Principles of Financial Computing

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References

- Yuh-Dauh Lyuu. *Financial Engineering & Computation: Principles, Mathematics, Algorithms*. Cambridge University Press. 2002.
- Official Web page is

  `www.csie.ntu.edu.tw/~lyuu/finance1.html`

- Check

  `www.csie.ntu.edu.tw/~lyuu/capitals.html`

  for some of the software.

Useful Journals

- *Applied Mathematical Finance.*
- *Finance and Stochastics.*
- *Financial Analysts Journal.*
- *Journal of Computational Finance.*
- *Journal of Derivatives.*
- *Journal of Economic Dynamics & Control.*
- *Journal of Finance.*
- *Journal of Financial Economics.*
- *Journal of Fixed Income.*

Useful Journals (continued)

- *Journal of Futures Markets.*
- *Journal of Financial and Quantitative Analysis.*
- *Journal of Portfolio Management.*
- *Journal of Real Estate Finance and Economics.*
- *Management Science.*
- *Mathematical Finance.*

Useful Journals (concluded)

- *Quantitative Finance.*
- *Review of Financial Studies.*
- *Review of Derivatives Research.*
- *Risk Magazine.*
- *Stochastics and Stochastics Reports.*

Introduction

[An] investment bank could be
more collegial than a college.
— Emanuel Derman,
My Life as a Quant (2004)

A Very Brief History of Modern Finance

- 1900: Ph.D. thesis *Mathematical Theory of Speculation* of Bachelier (1870–1946).
- 1950s: modern portfolio theory (MPT) of Markowitz.
- 1960s: the Capital Asset Pricing Model (CAPM) of Treynor, Sharpe, Lintner (1916–1984), and Mossin.
- 1960s: the efficient markets hypothesis of Samuelson and Fama.
- 1970s: theory of option pricing of Black (1938–1995) and Scholes.
- 1970s–present: new instruments and pricing methods.

A Very Brief and Biased History of Modern Computers

- 1930s: theory of Gödel (1906–1978), Turing (1912–1954), and Church (1903–1995).
- 1940s: first computers (Z3, ENIAC, etc.) and birth of solid-state transistor (Bell Labs).
- 1950s: Texas Instruments patented integrated circuits; Backus (IBM) invented FORTRAN.
- 1960s: Internet (ARPA) and mainframes (IBM).
- 1970s: relational database (Codd) and PCs (Apple).
- 1980s: IBM PC and Lotus 1-2-3.
- 1990s: Windows 3.1 (Microsoft) and World Wide Web (Berners-Lee).

What This Course Is About

- Financial theories in pricing.
- Mathematical backgrounds.
- Derivative securities.
- Pricing models.
- Efficient algorithms in pricing financial instruments.
- Research problems.
- Finding your thesis directions.

What This Course Is *Not* About

- How to program.
- Basic mathematics in calculus, probability, and algebra.
- Details of the financial markets.
- How to be rich.
- How the markets will perform tomorrow.

Outstanding U.S. Debts (bln)

Year	Municipal	Treasury	Mortgage—related	U.S. corporate	Fed agencies	Money market	Asset—backed	Total
85	859.5	1,437.7	372.1	776.5	293.9	847.0	0.9	4,587.6
86	920.4	1,619.0	534.4	959.6	307.4	877.0	7.2	5,225.0
87	1,010.4	1,724.7	672.1	1,074.9	341.4	979.8	12.9	5,816.2
88	1,082.3	1,821.3	772.4	1,195.7	381.5	1,108.5	29.3	6,391.0
89	1,135.2	1,945.4	971.5	1,292.5	411.8	1,192.3	51.3	7,000.0
90	1,184.4	2,195.8	1,333.4	1,350.4	434.7	1,156.8	89.9	7,745.4
91	1,272.2	2,471.6	1,636.9	1,454.7	442.8	1,054.3	129.9	8,462.4
92	1,302.8	2,754.1	1,937.0	1,557.0	484.0	994.2	163.7	9,192.8
93	1,377.5	2,989.5	2,144.7	1,674.7	570.7	971.8	199.9	9,928.8
94	1,341.7	3,126.0	2,251.6	1,755.6	738.9	1,034.7	257.3	10,505.8
95	1,293.5	3,307.2	2,352.1	1,937.5	844.6	1,177.3	316.3	11,228.5
96	1,296.0	3,459.7	2,486.1	2,122.2	925.8	1,393.9	404.4	12,088.1
97	1,367.5	3,456.8	2,680.2	2,346.3	1,022.6	1,692.8	535.8	13,102.0
98	1,464.3	3,355.5	2,955.2	2,666.2	1,296.5	1,978.0	731.5	14,447.2
99	1,532.5	3,281.0	3,334.2	3,022.9	1,616.5	2,338.2	900.8	16,026.4
00	1,567.8	2,966.9	3,564.7	3,372.0	1,851.9	2,661.0	1,071.8	17,056.1
01	1,688.4	2,967.5	4,125.5	3,817.4	2,143.0	2,542.4	1,281.1	18,565.3
02	1,783.8	3,204.9	4,704.9	3,997.2	2,358.5	2,577.5	1,543.3	20,170.1

Analysis of Algorithms

It is unworthy of excellent men
to lose hours like slaves
in the labor of computation.
— Gottfried Wilhelm Leibniz (1646–1716)

Computability and Algorithms

- Algorithms are precise procedures that can be turned into computer programs.
- Uncomputable problems.
  - Does this program have infinite loops?
  - Is this program bug free?
- Computable problems.
  - Intractable problems.
  - Tractable problems.

Complexity

- Start with a set of basic operations which will be assumed to take one unit of time.
- The total number of these operations is the total work done by an algorithm (its computational complexity).
- The space complexity is the amount of memory space used by an algorithm.
- Concentrate on the abstract complexity of an algorithm instead of its detailed implementation.
- Complexity is a good guide to an algorithm's *actual* running time.

Asymptotics

- Consider the search algorithm on p. 18.
- The worst-case complexity is n comparisons (why?).
- There are operations besides comparison.
- We care only about the asymptotic growth rate not the exact number of operations.
  - So the complexity of maintaining the loop is subsumed by the complexity of the body of the loop.
- The complexity is hence $O(n)$.

Algorithm for Searching an Element

```
for k = 1, 2, 3, . . . , n do
    if x = A_k then
        return k;
    end if
end for
return not-found;
```

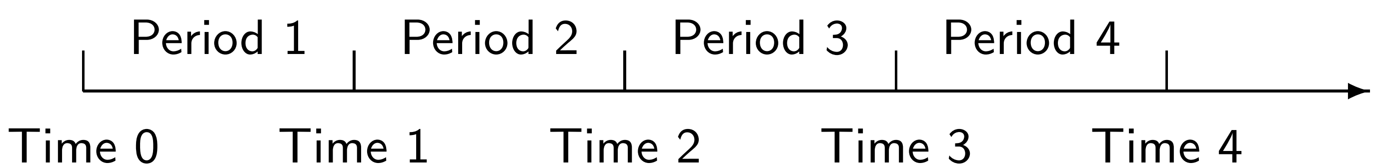
Common Complexities

- Let n stand for the "size" of the problem.
  - Number of elements, number of cash flows, etc.
- Linear time if the complexity is $O(n)$.
- Quadratic time if the complexity is $O(n^2)$.
- Cubic time if the complexity is $O(n^3)$.
- Exponential time if the complexity is $2^{O(n)}$.
- Superpolynomial if the complexity is less than exponential but higher than polynomials, say $2^{O(\sqrt{n})}$.
- It is possible for an exponential-time algorithm to perform well on "typical" inputs.

Basic Financial Mathematics

In the fifteenth century
mathematics was mainly concerned with
questions of commercial arithmetic and
the problems of the architect.
— Joseph Alois Schumpeter (1883–1950)

The Time Line



Time Value of Money

$$\begin{aligned} \text{FV} &= \text{PV}(1+r)^n, \\ \text{PV} &= \text{FV} \times (1+r)^{-n}. \end{aligned}$$

- FV (future value).
- PV (present value).
- r: interest rate.

Periodic Compounding

If interest is compounded m times per annum,

$$\text{FV} = \text{PV}\left(1 + \frac{r}{m}\right)^{nm}. \tag{1}$$

Common Compounding Methods

- Annual compounding: $m = 1$.
- Semiannual compounding: $m = 2$.
- Quarterly compounding: $m = 4$.
- Monthly compounding: $m = 12$.
- Weekly compounding: $m = 52$.
- Daily compounding: $m = 365$.

Easy Translations

- An interest rate of r compounded m times a year is "equivalent to" an interest rate of r/m per $1/m$ year.
- If a loan asks for a return of 1% per month, the annual interest rate will be 12% *with monthly compounding*.

Example

- Annual interest rate is 10% compounded twice per annum.
- Each dollar will grow to be

$$[\,1 + (0.1/2)\,]^2 = 1.1025$$

  one year from now.
- The rate is equivalent to an interest rate of 10.25% compounded once *per annum*.

Continuous Compounding

- Let $m \to \infty$ so that

$$\left(1 + \frac{r}{m}\right)^m \to e^r$$

in Eq. (1) on p. 24.

- Then

$$\text{FV} = \text{PV} \times e^{rn},$$

where $e = 2.71828\ldots$.

Continuous Compounding (concluded)

- Continuous compounding is easier to work with.
- Suppose the annual interest rate is r_1 for n_1 years and r_2 for the following n_2 years.
- Then the FV of one dollar will be

$$e^{r_1 n_1 + r_2 n_2}.$$

Efficient Algorithms for PV and FV

- The PV of the cash flow $C_1, C_2, \ldots, C_n$ at times $1, 2, \ldots, n$ is

$$\frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_n}{(1+y)^n}.$$

- This formula and its variations are the engine behind most of financial calculations.[a]
  - What is y?
  - What are C_i?
  - What is n?

[a] "Asset pricing theory all stems from one simple concept [...]: price equals expected discounted payoff" (see Cochrane (2005)).

Algorithm for Evaluating PV

$x := 0;$

$d := 1 + y;$

for $i = n, n-1, \ldots, 1$ **do**

  $x := (x + C_i)/d;$

end for

`return` $x;$

Horner's Rule: The Idea Behind p. 31

p. 31

- This idea is

$$\left(\cdots \left(\left(\frac{C_n}{1+y} + C_{n-1} \right) \frac{1}{1+y} + C_{n-2} \right) \frac{1}{1+y} + \cdots \right) \frac{1}{1+y}.$$

  - Due to Horner (1786–1837) in 1819.

- The algorithm takes $O(n)$ time.

- It is the most efficient possible in terms of the absolute number of arithmetic operations.[a]

[a] Borodin and Munro (1975).

Conversion between Compounding Methods

- Suppose r_1 is the annual rate with continuous compounding.
- Suppose r_2 is the equivalent rate compounded m times per annum.
- How are they related?

Conversion between Compounding Methods (concluded)

- Both interest rates must produce the same amount of money after one year.
- That is,

$$\left(1+\frac{r_2}{m}\right)^m = e^{r_1}.$$

- Therefore,

$$\begin{aligned} r_1 &= m \ln\left(1+\frac{r_2}{m}\right), \\ r_2 &= m\left(e^{r_1/m}-1\right). \end{aligned}$$

Annuities

- An annuity pays out the same C dollars at the end of each year for n years.
- With a rate of r, the FV at the end of the nth year is

$$\sum_{i=0}^{n-1} C(1+r)^i = C\,\frac{(1+r)^n - 1}{r}. \tag{2}$$

General Annuities

- If m payments of C dollars each are received per year (the general annuity), then Eq. (2) becomes

$$C\,\frac{\left(1+\frac{r}{m}\right)^{nm}-1}{\frac{r}{m}}.$$

- The PV of a general annuity is

$$\sum_{i=1}^{nm} C\left(1+\frac{r}{m}\right)^{-i} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-nm}}{\frac{r}{m}}. \tag{3}$$

Amortization

- It is a method of repaying a loan through regular payments of interest *and* principal.
- The size of the loan (the original balance) is reduced by the principal part of each payment.
- The interest part of each payment pays the interest incurred on the remaining principal balance.
- As the principal gets paid down over the term of the loan, the interest part of the payment diminishes.

Example: Home Mortgage

- By paying down the principal consistently, the risk to the lender is lowered.
- When the borrower sells the house, the remaining principal is due the lender.
- Consider the equal-payment case, i.e., fixed-rate, level-payment, fully amortized mortgages.
  - They are called traditional mortgages in the U.S.

A Numerical Example

- Consider a 15-year, \$250,000 loan at 8.0% interest rate.
- Solving Eq. (3) on p. 36 with $\text{PV} = 250000$, $n = 15$, $m = 12$, and $r = 0.08$ gives a monthly payment of $C = 2389.13$.
- The amortization schedule is shown on p. 40.
- In every month (1) the principal and interest parts add up to \$2,389.13, (2) the remaining principal is reduced by the amount indicated under the Principal heading, and (3) the interest is computed by multiplying the remaining balance of the previous month by 0.08/12.

Month	Payment	Interest	Principal	Remaining principal
				250,000.000
1	2,389.13	1,666.667	722.464	249,277.536
2	2,389.13	1,661.850	727.280	248,550.256
3	2,389.13	1,657.002	732.129	247,818.128
		. . .		
178	2,389.13	47.153	2,341.980	4,730.899
179	2,389.13	31.539	2,357.591	2,373.308
180	2,389.13	15.822	2,373.308	0.000
Total	430,043.438	180,043.438	250,000.000	

Method 1 of Calculating the Remaining Principal

- Go down the amortization schedule until you reach the particular month you are interested in.
  - A month's principal payment equals the monthly payment subtracted by the previous month's remaining principal times the monthly interest rate.
  - A month's remaining principal equals the previous month's remaining principal subtracted by the principal payment calculated above.

Method 1 of Calculating the Remaining Principal (concluded)

- This method is relatively slow but is universal in its applicability.
- It can, for example, accommodate prepayment and variable interest rates.

Method 2 of Calculating the Remaining Principal

- Right after the kth payment, the remaining principal is the PV of the future $nm - k$ cash flows,

$$\sum_{i=1}^{nm-k} C \left(1 + \frac{r}{m}\right)^{-i} = C \, \frac{1 - \left(1 + \frac{r}{m}\right)^{-nm+k}}{\frac{r}{m}}.$$

- This method is faster but more limited in applications.

Yields

- The term yield denotes the return of investment.
- Two widely used yields are the bond equivalent yield (BEY) and the mortgage equivalent yield (MEY).
- BEY corresponds to the r in Eq. (1) on p. 24 that equates PV with FV when $m = 2$.
- MEY corresponds to the r in Eq. (1) on p. 24 that equates PV with FV when $m = 12$.

Internal Rate of Return (IRR)

- It is the interest rate which equates an investment's PV with its price P,

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_n}{(1+y)^n}.$$

- The above formula is the foundation upon which pricing methodologies are built.

Numerical Methods for Yields

- Solve $f(y) = 0$ for $y \geq -1$, where

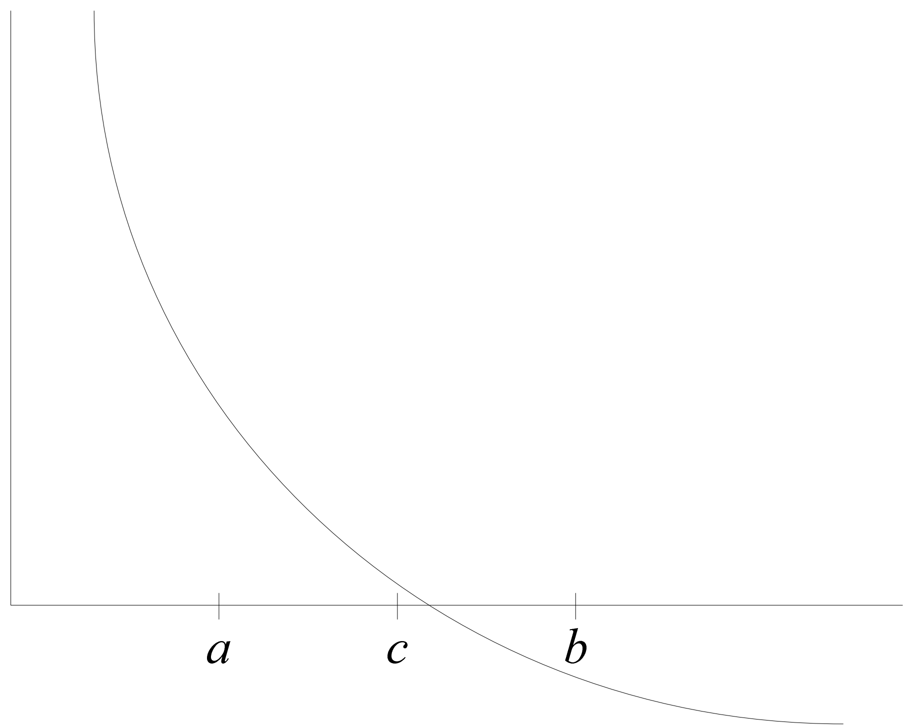
$$f(y) \equiv \sum_{t=1}^{n} \frac{C_t}{(1+y)^t} - P.$$

  - P is the market price.

- The function $f(y)$ is monotonic in y if $C_t > 0$ for all t.
- A unique solution exists for a monotonic $f(y)$.

The Bisection Method

- Start with a and b where $a < b$ and $f(a)\,f(b) < 0$.
- Then $f(\xi)$ must be zero for some $\xi \in [\,a, b\,]$.
- If we evaluate f at the midpoint $c \equiv (a+b)/2$, either (1) $f(c) = 0$, (2) $f(a)\,f(c) < 0$, or (3) $f(c)\,f(b) < 0$.
- In the first case we are done, in the second case we continue the process with the new bracket $[\,a, c\,]$, and in the third case we continue with $[\,c, b\,]$.
- The bracket is halved in the latter two cases.
- After n steps, we will have confined ξ within a bracket of length $(b-a)/2^n$.


a
c
b

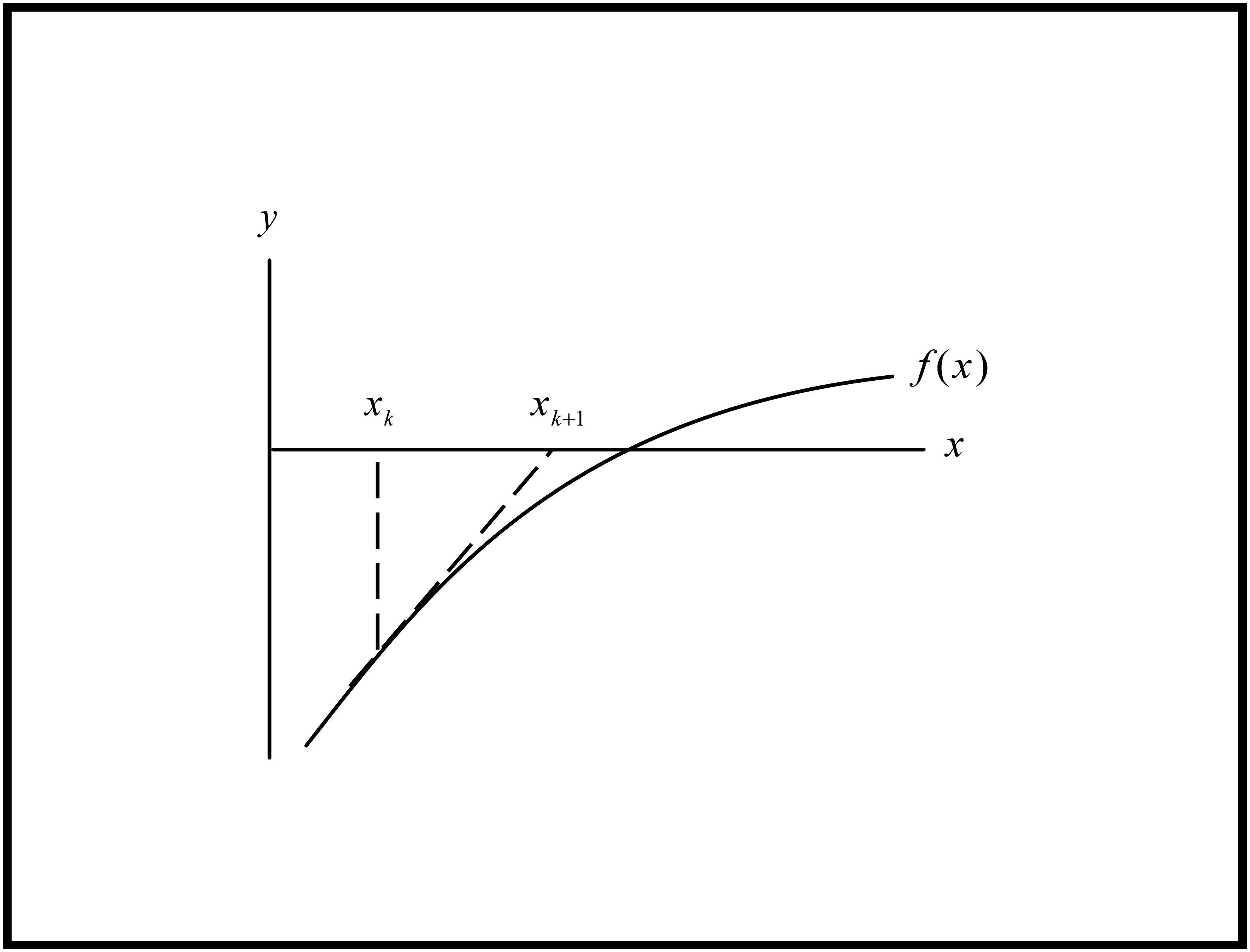
The Newton-Raphson Method

- Converges faster than the bisection method.
- Start with a first approximation x_0 to a root of $f(x) = 0$.
- Then

$$x_{k+1} \equiv x_k - \frac{f(x_k)}{f'(x_k)}.$$

- When computing yields,

$$f'(x) = -\sum_{t=1}^{n} \frac{tC_t}{(1+x)^{t+1}}.$$



y
f(x)
x_k
x_{k+1}
x

The Secant Method

- A variant of the Newton-Raphson method.
- Replace differentiation with difference.
- Start with two approximations x_0 and x_1.
- Then compute the $(k+1)$st approximation with

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

The Secant Method (concluded)

- Its convergence rate, 1.618.
- This is slightly worse than the Newton-Raphson method's 2.
- But the secant method does not need to evaluate $f'(x_k)$ needed by the Newton-Raphson method.
- This saves about 50% in computation efforts per iteration.
- The convergence rate of the bisection method is 1.

Solving Systems of Nonlinear Equations

- It is not easy to extend the bisection method to higher dimensions.
- But the Newton-Raphson method can be extended to higher dimensions.
- Let (x_k, y_k) be the kth approximation to the solution of the two simultaneous equations,

$$\begin{aligned} f(x,y) &= 0, \\ g(x,y) &= 0. \end{aligned}$$

Solving Systems of Nonlinear Equations (concluded)

- The $(k+1)$st approximation (x_{k+1}, y_{k+1}) satisfies the following linear equations,

$$\begin{bmatrix} \frac{\partial f(x_k, y_k)}{\partial x} & \frac{\partial f(x_k, y_k)}{\partial y} \\ \frac{\partial g(x_k, y_k)}{\partial x} & \frac{\partial g(x_k, y_k)}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{bmatrix} = - \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix},$$

  where $\Delta x_{k+1} \equiv x_{k+1} - x_k$ and $\Delta y_{k+1} \equiv y_{k+1} - y_k$.

- The above has a unique solution for $(\Delta x_{k+1}, \Delta y_{k+1})$ when the 2×2 matrix is invertible.

- Set $(x_{k+1}, y_{k+1}) = (x_k + \Delta x_{k+1}, y_k + \Delta y_{k+1})$.

Zero-Coupon Bonds (Pure Discount Bonds)

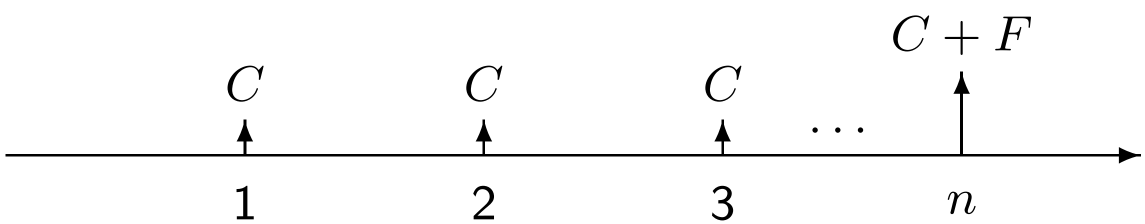
- The price of a zero-coupon bond that pays F dollars in n periods is
$$F/(1+r)^n,$$
where r is the interest rate per period.
- Can meet future obligations without reinvestment risk.

Example

- The interest rate is 8% compounded semiannually.
- A zero-coupon bond that pays the par value 20 years from now will be priced at $1/(1.04)^{40}$, or 20.83%, of its par value.
- It will be quoted as 20.83.
- If the bond matures in 10 years instead of 20, its price would be 45.64.

Level-Coupon Bonds

- Coupon rate.
- Par value, paid at maturity.
- F denotes the par value, and C denotes the coupon.
- Cash flow:



- Coupon bonds can be thought of as a matching package of zero-coupon bonds, at least theoretically.

Pricing Formula

$$
\begin{aligned}
P &= \sum_{i=1}^{n} \frac{C}{\left(1+\frac{r}{m}\right)^{i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}} \\
&= C\, \frac{1-\left(1+\frac{r}{m}\right)^{-n}}{\frac{r}{m}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}}. \qquad (4)
\end{aligned}
$$

- n: number of cash flows.
- m: number of payments per year.
- r: annual rate compounded m times per annum.
- $C = Fc/m$ when c is the annual coupon rate.
- Price P can be computed in $O(1)$ time.

Yields to Maturity

- It is the r that satisfies Eq. (4) on p. 58 with P being the bond price.
- For a 15% BEY, a 10-year bond with a coupon rate of 10% paid semiannually sells for

$$\begin{aligned} & 5 \times \frac{1-[\,1+(0.15/2)\,]^{-2\times 10}}{0.15/2} + \frac{100}{[\,1+(0.15/2)\,]^{2\times 10}} \\ = \ & 74.5138 \end{aligned}$$

percent of par.

Price Behavior (1)

- Bond prices fall when interest rates rise, and vice versa.
- "Only 24 percent answered the question correctly."[a]

[a]CNN, December 21, 2001.