), \\
v(t,T)^2 &\equiv \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases} .
\end{aligned}
$$

- By the put-call parity, the price of a European put is

$$
X P(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).
$$

# Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[0, T]$ divided into $n$ identical pieces.
- Let $\Delta t \equiv T/n$ and

$$p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[a]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\,\xi(k), \quad 0 \leq k < n.$$

---

[a]Nelson and Ramaswamy (1990).

## Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases}.$$

- Observe that the probability of an up move, $p$, is a decreasing function of the interest rate $r$.
- This is consistent with mean reversion.

## Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, $\sigma$.
- For a general process $Y$ with nonconstant volatility, the resulting binomial tree may not combine.

# The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\, dt + \sigma\sqrt{r}\, dW. \tag{94}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter $\beta$ determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.
- See text for the bond pricing formula.

---

[a] Cox, Ingersoll, and Ross (1985).

## Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into $n$ periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

## Binomial CIR (continued)

- Instead, consider the transformed process

$$x(r) \equiv 2\sqrt{r}/\sigma.$$

- It follows

$$dx = m(x)\, dt + dW,$$

where

$$m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- Since this new process has a constant volatility, its associated binomial tree combines.

## Binomial CIR (continued)

- Construct the combining tree for $r$ as follows.
- First, construct a tree for $x$.
- Then transform each node of the tree into one for $r$ via the inverse transformation $r = f(x) \equiv x^2 \sigma^2 / 4$ (p. 853).

$$
\begin{array}{cccccccccc}
 & & & & x + 2\sqrt{\Delta t} & & & & & f(x + 2\sqrt{\Delta t}) \\
 & & & \nearrow & & & & & \nearrow & \\
 & & x + \sqrt{\Delta t} & & & & & f(x + \sqrt{\Delta t}) & & \\
 & \nearrow & & \searrow & & & \nearrow & & \searrow & \\
x & & & & x & f(x) & & & & f(x) \\
 & \searrow & & \nearrow & & & \searrow & & \nearrow & \\
 & & x - \sqrt{\Delta t} & & & & & f(x - \sqrt{\Delta t}) & & \\
 & & & \searrow & & & & & \searrow & \\
 & & & & x - 2\sqrt{\Delta t} & & & & & f(x - 2\sqrt{\Delta t})
\end{array}
$$

## Binomial CIR (concluded)

- The probability of an up move at each node $r$ is
$$p(r) \equiv \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}. \tag{95}$$
  - $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.
  - $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.
- Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.

# Numerical Examples

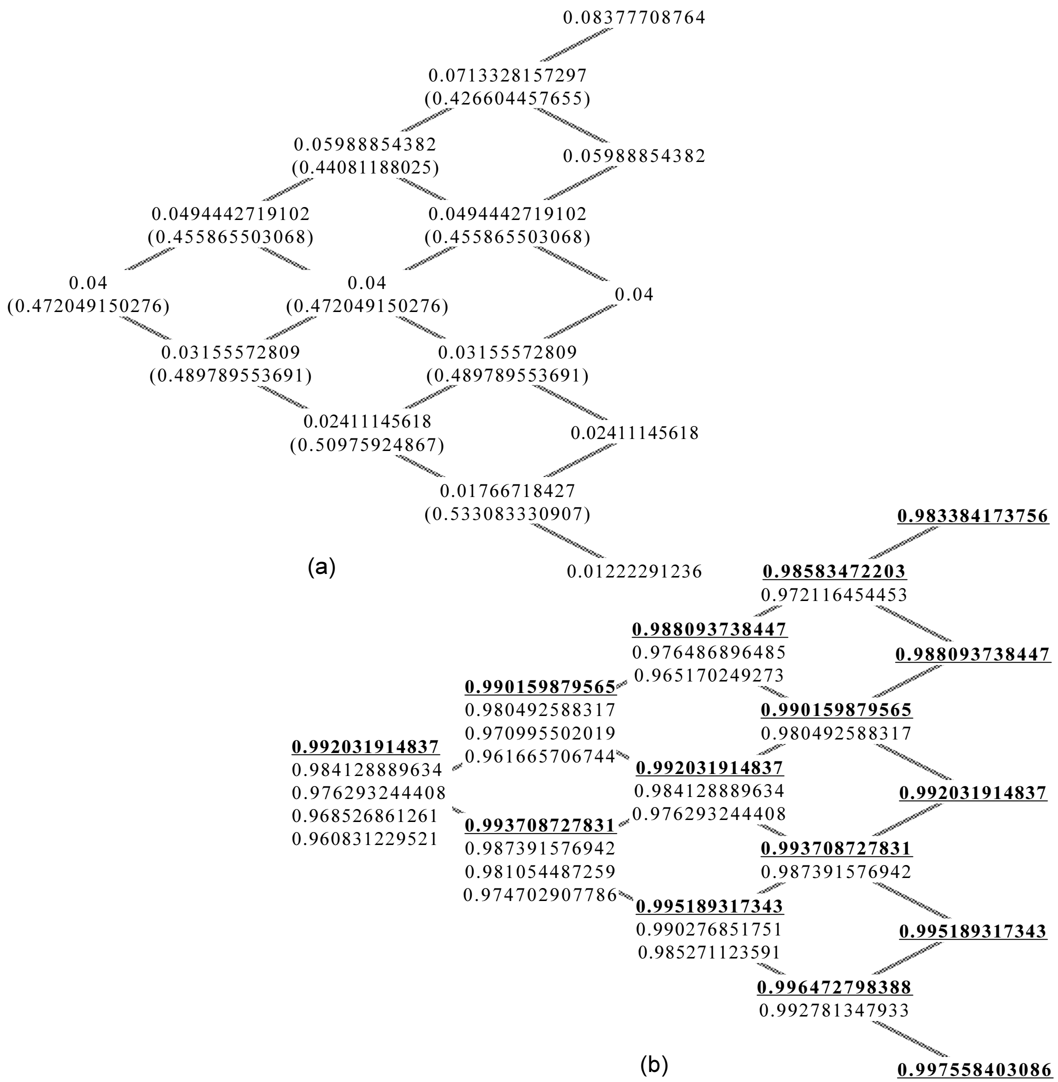
- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.

- See p. 856(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.


0.08377708764
0.0713328157297
(0.426604457655)
0.05988854382
(0.44081188025)
0.05988854382
0.0494442719102
(0.455865503068)
0.0494442719102
(0.455865503068)
0.04
(0.472049150276)
0.04
(0.472049150276)
0.04
0.03155572809
(0.489789553691)
0.03155572809
(0.489789553691)
0.02411145618
(0.50975924867)
0.02411145618
0.01766718427
(0.533083330907)
0.01222291236
(a)
0.983384173756
0.98583472203
0.972116454453
0.988093738447
0.976486896485
0.965170249273
0.988093738447
0.990159879565
0.980492588317
0.970995502019
0.961665706744
0.990159879565
0.980492588317
0.992031914837
0.984128889634
0.976293244408
0.968526861261
0.960831229521
0.992031914837
0.984128889634
0.976293244408
0.992031914837
0.993708727831
0.987391576942
0.981054487259
0.974702907786
0.993708727831
0.987391576942
0.995189317343
0.990276851751
0.985271123591
0.995189317343
0.996472798388
0.992781347933
(b)
0.997558403086

## Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's $x$ value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$.

## Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).

## A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process $dy = \alpha(y, t)\, dt + \sigma(y, t)\, dW$.
- Make sure the binomial model's drift and diffusion converge to the above process by setting the probability of an up move to

$$\frac{\alpha(y, t)\, \Delta t + y - y_{\mathrm{u}}}{y_{\mathrm{u}} - y_{\mathrm{d}}}.$$

- Here $y_{\mathrm{u}} \equiv y + \sigma(y, t)\sqrt{\Delta t}$ and $y_{\mathrm{d}} \equiv y - \sigma(y, t)\sqrt{\Delta t}$ represent the two rates that follow the current rate $y$.
- The displacements are identical, at $\sigma(y, t)\sqrt{\Delta t}$.

[a]Nelson and Ramaswamy (1990).

# A General Method (continued)

- But the binomial tree may not combine:

$$\sigma(y,t)\sqrt{\Delta t}-\sigma(y_{\mathrm{u}},t)\sqrt{\Delta t}\neq-\sigma(y,t)\sqrt{\Delta t}+\sigma(y_{\mathrm{d}},t)\sqrt{\Delta t}$$

  in general.

- When $\sigma(y,t)$ is a constant independent of $y$, equality holds and the tree combines.

- To achieve this, define the transformation

$$x(y,t)\equiv\int^{y}\sigma(z,t)^{-1}\,dz.$$

- Then $x$ follows $dx=m(y,t)\,dt+dW$ for some $m(y,t)$ (see text).

## A General Method (continued)

- The key is that the diffusion term is now a constant, and the binomial tree for $x$ combines.
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from $x$ back to $y$.
- Note that $y_{\mathrm{u}}(x,t) \equiv y(x+\sqrt{\Delta t}, t+\Delta t)$ and $y_{\mathrm{d}}(x,t) \equiv y(x-\sqrt{\Delta t}, t+\Delta t)$.

## A General Method (concluded)

- The transformation is

$$\int^{r} (\sigma\sqrt{z})^{-1}\, dz = 2\sqrt{r}/\sigma$$

  for the CIR model.

- The transformation is

$$\int^{S} (\sigma z)^{-1}\, dz = (1/\sigma) \ln S$$

  for the Black-Scholes model.

- The familiar binomial option pricing model in fact discretizes $\ln S$ not $S$.

*Finis*