American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$.
  - When $hS + B \geq S - X$, the call should not be exercised immediately.
  - When $hS + B < S - X$, the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 3 (p. 186).
- So $C = hS + B$.

Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where

$$\begin{aligned} P_u &= \max(0, X - Su), \\ P_d &= \max(0, X - Sd). \end{aligned}$$

- Let $B = \frac{uP_d - dP_u}{(u-d)\,R}$.
- The European put is worth $hS + B$.
- The American put is worth $\max(hS + B, X - S)$.
  - Early exercise is always possible with American puts.

Risk

- Surprisingly, the option value is independent of q.
- Hence it is independent of the expected gross return of the stock, $qSu + (1 - q)\, Sd$.
- It therefore does not directly depend on investors' risk preferences.
- The option value depends on the sizes of price changes, u and d, which the investors must agree upon.
- Note that the possible stock prices are the same whether under q or p.

Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$

- Rewrite it as

$$hS + B = \frac{pC_u + (1-p)\,C_d}{R},$$

where

$$p \equiv \frac{R-d}{u-d}.$$

- As $0 < p < 1$, it may be interpreted as a probability.

Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under p as $pSu + (1-p)\,Sd = RS$.
- Risk-neutral investors care only about expected returns.
- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, p is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the FV is the riskless rate in a risk-neutral economy.

Binomial Distribution

- Denote the binomial distribution with parameters n and p by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

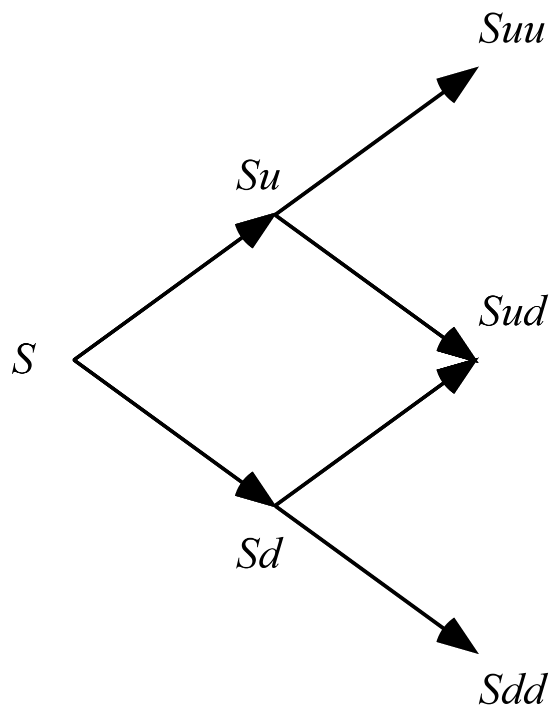
  - $n! = n \times (n-1) \cdots 2 \times 1$ with the convention $0! = 1$.

- Suppose you toss a coin n times with p being the probability of getting heads.

- Then $b(j; n, p)$ is the probability of getting j heads.

Option on a Non-Dividend-Paying Stock: Multi-Period

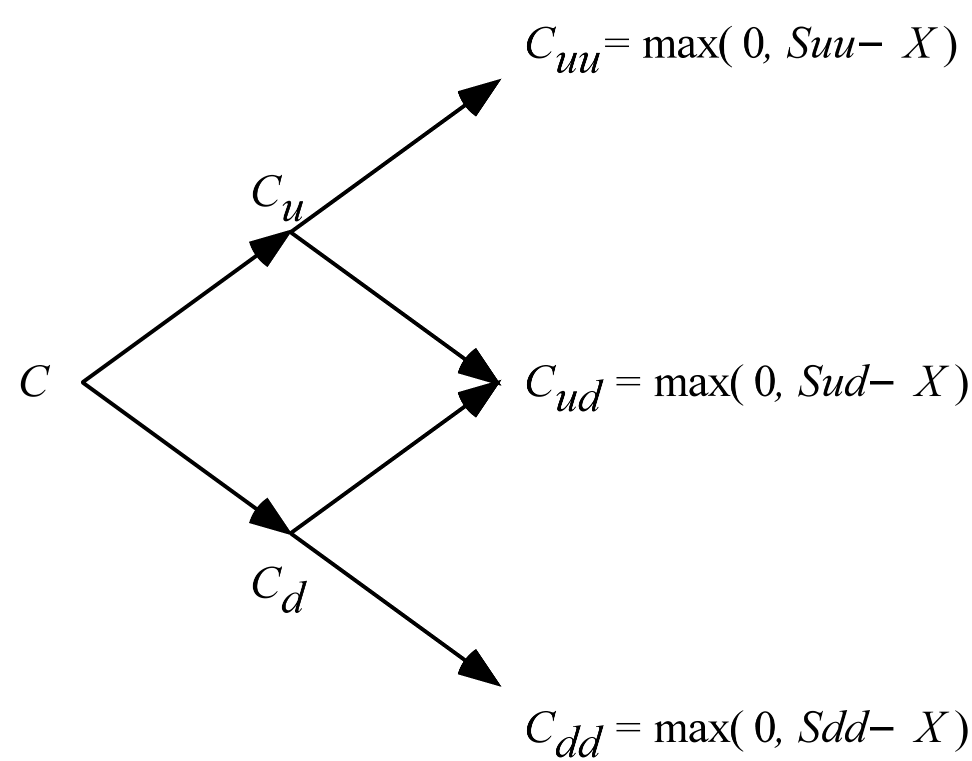
- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on three possible prices at time two: Suu, Sud, and Sdd.
  - There are 4 paths.
  - But the tree combines.
- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.



Suu
Su
Sud
S
Sd
Sdd

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let C_{uu} be the call's value at time two if the stock price is Suu.
- Thus,
$$C_{uu} = \max(0, Suu - X).$$
- C_{ud} and C_{dd} can be calculated analogously,
$$\begin{aligned} C_{ud} &= \max(0, Sud - X), \\ C_{dd} &= \max(0, Sdd - X). \end{aligned}$$



C
Cu
Cd
Cuu = max(0, Suu − X)
Cud = max(0, Sud − X)
Cdd = max(0, Sdd − X)

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time one can be obtained by applying the same logic:

$$
\begin{aligned}
C_u &= \frac{pC_{uu} + (1-p)\, C_{ud}}{R}, \qquad (21)\\
C_d &= \frac{pC_{ud} + (1-p)\, C_{dd}}{R}.
\end{aligned}
$$

- Deltas can be derived from Eq. (19) on p. 200.
- For example, the delta at C_u is

$$
\frac{C_{uu} - C_{ud}}{Suu - Sud}.
$$

Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- An equivalent portfolio of h shares of stock and $\$B$ riskless bonds can be set up for the call that costs C_u (C_d, resp.) if the stock price goes to Su (Sd, resp.).
- The values of h and B can be derived from Eqs. (19)–(20) on p. 200.
- Or, we can just compute

$$\frac{pC_u + (1-p)\,C_d}{R}$$

as the price.

Early Exercise

- Since the call will not be exercised at time one even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.
- Therefore,

$$\begin{aligned} hS + B &= \frac{pC_u + (1-p)\, C_d}{R} \geq \frac{[\, pu + (1-p)\, d\,]\, S - X}{R} \\ &= S - \frac{X}{R} > S - X. \end{aligned}$$

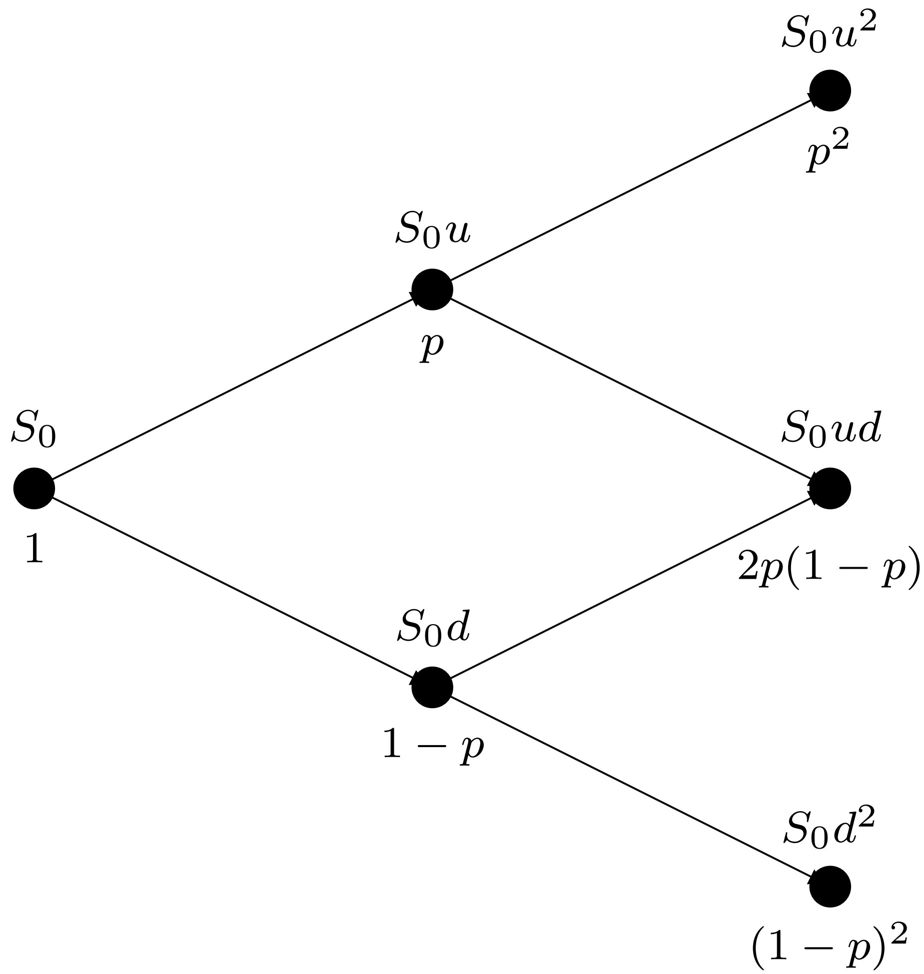
- The call again will not be exercised at present.
- So

$$C = hS + B = \frac{pC_u + (1-p)\, C_d}{R}.$$

Backward Induction of Zermelo (1871–1953)

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happens at C_u and C_d, too, as demonstrated in Eq. (21) on p. 211.
- This recursive procedure is called backward induction.
- Now, C equals

$$
\begin{aligned}
& [\, p^2 C_{uu} + 2p(1-p)\, C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\
= \;& [\, p^2 \max\big(0, Su^2 - X\big) + 2p(1-p) \max\,(0, Sud - X) \\
& +(1-p)^2 \max\big(0, Sd^2 - X\big)\,]/R^2.
\end{aligned}
$$


$S_0 u^2$
p^2
$S_0 u$
p
S_0
1
$S_0 ud$
$2p(1-p)$
$S_0 d$
$1-p$
$S_0 d^2$
$(1-p)^2$

Backward Induction (concluded)

- In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, Su^j d^{n-j} - X\right)}{R^n}.$$

  - The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- The value of a European put is

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, X - Su^j d^{n-j}\right)}{R^n}.$$

Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-\hat{r}n} E^{\pi}[\,\mathcal{D}\,].$$

  - E^{π} means the expectation is taken under the risk-neutral probability.
- The "equivalence" between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.

Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- The maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.
  - Changes in value are due entirely to capital gains.

The Binomial Option Pricing Formula

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer such that

$$Su^a d^{n-a} \geq X,$$

or

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

The Binomial Option Pricing Formula (concluded)

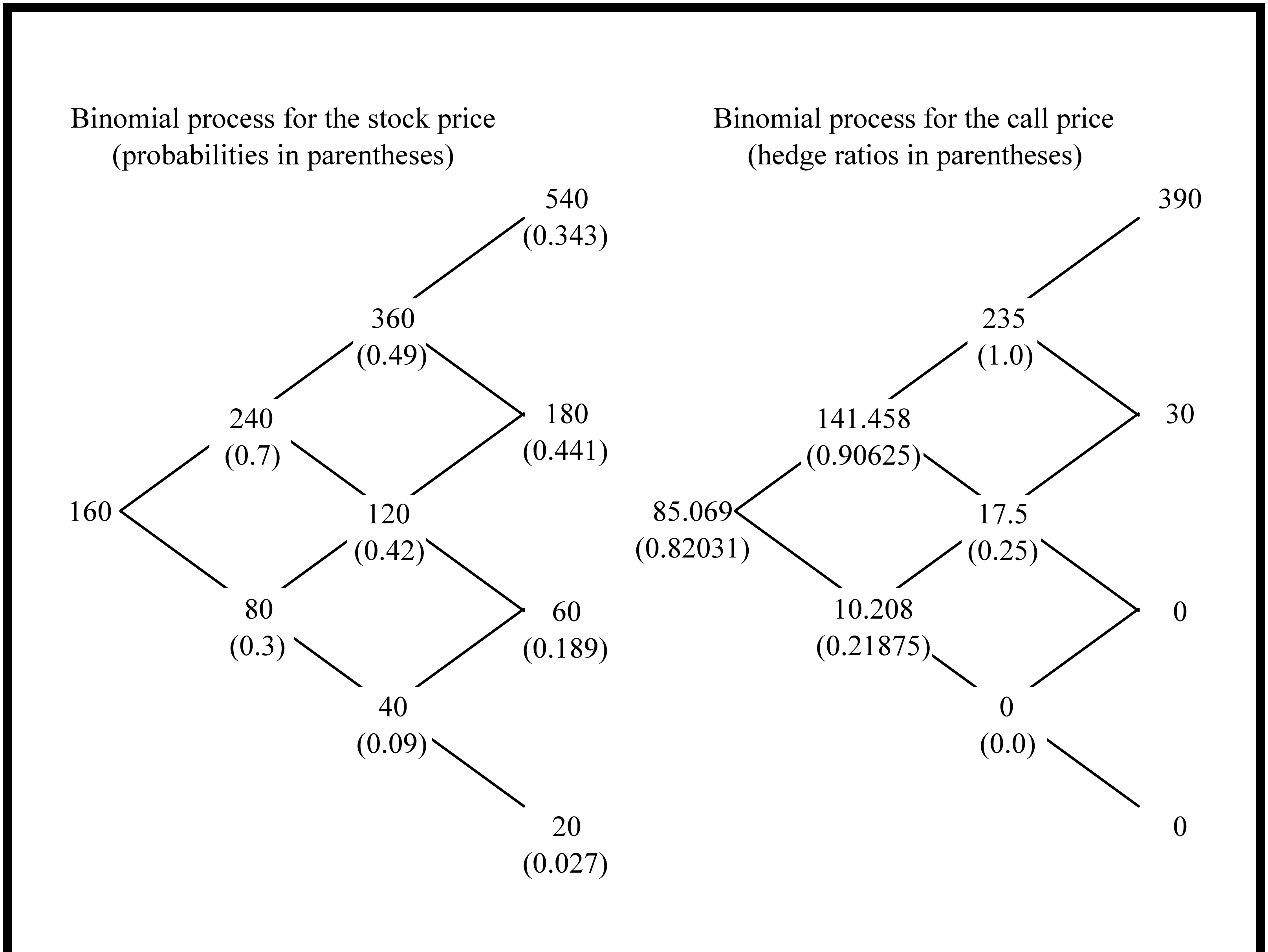
Hence,

$$
\begin{aligned}
& C \\
= \quad & \frac{\sum_{j=a}^{n} \binom{n}{j} p^{j} (1-p)^{n-j} \left(S u^{j} d^{n-j} - X \right)}{R^{n}} \qquad (22) \\
= \quad & S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^{j} [\,(1-p)\,d\,]^{n-j}}{R^{n}} - \frac{X}{R^{n}} \sum_{j=a}^{n} \binom{n}{j} p^{j} (1-p)^{n-j} \\
= \quad & S \sum_{j=a}^{n} b\left(j; n, pue^{-\hat{r}} \right) - X e^{-\hat{r} n} \sum_{j=a}^{n} b(j; n, p).
\end{aligned}
$$

Numerical Examples

- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($e^{0.18232} = 1.2$).
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Also the PV of the expected payoff at expiration,

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$


Binomial process for the stock price
(probabilities in parentheses)
540
(0.343)
360
(0.49)
240
(0.7)
180
(0.441)
160
120
(0.42)
80
(0.3)
60
(0.189)
40
(0.09)
20
(0.027)
Binomial process for the call price
(hedge ratios in parentheses)
390
235
(1.0)
141.458
(0.90625)
30
85.069
(0.82031)
17.5
(0.25)
10.208
(0.21875)
0
0
(0.0)
0

Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90 and invest \$85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars,}$$

is the arbitrage profit as we will see.

Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy $0.90625 - 0.82031 = 0.08594$ more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to $76.04232 \times 1.2 - 78.75 = 12.5$ dollars.

Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to \$180.
- The call we wrote finishes in the money.
- For a loss of $180 - 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

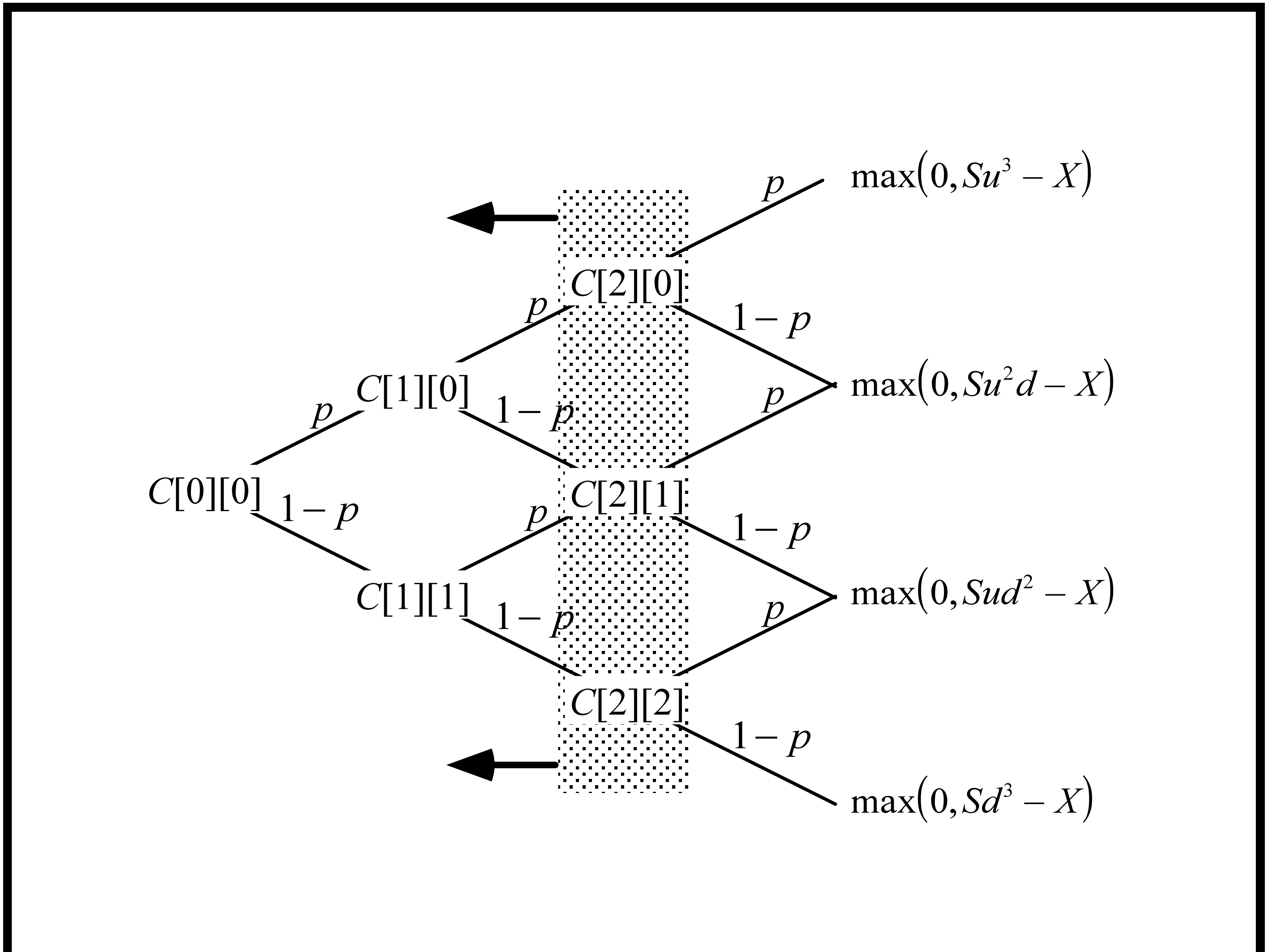
Numerical Examples (concluded)

Time 3 (the case of declining price):

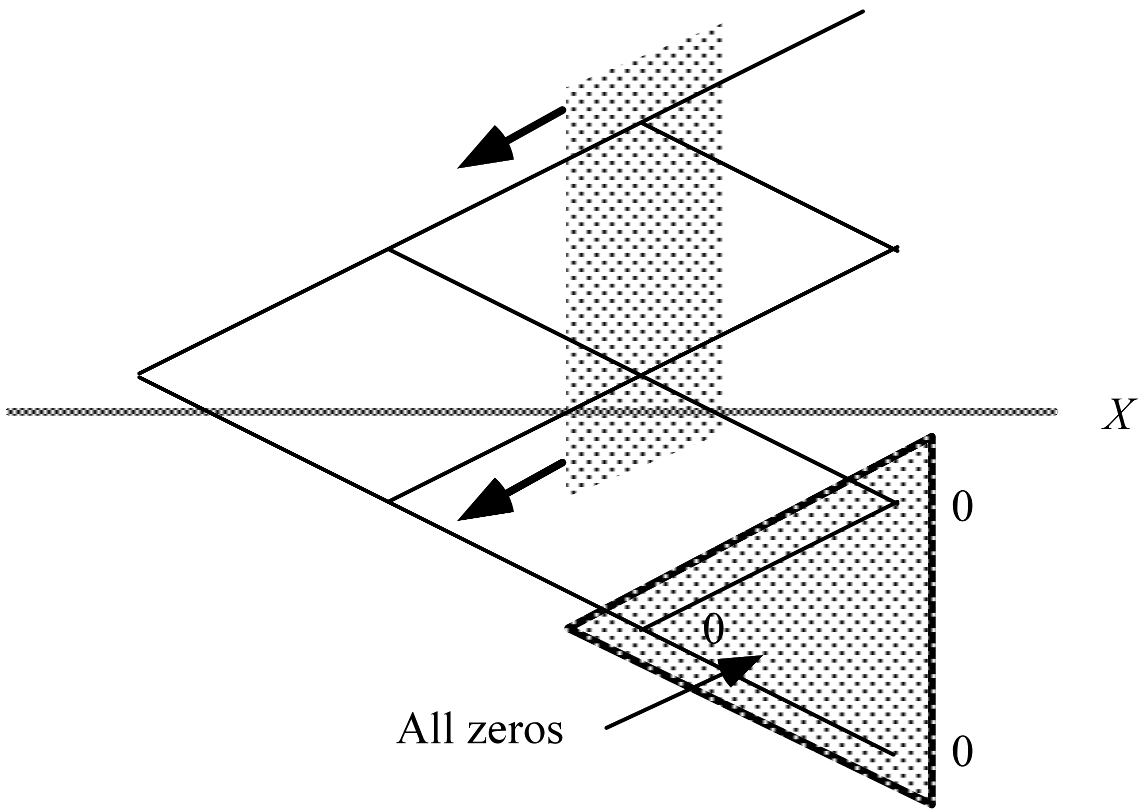
- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of $0.25 \times 60 = 15$ dollars.
- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.

Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$.
- The memory requirement is $O(n^2)$.
  - Can be further reduced to $O(n)$ by reusing space
- To price European puts, simply replace the payoff.


max(0, Su^3 − X)
C[2][0]
max(0, Su^2 d − X)
C[1][0]
C[0][0]
C[2][1]
max(0, Sud^2 − X)
C[1][1]
C[2][2]
max(0, Sd^3 − X)
p
1 − p

Further Improvement for Calls



Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j;n,p) = \frac{p(n-j+1)}{(1-p)\,j}\, b(j-1;n,p).$$

Optimal Algorithm (continued)

- The following program computes $b(j; n, p)$ in $b[\,j\,]$:

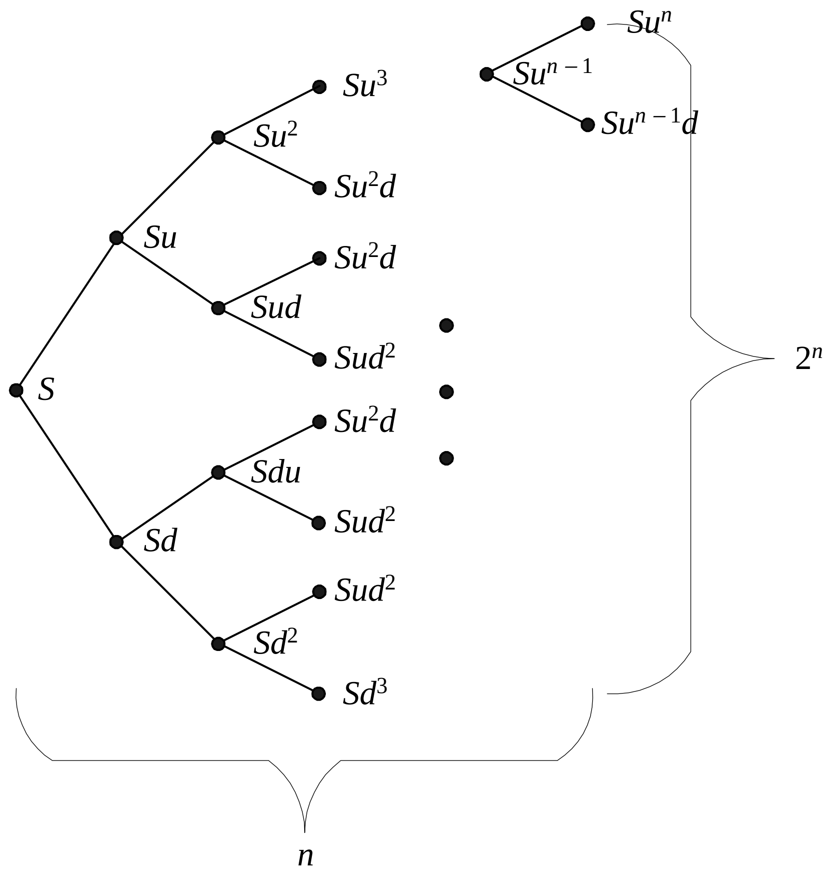
$b[\,a\,] := \binom{n}{a} p^a (1-p)^{n-a}$;
for $j = a+1, a+2, \ldots, n$ **do**
  $b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j)$;
end for

- It runs in $O(n)$ steps.

Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (22) on p. 220 is trivial to compute.
- We only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique cannot be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

On the Bushy Tree



Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As the number of periods increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.
- Any proper calibration of the model parameters makes the BOPM converge to the continuous-time model.
- We now skim through the proof.

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as n goes to infinity.
- First, $\hat{r} = r\tau/n$.
  - The period gross return $R = e^{\hat{r}}$.

Toward the Black-Scholes Formula (continued)

- Use

$$\widehat{\mu} \equiv \frac{1}{n}\, E\left[\ln \frac{S_\tau}{S}\right] \quad \text{and} \quad \widehat{\sigma}^2 \equiv \frac{1}{n}\, \text{Var}\left[\ln \frac{S_\tau}{S}\right]$$

to denote, resp., the expected value and variance of the period continuously compounded rate of return.

- Under the BOPM, it is not hard to show that

$$\begin{aligned} \widehat{\mu} &= q \ln(u/d) + \ln d, \\ \widehat{\sigma}^2 &= q(1-q) \ln^2(u/d). \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- Assume the stock's true continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
  - Call σ the stock's (annualized) volatility.
- The BOPM converges to the distribution only if

$$
\begin{aligned}
n\widehat{\mu} &= n(q\ln(u/d) + \ln d) \to \mu\tau, \\
n\widehat{\sigma}^2 &= nq(1-q)\ln^2(u/d) \to \sigma^2\tau.
\end{aligned}
$$

- Impose $ud = 1$ to make nodes at the same horizontal level of the tree have identical price (review p. 230).
  - Other choices are possible (see text).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\,\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}\,. \tag{23}$$

- With Eqs. (23),

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right]\sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $u > R > d$ may not hold under Eqs. (23) on p. 239.
  - If this happens, the risk-neutral probability may lie outside $[\,0, 1\,]$.
- The problem disappears when n satisfies

$$e^{\sigma \sqrt{\tau/n}} > e^{r\tau/n},$$

in other words, when $n > r^2 \tau / \sigma^2$ (check it).

  - So it goes away if n is large enough.
  - Other solutions will be presented later.

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$?
- The central limit theorem says $\ln(S_\tau/S)$ converges to the normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$.
- So $\ln S_\tau$ approaches the normal distribution with mean $\mu\tau + \ln S$ and variance $\sigma^2\tau$.
- S_τ has a lognormal distribution in the limit.

Toward the Black-Scholes Formula (continued)

Lemma 7 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r-\sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability $p \equiv (e^{r\tau/n}-d)/(u-d)$.
- Let $n \to \infty$.

Toward the Black-Scholes Formula (continued)

- By Lemma 7 (p. 242) and Eq. (17) on p. 147, the expected stock price at expiration in a risk-neutral economy is $Se^{r\tau}$.
- The stock's expected annual rate of return[a] is thus the riskless rate r.

---

[a] In the sense of $(1/\tau) \ln E[\, S_\tau / S \,]$ not $(1/\tau) E[\, \ln(S_{\tau/S}) \,]$.

Toward the Black-Scholes Formula (concluded)

Theorem 8 (The Black-Scholes Formula)

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x), \end{aligned}$$

where

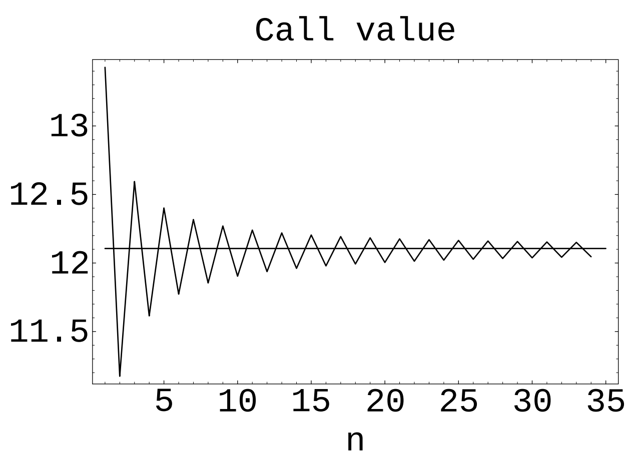
$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

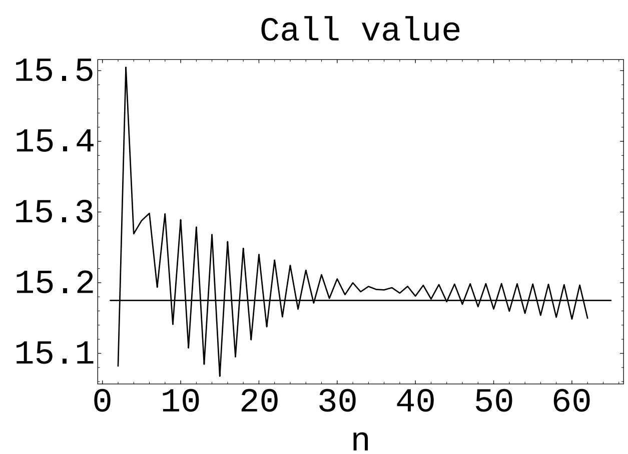
BOPM and Black-Scholes Model

- The Black-Scholes formula needs five parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take six inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$u = e^{\sigma\sqrt{\tau/n}},\ d = e^{-\sigma\sqrt{\tau/n}},\ \hat{r} = r\tau/n.$$

- The binomial tree algorithms converge reasonably fast.
- Oscillations can be dealt with by the judicious choices of u and d (see text).





$S = 100$, $X = 100$ (left), and $X = 95$ (right).

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?
- This volatility is called the implied volatility.
- Implied volatility is often preferred to historical volatility in practice.[a]

[a] It is like driving a car with your eyes on the rearview mirror?