Gamma

- The finite-difference formula for gamma is

$$e^{-r\tau}\, E\left[\frac{P(S+\epsilon)-2\times P(S)+P(S-\epsilon)}{\epsilon^2}\right].$$

- For a correlation option with multiple underlying assets, the finite-difference formula for the cross gammas $\partial^2 P(S_1, S_2, \dots)/(\partial S_1 \partial S_2)$ is:

$$e^{-r\tau}\, E\left[\frac{P(S_1+\epsilon_1, S_2+\epsilon_2)-P(S_1-\epsilon_1, S_2+\epsilon_2)}{4\epsilon_1\epsilon_2}\right.$$

$$\left.\frac{-P(S_1+\epsilon_1, S_2-\epsilon_2)+P(S_1-\epsilon_1, S_2-\epsilon_2)}{}\right].$$

Gamma (concluded)

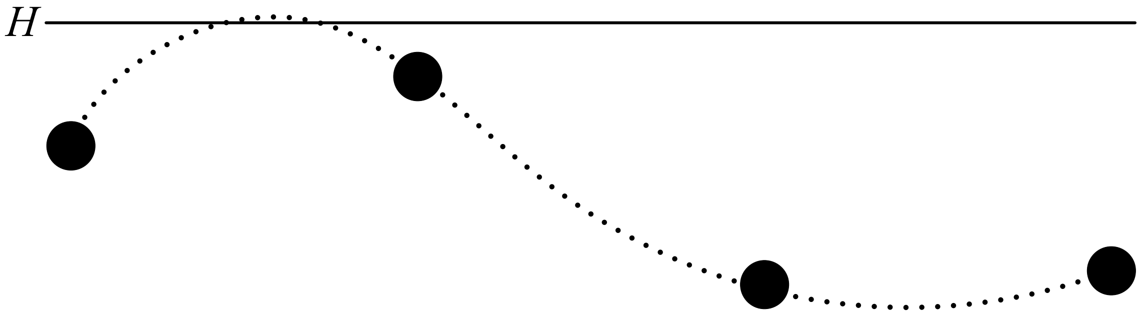
- Choosing an ϵ of the right magnitude can be challenging.
  - If ϵ is too large, inaccurate Greeks result.
  - If ϵ is too small, unstable Greeks result.

Biases in Pricing Continuously Monitored Options with Monte Carlo

- We are asked to price a continuously monitored up-and-out call with barrier H.
- The Monte Carlo method samples the stock price at n discrete time points $t_1, t_2, \ldots, t_n$.
- A sample path $S(t_0), S(t_1), \ldots, S(t_n)$ is produced.
  - Here, $t_0 = 0$ is the current time, and $t_n = T$ is the expiration time of the option.
- If all of the sampled prices are below the barrier, this sample path pays $\max(S(t_n) - X, 0)$.

Biases in Pricing Continuously Monitored Options with Monte Carlo (continued)

- Repeating these steps and averaging the payoffs yield a Monte Carlo estimate.
- This estimate is biased.
  - Suppose none of the sampled prices on a sample path equals or exceeds the barrier H.
  - It remains possible for the continuous sample path that passes through them to hit the barrier between sampled time points (see plot on next page).



H

Biases in Pricing Continuously Monitored Options with Monte Carlo (concluded)

- The bias can certainly be lowered by increasing the number of observations along the sample path.
- However, even daily sampling may not suffice.
- The computational cost also rises as a result.

Brownian Bridge Approach to Pricing Barrier Options

- We desire an unbiased estimate efficiently.
- So the above-mentioned payoff should be multiplied by the probability p that a continuous sample path does not hit the barrier conditional on the sampled prices.
- This methodology is called the Brownian bridge approach.
- Formally, we have

$$p \equiv \text{Prob}[\, S(t) < H, 0 \leq t \leq T \mid S(t_0), S(t_1), \ldots, S(t_n)\,].$$

Brownian Bridge Approach to Pricing Barrier Options (continued)

- As a barrier is hit over a time interval if and only if the maximum stock price over that period is at least H,

$$p = \text{Prob}\left[\max_{0 \leq t \leq T} S(t) < H \mid S(t_0), S(t_1), \ldots, S(t_n)\right].$$

- Luckily, the conditional distribution of the maximum over a time interval given the beginning and ending stock prices is known.

Brownian Bridge Approach to Pricing Barrier Options (continued)

Lemma 19 *Assume S follows $dS/S = \mu\, dt + \sigma\, dW$ and define*

$$\zeta(x) \equiv \exp\left[-\frac{2\ln(x/S(t))\ln(x/S(t+\Delta t))}{\sigma^2 \Delta t} \right].$$

(1) If $H > \max(S(t), S(t+\Delta t))$, then

$$\text{Prob}\left[\max_{t \leq u \leq t+\Delta t} S(u) < H \,\middle|\, S(t), S(t+\Delta t) \right] = 1 - \zeta(H).$$

(2) If $h < \min(S(t), S(t+\Delta t))$, then

$$\text{Prob}\left[\min_{t \leq u \leq t+\Delta t} S(u) > h \,\middle|\, S(t), S(t+\Delta t) \right] = 1 - \zeta(h).$$

Brownian Bridge Approach to Pricing Barrier Options (continued)

- Lemma 19 gives the probability that the barrier is not hit in a time interval, given the starting and ending stock prices.
- For our up-and-out call, choose $n = 1$.
- As a result,

$$p = \begin{cases} 1 - \exp\left[-\frac{2 \ln(H/S(0)) \ln(H/S(T))}{\sigma^2 T} \right], & \text{if } H > \max(S(0), S(T)), \\ 0, & \text{otherwise.} \end{cases}$$

Brownian Bridge Approach to Pricing Barrier Options (continued)

$C := 0$;
for $i = 1, 2, 3, \ldots, m$ **do**
    $P := S \times e^{(r-q-\sigma^2/2)\,T + \sigma\sqrt{T}\;\xi(\,)}$;
    if $(S < H$ and $P < H)$ or $(S > H$ and $P > H)$ **then**
        $C := C + \max(P - X, 0) \times \left\{ 1 - \exp\left[-\frac{2\ln(H/S) \times \ln(H/P)}{\sigma^2 T} \right] \right\}$;
    end if
end for
`return` Ce^{-rT}/m;

Brownian Bridge Approach to Pricing Barrier Options (concluded)

- The idea can be generalized.
- For example, we can handle more complex barrier options.
- Consider an up-and-out call with barrier H_i for the time interval $(t_i, t_{i+1}]$, $0 \leq i < n$.
- This option thus contains n barriers.
- It is a simple matter of multiplying the probabilities for the n time intervals properly to obtain the desired probability adjustment term.

Variance Reduction: Antithetic Variates

- We are interested in estimating $E[\, g(X_1, X_2, \ldots, X_n)\,]$, where $X_1, X_2, \ldots, X_n$ are independent.
- Let Y_1 and Y_2 be random variables with the same distribution as $g(X_1, X_2, \ldots, X_n)$.
- Then

$$\mathrm{Var}\left[\frac{Y_1 + Y_2}{2}\right] = \frac{\mathrm{Var}[\, Y_1\,]}{2} + \frac{\mathrm{Cov}[\, Y_1, Y_2\,]}{2}.$$

  - $\mathrm{Var}[\, Y_1\,]/2$ is the variance of the Monte Carlo method with two (independent) replications.
- The variance $\mathrm{Var}[\, (Y_1 + Y_2)/2\,]$ is smaller than $\mathrm{Var}[\, Y_1\,]/2$ when Y_1 and Y_2 are negatively correlated.

Variance Reduction: Antithetic Variates (continued)

- For each simulated sample path X, a second one is obtained by reusing the random numbers on which the first path is based.
- This yields a second sample path Y.
- Two estimates are then obtained: One based on X and the other on Y.
- If N independent sample paths are generated, the antithetic-variates estimator averages over $2N$ estimates.

Variance Reduction: Antithetic Variates (continued)

- Consider process $dX = a_t \, dt + b_t \sqrt{dt} \, \xi$.
- Let g be a function of n samples $X_1, X_2, \ldots, X_n$ on the sample path.
- We are interested in $E[\, g(X_1, X_2, \ldots, X_n) \,]$.
- Suppose one simulation run has realizations $\xi_1, \xi_2, \ldots, \xi_n$ for the normally distributed fluctuation term ξ.
- This generates samples $x_1, x_2, \ldots, x_n$.
- The estimate is then $g(\boldsymbol{x})$, where $\boldsymbol{x} \equiv (x_1, x_2 \ldots, x_n)$.

Variance Reduction: Antithetic Variates (concluded)

- We do not sample n more numbers from ξ for the second estimate.
- The antithetic-variates method computes $g(\boldsymbol{x}')$ from the sample path $\boldsymbol{x}' \equiv (x_1', x_2' \ldots, x_n')$ generated by $-\xi_1, -\xi_2, \ldots, -\xi_n$.
- We then output $(g(\boldsymbol{x}) + g(\boldsymbol{x}'))/2$.
- Repeat the above steps for as many times as required by accuracy.

Variance Reduction: Conditioning

- We are interested in estimating $E[\,X\,]$.
- Suppose here is a random variable Z such that $E[\,X \mid Z = z\,]$ can be efficiently and precisely computed.
- $E[\,X\,] = E[\,E[\,X \mid Z\,]\,]$ by the law of iterated conditional expectations.
- Hence the random variable $E[\,X \mid Z\,]$ is also an unbiased estimator of $E[\,X\,]$.

Variance Reduction: Conditioning (concluded)

- As $\mathrm{Var}[\, E[\, X \mid Z \,]\,] \leq \mathrm{Var}[\, X \,]$, $E[\, X \mid Z \,]$ has a smaller variance than observing X directly.
- First obtain a random observation z on Z.
- Then calculate $E[\, X \mid Z = z \,]$ as our estimate.
  - There is no need to resort to simulation in computing $E[\, X \mid Z = z \,]$.
- The procedure can be repeated a few times to reduce the variance.

Control Variates

- Use the analytic solution of a similar yet simpler problem to improve the solution.
- Suppose we want to estimate $E[\,X\,]$ and there exists a random variable Y with a known mean $\mu \equiv E[\,Y\,]$.
- Then $W \equiv X + \beta(Y - \mu)$ can serve as a "controlled" estimator of $E[\,X\,]$ for any constant β.
  - β can scale the deviation $Y - \mu$ to arrive at an adjustment for X.
  - However β is chosen, W remains an unbiased estimator of $E[\,X\,]$ as

$$E[\,W\,] = E[\,X\,] + \beta E[\,Y - \mu\,] = E[\,X\,].$$

Control Variates (continued)

- Note that

$$\mathrm{Var}[\,W\,] = \mathrm{Var}[\,X\,] + \beta^2\,\mathrm{Var}[\,Y\,] + 2\beta\,\mathrm{Cov}[\,X, Y\,], \tag{64}$$

- Hence W is less variable than X if and only if

$$\beta^2\,\mathrm{Var}[\,Y\,] + 2\beta\,\mathrm{Cov}[\,X, Y\,] < 0. \tag{65}$$

- The success of the scheme clearly depends on both β and the choice of Y.

Control Variates (concluded)

- For example, arithmetic average-rate options can be priced by choosing Y to be the otherwise identical geometric average-rate option's price and $\beta = -1$.
- This approach is much more effective than the antithetic-variates method.

Choice of Y

- In general, the choice of Y is ad hoc, and experiments must be performed to confirm the wisdom of the choice.
- Try to match calls with calls and puts with puts.[a]
- On many occasions, Y is a discretized version of the derivative that gives μ.
  - Discretely monitored geometric average-rate option vs. the continuously monitored geometric average-rate option given by formulas (29) on p. 327.
- For some choices, the discrepancy can be significant, such as the lookback option.[b]

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[a]Contributed by Ms. Teng, Huei-Wen (`R91723054`) on May 25, 2004.
[b]Contributed by Mr. Tsai, Hwai (`R92723049`) on May 12, 2004.

Optimal Choice of β

- Equation (64) on p. 618 is minimized when

$$\beta = -\mathrm{Cov}[\,X, Y\,]/\mathrm{Var}[\,Y\,],$$

which was called beta earlier in the book.

- For this specific β,

$$\mathrm{Var}[\,W\,] = \mathrm{Var}[\,X\,] - \frac{\mathrm{Cov}[\,X, Y\,]^2}{\mathrm{Var}[\,Y\,]} = \left(1 - \rho_{X,Y}^2\right) \mathrm{Var}[\,X\,],$$

where $\rho_{X,Y}$ is the correlation between X and Y.

- The stronger X and Y are correlated, the greater the reduction in variance.

Optimal Choice of β (continued)

- For example, if this correlation is nearly perfect (± 1), we could control X almost exactly, eliminating practically all of its variance.
- Typically, neither $\text{Var}[\,Y\,]$ nor $\text{Cov}[\,X, Y\,]$ is known.
- Therefore, we cannot obtain the maximum reduction in variance.
- We can guess these values and hope that the resulting W does indeed have a smaller variance than X.
- A second possibility is to use the simulated data to estimate these quantities.

Optimal Choice of β (concluded)

- Observe that $-\beta$ has the same sign as the correlation between X and Y.
- Hence, if X and Y are positively correlated, $\beta < 0$, then X is adjusted downward whenever $Y > \mu$ and upward otherwise.
- The opposite is true when X and Y are negatively correlated, in which case $\beta > 0$.

Problems with the Monte Carlo Method

- The error bound is only probabilistic.
- The probabilistic error bound of $\sqrt{N}$ does not benefit from regularity of the integrand function.
- The requirement that the points be independent random samples are wasteful because of clustering.
- In reality, pseudorandom numbers generated by completely deterministic means are used.
- Monte Carlo simulation exhibits a great sensitivity on the seed of the pseudorandom-number generator.