Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.

Problems; the Smile (concluded)

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.[a]
  - σ measures the volatility of stock price one year from now (regardless of what happens in between).
- How to incorporate these two different ways of day count into the Black-Scholes formula and binomial tree algorithms?

[a]Fama (1965); French (1980); French and Roll (1986).

Trading Days and Calendar Days (concluded)

- Suppose a year has 260 trading days.
- A quick and dirty way is to replace σ with[a]

$$\sigma\sqrt{\frac{365}{260}\,\frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}\,.$$

- How about binomial tree algorithms?

---

[a]French (1984).

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
$$\max(0, X - Su^j d^{n-j})$$
and applies backward induction.
- At each intermediate node, it checks for early exercise by comparing the payoff if exercised with the continuation value.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the prevailing stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\,u$, $(Su - D)\,d$, $(Sd - D)\,u$, $(Sd - D)\,d$.
  - The binomial tree no longer combines.

$$
\begin{array}{ccccc}
 & & & & (Su-D)\,u \\
 & & & \nearrow & \\
 & & Su-D & & \\
 & \nearrow & & \searrow & \\
 & & & & (Su-D)\,d \\
S & & & & \\
 & & & & (Sd-D)\,u \\
 & \searrow & & \nearrow & \\
 & & Sd-D & & \\
 & & & \searrow & \\
 & & & & (Sd-D)\,d
\end{array}
$$

An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends (Roll, 1977).
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - σ equal to the volatility of the process followed by the risky component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all future dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

An Uncompromising Approach[a]

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases.

[a]Dai and Lyuu (2004).

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.

Continuous Dividend Yields (continued)

- The Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$ (Merton, 1973):

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{24}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{24'}$$

where

$$x \equiv \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (24) and (24') remain valid as long as the dividend yield is predictable.

- Replace q with the average annualized dividend yield.

Continuous Dividend Yields (concluded)

- To run binomial tree algorithms, pick the risk-neutral probability as

$$\frac{e^{(r-q)\,\Delta t} - d}{u-d}, \tag{25}$$

where $\Delta t \equiv \tau/n$.

  – Because the stock price grows at an expected rate of $r-q$ in a risk-neutral economy.

- The u and d remain unchanged.
- Other than the change in Eq. (25), binomial tree algorithms stay the same.

Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
— Blaise Pascal (1623–1662)

Sensitivity Measures ("The Greeks")

- Understanding how the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- Let $x \equiv \frac{\ln(S/X)+(r+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$ (recall p. 244).
- Note that

$$N'(y) = (1/\sqrt{2\pi}\,)\,e^{-y^2/2} > 0,$$

the density function of standard normal distribution.

Delta

- Defined as $\Delta \equiv \partial f / \partial S$.
  - f is the price of the derivative.
  - S is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.
- The delta used in the BOPM is the discrete analog.

Delta (concluded)

- The delta of a European call on a non-dividend-paying stock equals
$$\frac{\partial C}{\partial S} = N(x) > 0.$$
- The delta of a European put equals
$$\frac{\partial P}{\partial S} = N(x) - 1 < 0.$$
- The delta of a long stock is 1.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
- A delta-neutral portfolio is immune to small price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  - Short Δ shares of stock to hedge a long call.
- In general, hedge a position in a security with a delta of Δ_1 by shorting Δ_1/Δ_2 units of a security with a delta of Δ_2.

Theta (Time Decay)

- Defined as the rate of change of a security's value with respect to time, or $\Theta \equiv -\partial f / \partial \tau$.
- For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau} N(x - \sigma\sqrt{\tau}) < 0.$$

  – The call loses value with the passage of time.
- For a European put,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau} N(-x + \sigma\sqrt{\tau}).$$

  – Can be negative or positive.

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \equiv \partial^2\Pi/\partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- A portfolio with a high gamma needs in practice be rebalanced more often to maintain delta neutrality.
- Delta $\sim$ duration; gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$

Vega[a] (Lambda, Kappa, Sigma)

- Defined as the rate of change of its value with respect to the volatility of the underlying asset $\Lambda \equiv \partial \Pi / \partial \sigma$.
- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\, N'(x) > 0$.
  - Higher volatility increases option value.

[a]Vega is not Greek.

Rho

- Defined as the rate of change in its value with respect to interest rates $\rho \equiv \partial\Pi/\partial r$.
- The rho of a European call on a non-dividend-paying stock is
$$X\tau e^{-r\tau} N(x - \sigma\sqrt{\tau}) > 0.$$
- The rho of a European put on a non-dividend-paying stock is
$$-X\tau e^{-r\tau} N(-x + \sigma\sqrt{\tau}) < 0.$$

Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,
$$\frac{f(S + \Delta S) - f(S - \Delta S)}{2\Delta S}.$$
- The computation time roughly doubles that for evaluating the derivative security itself.

An Alternative Numerical Delta[a]

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, f_u and f_d are computed.
- These values correspond to derivative values at stock prices Su and Sd, respectively.
- Delta is approximated by

$$\frac{f_u - f_d}{Su - Sd}.$$

- Almost zero extra computational effort.

[a]Pelsser and Vorst (1994).

Numerical Gamma

- At the stock price $(Suu + Sud)/2$, delta is approximately $(f_{uu} - f_{ud})/(Suu - Sud)$.
- At the stock price $(Sud + Sdd)/2$, delta is approximately $(f_{ud} - f_{dd})/(Sud - Sdd)$.
- Gamma is the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,

$$\frac{\frac{f_{uu}-f_{ud}}{Suu-Sud} - \frac{f_{ud}-f_{dd}}{Sud-Sdd}}{(Suu - Sdd)/2}.$$

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

$$\frac{f(S+\Delta S)-2f(S)+f(S-\Delta S)}{(\Delta S)^2}.$$

- It does not work (see text).
- Why did the binomial tree version work?

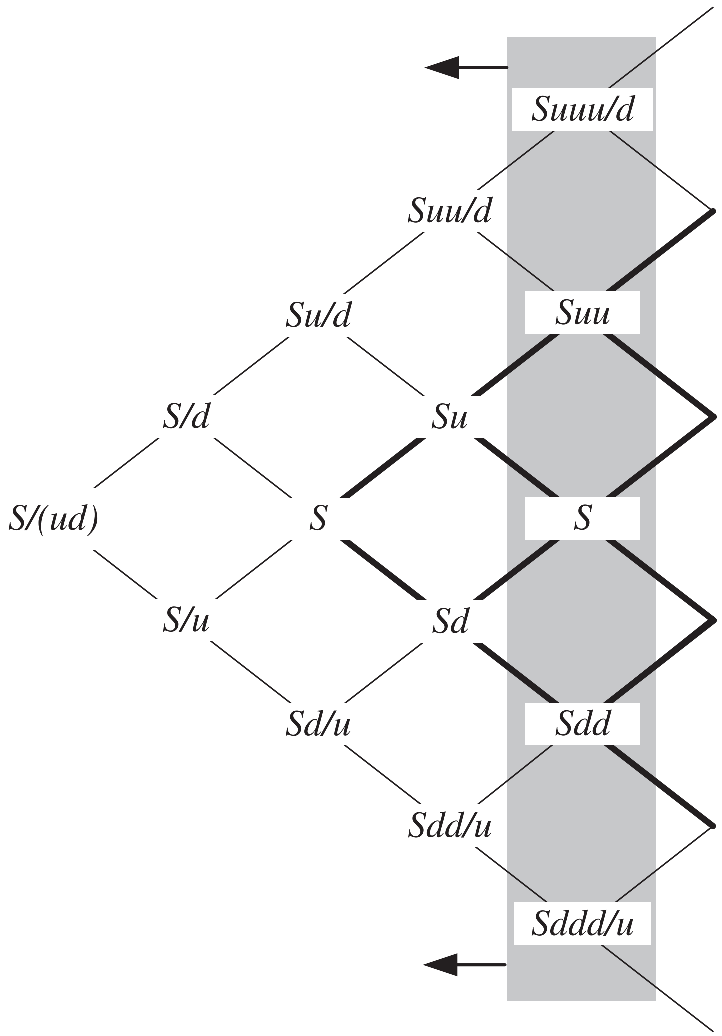
Other Numerical Greeks

- The theta can be computed as

$$\frac{f_{ud} - f}{2(\tau/n)}.$$

  - In fact, the theta of a European option will be shown to be computable from delta and gamma (see p. 488).

- For vega and rho, there is no alternative but to run the binomial tree algorithm twice.



Suuu/d
Suu/d
Su/d
Suu
S/d
Su
S/(ud)
S
S
S/u
Sd
Sd/u
Sdd
Sdd/u
Sddd/u

Extensions of Options Theory

As I never learnt mathematics,
so I have had to think.
— Joan Robinson (1903–1983)

Pricing Corporate Securities[a]

- Interpret the underlying asset as the total value of the firm.
- The option pricing methodology can be applied to pricing corporate securities.
- Assume:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

[a]Black and Scholes (1973).

Risky Zero-Coupon Bonds and Stock

- Consider XYZ.com.
- Capital structure:
  - n shares of its own common stock, S.
  - Zero-coupon bonds with an aggregate par value of X.
- What is the value of the bonds, B?
- What is the value of the XYZ.com stock?

Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds' maturity date, suppose the total value of the firm V^* is less than the bondholders' claim X.
- Then the firm declares bankruptcy and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain X and the stockholders $V^* - X$.

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

Risky Zero-Coupon Bonds and Stock (continued)

- The stock is a call on the total value of the firm with a strike price of X and an expiration date equal to the bonds'.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call on the total value of the firm.
- Let V stand for the total value of the firm.
- Let C stand for the call.

Risky Zero-Coupon Bonds and Stock (continued)

- Thus $nS = C$ and $B = V - C$.
- Knowing C amounts to knowing how the value of the firm is divided between stockholders and bondholders.
- Whatever the value of C, the total value of the stock and bonds at maturity remains V^*.
- The relative size of debt and equity is irrelevant to the firm's current value V.

Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 8 (p. 244) and the put-call parity,

$$
\begin{aligned}
nS &= VN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\
B &= VN(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}).
\end{aligned}
$$

  - Above,

$$x \equiv \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

- The continuously compounded yield to maturity of the firm's bond is

$$\frac{\ln(X/B)}{\tau}.$$

Risky Zero-Coupon Bonds and Stock (concluded)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

$$
\begin{aligned}
&\frac{\ln(X/B)}{\tau} - r \\
= \quad & -\frac{1}{\tau} \ln\left(N(-z) + \frac{1}{\omega} N(z - \sigma\sqrt{\tau}) \right).
\end{aligned}
$$

  - $\omega \equiv X e^{-r\tau}/V$.
  - $z \equiv (\ln \omega)/(\sigma\sqrt{\tau}) + (1/2)\, \sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}$.
  - Note that ω is the debt-to-total-value ratio.

A Numerical Example

- XYZ.com's assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck's market value per share is $44.5.
- XYZ.com's securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay $1,000 at maturity.
- $n = 1000$, $V = 44.5 \times n = 44500$, and $X = 30 \times n = 30000$.

			—Call—		—Put—	
Option	Strike	Exp.	Vol.	Last	Vol.	Last
Merck	30	Jul	328	15 1/4	...	...
44 1/2	35	Jul	150	9 1/2	10	1/16
44 1/2	40	Apr	887	4 3/4	136	1/16
44 1/2	40	Jul	220	5 1/2	297	1/4
44 1/2	40	Oct	58	6	10	1/2
44 1/2	45	Apr	3050	7/8	100	1 1/8
44 1/2	45	May	462	1 3/8	50	1 3/8
44 1/2	45	Jul	883	1 15/16	147	1 3/4
44 1/2	45	Oct	367	2 3/4	188	2 1/16

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for \$15.25.
- So XYZ.com's stock is worth $15.25 \times n = 15250$ dollars.
- The entire bond issue is worth $B = 44500 - 15250 = 29250$ dollars.
  - Or \$975 per bond.

A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $\$X$ par value plus n written European puts on Merck at a strike price of \$30.
  - By the put-call parity.
- The difference between B and the price of the default-free bond is the value of these puts.
- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts X.

Promised payment to bondholders X	Current market value of bonds B	Current market value of stock nS	Current total value of firm V
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

A Numerical Example (continued)

- Suppose the promised payment to bondholders is \$45,000.
- Then the relevant option is the July call with a strike price of $45000/n = 45$ dollars.
- Since that option is selling for \$1 15/16, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option's terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
- So parameters such volatility, dividend, and strike price are under partial control of the stockholders.

A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now $X = 45{,}000$ dollars.
- The table on p. 291 says the total market value of the bonds should be \$42,562.5.
- The *new* bondholders pay $42562.5 \times (15/45) = 14187.5$ dollars.
- The remaining stock is worth \$1,937.5.

A Numerical Example (continued)

- The stockholders therefore gain

$$14187.5 + 1937.5 - 15250 = 875$$

dollars.

- The *original* bondholders lose an equal amount,

$$29250 - \frac{30}{45} \times 42562.5 = 875. \tag{26}$$

A Numerical Example (continued)

- Suppose the stockholders distribute \$14,833.3 cash dividends by selling $(1/3) \times n$ Merck shares.
- They now have \$14,833.3 in cash plus a call on $(2/3) \times n$ Merck shares.
- The strike price remains $X = 30000$.
- This is equivalent to owning two-thirds of a call on n Merck shares with a total strike price of \$45,000.
- n such calls are worth \$1,937.5 (p. 291).
- So the total market value of the XYZ.com stock is $(2/3) \times 1937.5 = 1291.67$ dollars.

A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence $(2/3) \times n \times 44.5 - 1291.67 = 28375$ dollars.
- Hence the stockholders gain

$$14833.3 + 1291.67 - 15250 \approx 875$$

  dollars.
- The bondholders watch their value drop from \$29,250 to \$28,375, a loss of \$875.

Other Examples

- Subordinated debts as bull call spreads.
- Warrants as calls.
- Callable bonds as American calls with 2 strike prices.
- Convertible bonds.

Barrier Options[a]

- Their payoff depends on whether the underlying asset's price reaches a certain price level H.
- A knock-out option is an ordinary European option which ceases to exist if the barrier H is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if $H < S$.
- A put knock-out option is sometimes called an up-and-out option when $H > S$.

---

[a] A former student told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank. She is working for Lehman Brothers in HK as of April, 2006.

Barrier Options (concluded)

- A knock-in option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.
- Formulas exist for all kinds of barrier options.

A Formula for Down-and-In Calls[a]

- Assume $X \geq H$.
- The value of a European down-and-in call on a stock paying a dividend yield of q is

$$Se^{-q\tau} \left(\frac{H}{S}\right)^{2\lambda} N(x) - Xe^{-r\tau} \left(\frac{H}{S}\right)^{2\lambda-2} N(x - \sigma\sqrt{\tau}), \tag{27}$$

  - $x \equiv \frac{\ln(H^2/(SX))+(r-q+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$.
  - $\lambda \equiv (r - q + \sigma^2/2)/\sigma^2$.
- A European down-and-out call can be priced via the in-out parity.

---

[a]Merton (1973).

A Formula for Down-and-Out Calls[a]

- Assume $X \leq H$.
- The value of a European up-and-in put is

$$Xe^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N(-x+\sigma\sqrt{\tau}) - Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda} N(-x).$$

- A European up-and-out call can be priced via the in-out parity.

---

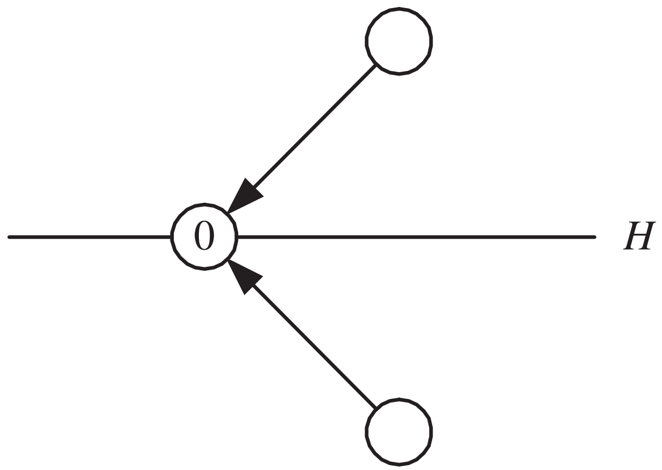
[a] Merton (1973).

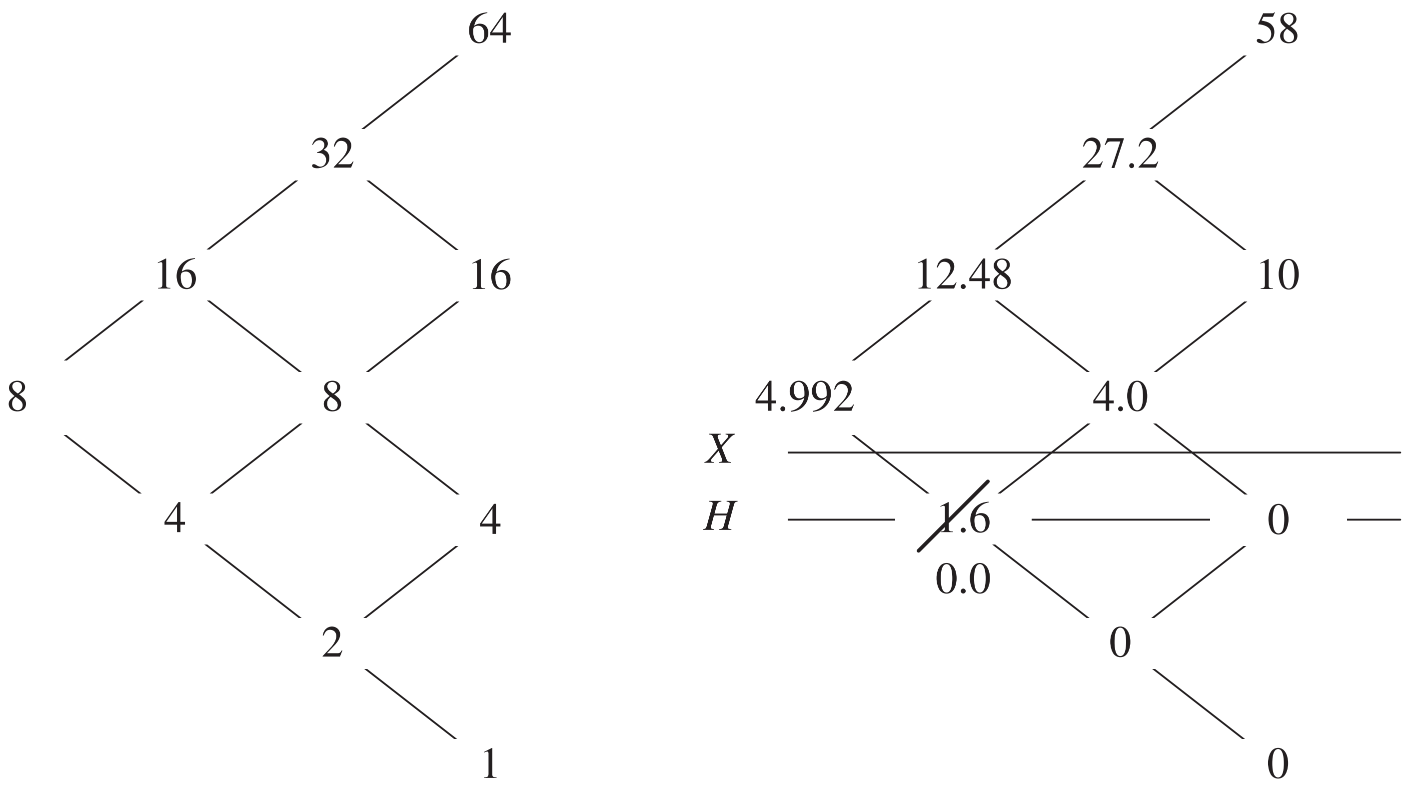
Interesting Observations

- Assume $H < X$.
- Replace S in the pricing formula for the down-and-in call, Eq. (27) on p. 301, with H^2/S.
- Equation (27) becomes Eq. (24) on p. 260 when $r - q = \sigma^2/2$.
- Equation (27) becomes S/H times Eq. (24) on p. 260 when $r - q = 0$.

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.



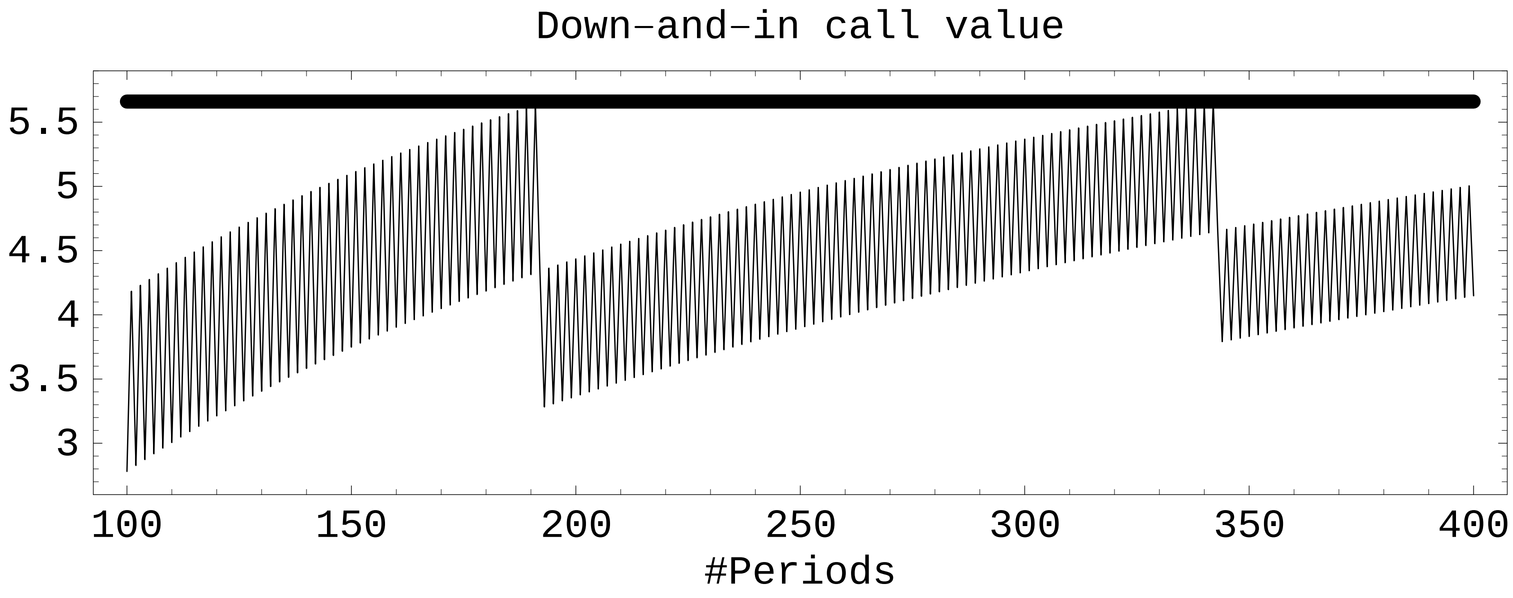


$S = 8$, $X = 6$, $H = 4$, $R = 1.25$, $u = 2$, and $d = 0.5$.

Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25$.

Binomial Tree Algorithms (concluded)

- But convergence is erratic because H is not at a price level on the tree (see plot on next page).
  - Typically, the barrier has to be adjusted to be at a price level.
- Solutions will be presented later.


Down-and-in call value
5.5
5
4.5
4
3.5
3
100
150
200
250
300
350
400
#Periods