Delta Hedge

- The delta (hedge ratio) of a derivative $f$ is defined as $\Delta \equiv \partial f / \partial S$.

- Thus $\Delta f \approx \Delta \times \Delta S$ for relatively small changes in the stock price, $\Delta S$.

- A delta-neutral portfolio is hedged in the sense that it is immunized against small changes in the stock price.

- A trading strategy that dynamically maintains a delta-neutral portfolio is called delta hedge.
Delta Hedge (concluded)

• Delta changes with the stock price.

• A delta hedge needs to be rebalanced periodically in order to maintain delta neutrality.

• In the limit where the portfolio is adjusted continuously, perfect hedge is achieved and the strategy becomes self-financing.

• This was the gist of the Black-Scholes-Merton argument.
Implementing Delta Hedge

• We want to hedge $N$ short derivatives.
• Assume the stock pays no dividends.
• The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus $B$ borrowed dollars such that

$$-N \times f + N \times \Delta \times S - B = 0.$$  

• At next rebalancing point when the delta is $\Delta'$, buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares with a total borrowing of $B' = N \times \Delta' \times S' - N \times f'$.

• Delta hedge is the discrete-time analog of the continuous-time limit and will rarely be self-financing.
Example

- A hedger is short 10,000 European calls.
- $\sigma = 30\%$ and $r = 6\%$.
- This call’s expiration is four weeks away, its strike price is $\$50$, and each call has a current value of $f = 1.76791$.
- As an option covers 100 shares of stock, $N = 1,000,000$.
- The trader adjusts the portfolio weekly.
- The calls are replicated\(^a\) well if the cumulative cost of trading stock is close to the call premium’s FV.

\(^a\)This example takes the replication viewpoint.
Example (continued)

• As \( \Delta = 0.538560 \), \( N \times \Delta = 538,560 \) shares are purchased for a total cost of \( 538,560 \times 50 = 26,928,000 \) dollars to make the portfolio delta-neutral.

• The trader finances the purchase by borrowing

\[
B = N \times \Delta \times S - N \times f = 25,160,090
\]

dollars net.\(^a\)

• The portfolio has zero net value now.

\(^a\)This takes the hedging viewpoint — an alternative. See an exercise in the text.
Example (continued)

• At 3 weeks to expiration, the stock price rises to $51.
• The new call value is $f' = 2.10580$.
• So the portfolio is worth

\[-N \times f' + 538,560 \times 51 - Be^{0.06/52} = 171,622\]

before rebalancing.
Example (continued)

• A delta hedge does not replicate the calls perfectly; it is not self-financing as $171,622 can be withdrawn.

• The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.

• In fact, the tracking error over one rebalancing act is positive about 68% of the time, but its expected value is essentially zero.\(^a\)

• It is furthermore proportional to vega.

\(^a\)Boyle and Emanuel (1980).
Example (continued)

- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.

- With a higher delta $\Delta' = 0.640355$, the trader buys $N \times (\Delta' - \Delta) = 101,795$ shares for $5,191,545$.

- The number of shares is increased to $N \times \Delta' = 640,355$. 
Example (continued)

- The cumulative cost is
  \[26,928,000 \times e^{0.06/52} + 5,191,545 = 32,150,634.\]

- The total borrowed amount is
  \[B' = 640,355 \times 51 - N \times f' = 30,552,305.\]

- The portfolio is again delta-neutral with zero value.
<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$S$</th>
<th>$f$</th>
<th>$\Delta$</th>
<th>$N \times (5)$</th>
<th>$(1) \times (6)$</th>
<th>$FV(8') + (7)$</th>
<th>Cumulative cost</th>
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<td>0.16017</td>
<td>160,175</td>
<td>8,649,450</td>
<td>51,524,853</td>
</tr>
</tbody>
</table>

The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).
Example (concluded)

- At expiration, the trader has 1,000,000 shares.
- They are exercised against by the in-the-money calls for $50,000,000.
- The trader is left with an obligation of
  
  \[ 51,524,853 - 50,000,000 = 1,524,853, \]
  
  which represents the replication cost.
- Compared with the FV of the call premium,
  
  \[ 1,767,910 \times e^{0.06 \times 4/52} = 1,776,088, \]
  
  the net gain is \[ 1,776,088 - 1,524,853 = 251,235. \]
Tracking Error Revisited\textsuperscript{a}

- The tracking error $\epsilon_n$ over $n$ rebalancing acts (such as 251,235 above) has about the same probability of being positive as being negative.

- Subject to certain regularity conditions, the root-mean-square tracking error $\sqrt{E[\epsilon_n^2]}$ is $O(1/\sqrt{n})$.\textsuperscript{b}

- The root-mean-square tracking error increases with $\sigma$ at first and then decreases.

\textsuperscript{a}Bertsimas, Kogan, and Lo (2000).

\textsuperscript{b}See also Grannan and Swindle (1996).
Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, $\Delta f$, due to changes in the stock price, $\Delta S$.

- When $\Delta S$ is not small, the second-order term, gamma $\Gamma \equiv \partial^2 f/\partial S^2$, helps (theoretically).

- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.

- To meet this extra condition, one more security needs to be brought in.
Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call $f_2$ is brought in.
- To set up a delta-gamma hedge, we solve

\[-N \times f + n_1 \times S + n_2 \times f_2 - B = 0 \text{ (self-financing)},\]
\[-N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 = 0 \text{ (delta neutrality)},\]
\[-N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 = 0 \text{ (gamma neutrality)},\]

for $n_1$, $n_2$, and $B$.

- The gammas of the stock and bond are 0.
Other Hedges

- If volatility changes, delta-gamma hedge may not work well.

- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.

- To accomplish this, one more security has to be brought into the process.

- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.
Trees
I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

And though the holes were rather small,
they had to count them all.
The Combinatorial Method

• The combinatorial method can often cut the running time by an order of magnitude.

• The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.

• We first used this method in the linear-time algorithm for standard European option pricing on p. 231.
  – In general, it cannot apply to American options.

• We will now apply it to price barrier options.
The Reflection Principle\textsuperscript{a}

- Imagine a particle at position $(0, -a)$ on the integral lattice that is to reach $(n, -b)$.
- Without loss of generality, assume $a > 0$ and $b \geq 0$.
- This particle’s movement:
  
  \[
  (i, j) \rightarrow (i + 1, j + 1) \text{ up move } S \rightarrow Su
  \]
  
  \[
  (i, j) \rightarrow (i + 1, j - 1) \text{ down move } S \rightarrow Sd
  \]
- How many paths touch the $x$ axis?

\textsuperscript{a}André (1887).
The Reflection Principle (continued)

- For a path from $(0, -a)$ to $(n, -b)$ that touches the $x$ axis, let $J$ denote the first point this happens.
- Reflect the portion of the path from $(0, -a)$ to $J$.
- A path from $(0, a)$ to $(n, -b)$ is constructed.
- It also hits the $x$ axis at $J$ for the first time.
- The one-to-one mapping shows the number of paths from $(0, -a)$ to $(n, -b)$ that touch the $x$ axis equals the number of paths from $(0, a)$ to $(n, -b)$.
The Reflection Principle (concluded)

- A path of this kind has \((n + b + a)/2\) down moves and 
  \((n - b - a)/2\) up moves.
- Hence there are

\[
\binom{n}{n+a+b} \quad \text{such paths for even } n + a + b.
\]

  - Convention: \(\binom{n}{k} = 0\) for \(k < 0\) or \(k > n\).
Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define

\[
\begin{align*}
a & \equiv \left\lceil \frac{\ln \left( \frac{X}{(Sd^n)} \right)}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma \sqrt{\Delta t}} + \frac{n}{2} \right\rceil, \\
h & \equiv \left\lfloor \frac{\ln \left( \frac{H}{(Sd^n)} \right)}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma \sqrt{\Delta t}} + \frac{n}{2} \right\rfloor. 
\end{align*}
\]

- $h$ is such that $\tilde{H} \equiv Su^h d^{n-h}$ is the terminal price that is closest to, but does not exceed $H$.
- $a$ is such that $\tilde{X} \equiv Su^a d^{n-a}$ is the terminal price that is closest to, but is not exceeded by $X$. 
Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with $n$ moves hence ends up in the money if and only if the number of up moves is at least $a$.
- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price $Sd^n$ on the binomial tree.

\[
Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-2k}. \tag{55}
\]
Pricing Barrier Options (continued)

• The number of paths from $S$ to the terminal price $S_u^j d^{n-j}$ is $\binom{n}{j}$, each with probability $p^j(1-p)^{n-j}$.

• With reference to p. 525, the reflection principle can be applied with $a = n - 2h$ and $b = 2j - 2h$ in Eq. (54) on p. 522 by treating the $S$ line as the $x$ axis.

• Therefore,

$$\left(\frac{n}{2 + (n-2h)+(2j-2h)}\right) = \binom{n}{n-2h+j}$$

paths hit $\tilde{H}$ in the process for $h \leq n/2$. 
Pricing Barrier Options (concluded)

- The terminal price $S u^j d^{n-j}$ is reached by a path that hits the effective barrier with probability

\[
\binom{n}{n - 2h + j} p^j (1 - p)^{n-j}.
\]

- The option value equals

\[
\sum_{j=0}^{2h} \binom{n}{n - 2h + j} p^j (1 - p)^{n-j} \frac{(S u^j d^{n-j} - X)}{R^n}.
\]  \hspace{1cm} (56)

- $R \equiv e^{r \tau/n}$ is the riskless return per period.

- It implies a linear-time algorithm.
Convergence of BOPM

- Equation (56) results in the sawtooth-like convergence shown on p. 310.
- The reasons are not hard to see.
- The true barrier most likely does not equal the effective barrier.
- The same holds for the strike price and the effective strike price.
- The issue of the strike price is less critical.
- But the issue of the barrier is not negligible.
Convergence of BOPM (continued)

• Convergence is actually good if we limit $n$ to certain values—191, for example.

• These values make the true barrier coincide with or occur just above one of the stock price levels, that is, $H \approx Sd^j = Se^{-j\sigma\sqrt{\tau/n}}$ for some integer $j$.

• The preferred $n$’s are thus

$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

• There is only one minor technicality left.
Convergence of BOPM (continued)

- We picked the effective barrier to be one of the $n + 1$ possible terminal stock prices.
- However, the effective barrier above, $Sd^j$, corresponds to a terminal stock price only when $n - j$ is even by Eq. (55) on p. 524.\(^a\)
- To close this gap, we decrement $n$ by one, if necessary, to make $n - j$ an even number.

\(^a\)We could have adopted the form $Sd^j \ (-n \leq j \leq n)$ for the effective barrier.
Convergence of BOPM (concluded)

• The preferred \( n \)'s are now

\[
\begin{align*}
n &= \begin{cases} 
\ell & \text{if } \ell - j \text{ is even} \\
\ell - 1 & \text{otherwise}
\end{cases}, \\
\end{align*}
\]

\( j = 1, 2, 3, \ldots \), where

\[
\ell \equiv \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.
\]

• Evaluate pricing formula (56) on p. 527 only with the \( n \)'s above.
Down-and-in call value

#Periods
0 500 1000 1500 2000 2500 3000 3500
5.5
5.55
5.6
5.65
5.7

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Practical Implications

• Now that barrier options can be efficiently priced, we can afford to pick very large $n$’s (p. 534).

• This has profound consequences.
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<tr>
<td>7717</td>
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</table>
Practical Implications (concluded)

• Pricing is prohibitively time consuming when $S \approx H$ because $n \sim 1/\ln^2(S/H)$.

• This observation is indeed true of standard quadratic-time binomial tree algorithms.

• But it no longer applies to linear-time algorithms (p. 536).
<table>
<thead>
<tr>
<th>Barrier at 95.0</th>
<th>Barrier at 99.5</th>
<th>Barrier at 99.9</th>
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</table>

(All times in milliseconds.)
Trinomial Tree

• Set up a trinomial approximation to the geometric Brownian motion \( dS/S = r \, dt + \sigma \, dW \).\(^{a}\)

• The three stock prices at time \( \Delta t \) are \( S \), \( Su \), and \( Sd \), where \( ud = 1 \).

• Impose the matching of mean and that of variance:

\[
1 = p_u + p_m + p_d, \\
SM = (p_u u + p_m + (p_d/u)) \, S, \\
S^2 V = p_u (Su - SM)^2 + p_m (S - SM)^2 + p_d (Sd - SM)^2.
\]

\(^{a}\)Boyle (1988).
• Above,

\[ M \equiv e^{r\Delta t}, \]
\[ V \equiv M^2(e^{\sigma^2\Delta t} - 1), \]

by Eqs. (17) on p. 147.
Trinomial Tree (continued)

- Use linear algebra to verify that

\[
p_u = \frac{u (V + M^2 - M) - (M - 1)}{(u - 1) (u^2 - 1)},
\]

\[
p_d = \frac{u^2 (V + M^2 - M) - u^3 (M - 1)}{(u - 1) (u^2 - 1)}.
\]

- In practice, must make sure the probabilities lie between 0 and 1.

- Countless variations.
Trinomial Tree (concluded)

- Use $u = e^{\lambda \sigma \sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.

- Then

$$p_u \rightarrow \frac{1}{2\lambda^2} + \frac{(r + \sigma^2) \sqrt{\Delta t}}{2\lambda \sigma},$$

$$p_d \rightarrow \frac{1}{2\lambda^2} - \frac{(r - 2\sigma^2) \sqrt{\Delta t}}{2\lambda \sigma}.$$

- A nice choice for $\lambda$ is $\sqrt{\pi/2}$.\(^a\)

\(^a\)Omberg (1988).
Barrier Options Revisited

• BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.

• The trinomial model solves the problem by adjusting $\lambda$ so that the barrier is hit exactly.$^a$

• It takes

$$h = \frac{\ln(S/H)}{\lambda \sigma \sqrt{\Delta t}}$$

consecutive down moves to go from $S$ to $H$ if $h$ is an integer, which is easy to achieve by adjusting $\lambda$.

- This is because $Se^{-h\lambda \sigma \sqrt{\Delta t}} = H$.

Barrier Options Revisited (continued)

- Typically, we find the smallest $\lambda \geq 1$ such that $h$ is an integer.

- That is, we find the largest integer $j \geq 1$ that satisfies
  \[
  \ln\left(\frac{S}{H}\right) j \sigma \sqrt{\Delta t} \geq 1
  \]
  and then let
  \[
  \lambda = \frac{\ln\left(\frac{S}{H}\right)}{j \sigma \sqrt{\Delta t}}.
  \]

  - Such a $\lambda$ may not exist for very small $n$’s.
  - This is not hard to check.

- This done, one of the layers of the trinomial tree coincides with the barrier.
Barrier Options Revisited (concluded)

- The following probabilities may be used,

\[ p_u = \frac{1}{2\lambda^2} + \frac{\mu' \sqrt{\Delta t}}{2\lambda \sigma}, \]
\[ p_m = 1 - \frac{1}{\lambda^2}, \]
\[ p_d = \frac{1}{2\lambda^2} - \frac{\mu' \sqrt{\Delta t}}{2\lambda \sigma}. \]

\[ - \mu' \equiv r - \sigma^2/2. \]
Algorithms Comparison\textsuperscript{a}

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the $n$ value at which they converge.
  - The one with the smallest $n$ wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times.

\textsuperscript{a}Lyuu (1998).
Algorithms Comparison (concluded)

• Pages 310 and 545 show the trinomial model converges at a smaller \( n \) than BOPM.

• It is in this sense when people say trinomial models converge faster than binomial ones.

• But is the trinomial model better then?

• The linear-time binomial tree algorithm actually performs better than the trinomial one (see next page expanded from p. 534).

<table>
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<th>Combinatorial method</th>
<th>Trinomial tree algorithm</th>
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</tr>
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</table>

(All times in milliseconds.)
Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
- Assume $L < S < H$.
- The binomial model produces oscillating option values (see plot next page).a

---

aChao (1999); Dai and Lyuu (2005);
Double-Barrier Knock-Out Options

• We knew how to pick the $\lambda$ so that one of the layers of the trinomial tree coincides with one barrier, say $H$.

• This choice, however, does not guarantee that the other barrier, $L$, is also hit.

• One way to handle this problem is to lower the layer of the tree just above $L$ to coincide with $L$.$^a$
  
  – More general ways to make the trinomial model hit both barriers are available.$^b$

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Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above $L$ must be adjusted.

- Let $\ell$ be the positive integer such that

  $$Sd^{\ell+1} < L < Sd^\ell.$$ 

- Hence the layer of the tree just above $L$ has price $Sd^\ell$. 
Double-Barrier Knock-Out Options (concluded)

• Define $\gamma > 1$ as the number satisfying

\[ L = Sd^{\ell-1}e^{-\gamma \lambda \sigma \sqrt{\Delta t}}. \]

– The prices between the barriers are

\[ L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H. \]

• The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are

\[ p_u' = \frac{b + a\gamma}{1 + \gamma}, \quad p_d' = \frac{b - a}{\gamma + \gamma^2}, \quad \text{and} \quad p_m' = 1 - p_u' - p_d', \]

where $a \equiv \mu' \sqrt{\Delta t}/(\lambda \sigma)$ and $b \equiv 1/\lambda^2$. 

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Convergence: Binomial vs. Trinomial

\begin{figure}
\centering
\includegraphics[width=\textwidth]{convergence_graph.png}
\caption{Convergence of option values between Binomial and Trinomial models.}
\end{figure}