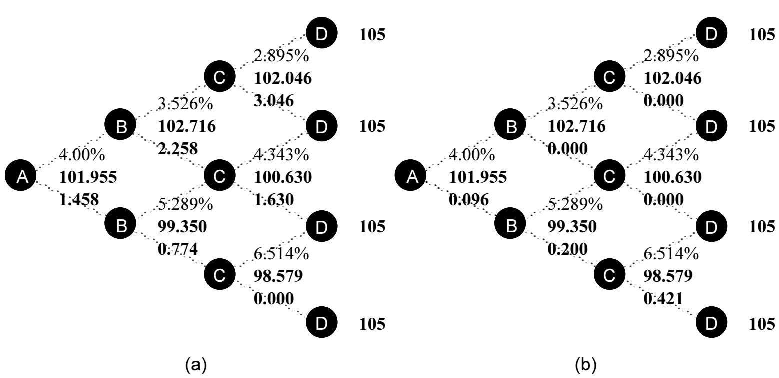
Fixed-Income Options

- Consider a two-year 99 European call on the three-year, 5% Treasury.
- Assume the Treasury pays annual interest.
- From p. 739 the three-year Treasury's price minus the \$5 interest could be \$102.046, \$100.630, or \$98.579 two years from now.
- Since these prices do not include the accrued interest, we should compare the strike price against them.
- The call is therefore in the money in the first two scenarios, with values of \$3.046 and \$1.630, and out of the money in the third scenario.



Fixed-Income Options (continued)

- The option value is calculated to be \$1.458 on p. 739(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only if the Treasury is worth \$98.579 without the accrued interest.
- The option value is computed to be \$0.096 on p. 739(b).

Fixed-Income Options (concluded)

- The present value of the strike price is $\text{PV}(X) = 99 \times 0.92101 = 91.18$.
- The Treasury is worth $B = 101.955$.
- The present value of the interest payments during the life of the options is

$$\text{PV}(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275.$$

- The call and the put are worth $C = 1.458$ and $P = 0.096$, respectively.
- Hence the put-call parity is preserved:

$$C = P + B - \text{PV}(I) - \text{PV}(X).$$

Delta or Hedge Ratio

- How much does the option price change in response to changes in the price of the underlying bond?
- This relation is called delta (or hedge ratio) defined as

$$\frac{O_{\rm h} - O_\ell}{P_{\rm h} - P_\ell}.$$

- In the above $P_{\rm h}$ and P_ℓ denote the bond prices if the short rate moves up and down, respectively.
- Similarly, $O_{\rm h}$ and O_ℓ denote the option values if the short rate moves up and down, respectively.

Delta or Hedge Ratio (concluded)

- Since delta measures the sensitivity of the option value to changes in the underlying bond price, it shows how to hedge one with the other.
- Take the call and put on p. 739 as examples.
- Their deltas are

$$\begin{aligned} \frac{0.774 - 2.258}{99.350 - 102.716} &= 0.441, \\ \frac{0.200 - 0.000}{99.350 - 102.716} &= -0.059, \end{aligned}$$

respectively.

Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an n-period zero-coupon bond.
- First find its yield to maturity $y_{\rm h}$ (y_ℓ, respectively) at the end of the initial period if the rate rises (declines, respectively).
- The yield volatility for our model is defined as $(1/2) \ln(y_{\rm h}/y_\ell)$.

Volatility Term Structures (continued)

- For example, based on the tree on p. 720, the two-year zero's yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.
- Its yield volatility is therefore

$$\frac{1}{2} \ln\left(\frac{0.05289}{0.03526}\right) = 20.273\%.$$

Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the rate rises, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.05289} \times \left(\frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.$$

- Thus its yield is $\sqrt{\frac{1}{0.90096}} - 1 = 0.053531$.
- If the rate declines, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.03526} \times \left(\frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.$$

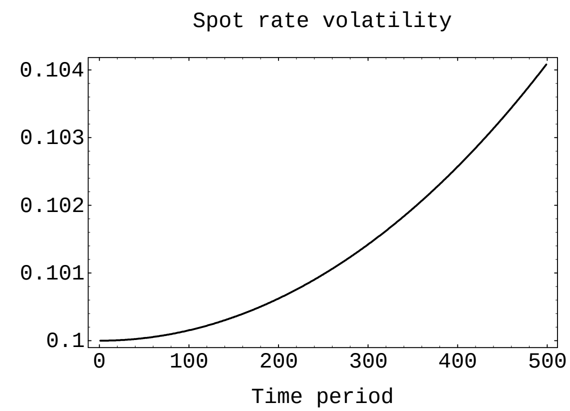
Volatility Term Structures (continued)

- Thus its yield is $\sqrt{\frac{1}{0.93225}} - 1 = 0.0357$.
- The yield volatility is hence

$$\frac{1}{2} \ln \left(\frac{0.053531}{0.0357} \right) = 20.256\%,$$

slightly less than the one-year yield volatility.

- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.



Short rate volatility given flat %10 volatility term structure.

Volatility Term Structures (continued)

- We started with v_i and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The v_i—hence the short rate volatilities via Eq. (77) on p. 700—and the r_i are then simultaneously determined.
- The result is the Black-Derman-Toy model.

Volatility Term Structures (concluded)

- Suppose the user supplies the volatility term structure which results in $(v_1, v_2, v_3, \dots)$ for the tree.
- The volatility term structure one period from now will be determined by $(v_2, v_3, v_4, \dots)$ not $(v_1, v_2, v_3, \dots)$.
- The volatility term structure supplied by the user is hence not maintained through time.
- This issue will be addressed by other types of (complex) models.

Foundations of Term Structure Modeling

> [Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader.
> — Roger Lowenstein,
> *When Genius Failed*

Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time t, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

t**:** a point in time.

$r(t)$**:** the one-period riskless rate prevailing at time t for repayment one period later (the instantaneous spot rate, or short rate, at time t).

$P(t,T)$**:** the present value at time t of one dollar at time T.

Standard Notations (continued)

$r(t,T)$**:** the $(T-t)$-period interest rate prevailing at time t stated on a per-period basis and compounded once per period—in other words, the $(T-t)$-period spot rate at time t.

- The long rate is defined as $r(t,\infty)$.

$F(t,T,M)$**:** the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \geq T$.

Standard Notations (concluded)

$f(t,T,L)$**:** the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.

$f(t,T)$**:** the one-period or instantaneous forward rate at time T as seen at time t stated on a per period basis and compounded once per period.

- It is $f(t,T,1)$ in the discrete-time model and $f(t,T,dt)$ in the continuous-time model.
- Note that $f(t,t)$ equals the short rate $r(t)$.

Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t,T) = \begin{cases} (1+r(t,T))^{-(T-t)} & \text{in discrete time,} \\ e^{-r(t,T)(T-t)} & \text{in continuous time.} \end{cases}$$

- $r(t,T)$ as a function of T defines the spot rate curve at time t.
- By definition,

$$f(t,t) = \begin{cases} r(t,t+1) & \text{in discrete time,} \\ r(t,t) & \text{in continuous time.} \end{cases}$$

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t,T,M) = \frac{P(t,M)}{P(t,T)}, \quad T \leq M. \tag{82}$$

  - The forward price equals the future value at time T of the underlying asset (see text for proof).

- Equation (82) holds whether the model is discrete-time or continuous-time, and it implies

$$F(t,T,M) = F(t,T,S)\, F(t,S,M), \quad T \leq S \leq M.$$

Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t,T,L) = \left(\frac{1}{F(t,T,T+L)}\right)^{1/L} - 1 = \left(\frac{P(t,T)}{P(t,T+L)}\right)^{1/L} - 1 \tag{83}$$

  in discrete time.

  - $f(t,T,L) = \frac{1}{L}\left(\frac{P(t,T)}{P(t,T+L)} - 1\right)$ is the analog to Eq. (83) under simple compounding.

Fundamental Relations (continued)

- In continuous time,

$$f(t,T,L) = -\frac{\ln F(t,T,T+L)}{L} = \frac{\ln(P(t,T)/P(t,T+L))}{L} \tag{84}$$

  by Eq. (82) on p. 758.

- Furthermore,

$$\begin{aligned} f(t,T,\Delta t) &= \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \rightarrow -\frac{\partial \ln P(t,T)}{\partial T} \\ &= -\frac{\partial P(t,T)/\partial T}{P(t,T)}. \end{aligned}$$

Fundamental Relations (continued)

- So

$$f(t,T) \equiv \lim_{\Delta t \to 0} f(t,T,\Delta t) = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \leq T. \tag{85}$$

- Because Eq. (85) is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s)\, ds}, \tag{86}$$

  the spot rate curve is

$$r(t,T) = \frac{1}{T-t} \int_t^T f(t,s)\, ds.$$

Fundamental Relations (concluded)

- The discrete analog to Eq. (86) is

$$P(t,T) = \frac{1}{(1+r(t))(1+f(t,t+1))\cdots(1+f(t,T-1))}. \tag{87}$$

- The short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}.$$

  – This can be verified with Eq. (85) on p. 761 and the observation that $P(t,t)=1$ and $r(t)=f(t,t)$.

Risk-Neutral Pricing

- Under the local expectations theory, the expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  – For all $t+1<T$,

$$\frac{E_t[\,P(t+1,T)\,]}{P(t,T)} = 1+r(t). \tag{88}$$

  – Relation (88) in fact follows from the risk-neutral valuation principle, Theorem 14 (p. 419).

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π.
- Rewrite Eq. (88) as

$$\frac{E_t^{\pi}[\,P(t+1,T)\,]}{1+r(t)} = P(t,T).$$

  – It says the current spot rate curve equals the expected spot rate curve one period from now discounted by the short rate.

Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

$$\begin{aligned} &P(t,T)\\ =\;& E_t^{\pi}\left[\frac{P(t+1,T)}{1+r(t)}\right]\\ =\;& E_t^{\pi}\left[\frac{E_{t+1}^{\pi}[\,P(t+2,T)\,]}{(1+r(t))(1+r(t+1))}\right] = \cdots\\ =\;& E_t^{\pi}\left[\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))}\right]. \end{aligned} \tag{89}$$

Risk-Neutral Pricing (concluded)

- Equation (88) on p. 763 can also be expressed as

$$E_t[\, P(t+1,T)\,] = F(t,t+1,T).$$

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

$$P(t,T) = E_t \left[e^{-\int_t^T r(s)\, ds} \right], \quad t < T. \qquad (90)$$

- Note that $e^{\int_t^T r(s)\, ds}$ is the bank account process, which denotes the rolled-over money market account.
- When the local expectations theory holds, riskless arbitrage opportunities are impossible.

Interest Rate Swaps

- Consider an interest rate swap made at time t with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is c per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant Δt for all i, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

Interest Rate Swaps (continued)

- The amount to be paid out at time t_{i+1} is $(f_i - c)\,\Delta t$ for the floating-rate payer.
  - Simple rates are adopted here.
- Hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

Interest Rate Swaps (continued)

- The value of the swap at time t is thus

$$
\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi} \left[e^{-\int_t^{t_i} r(s)\, ds} (f_{i-1} - c)\, \Delta t \right] \\
= & \sum_{i=1}^{n} E_t^{\pi} \left[e^{-\int_t^{t_i} r(s)\, ds} \left(\frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right] \\
= & \sum_{i=1}^{n} (P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i)) \\
= & \ P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}
$$

Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.

Swap Rate

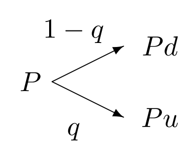
- The swap rate, which gives the swap zero value, equals

$$
S_n(t) \equiv \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i)\, \Delta t}. \tag{91}
$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t, t_0) = 1$.

The Binomial Model

- The analytical framework can be nicely illustrated with the binomial model.
- Suppose the bond price P can move with probability q to Pu and probability $1 - q$ to Pd, where $u > d$:



The Binomial Model (continued)

- Over the period, the bond's expected rate of return is

$$\widehat{\mu} \equiv \frac{qPu + (1-q)\,Pd}{P} - 1 = qu + (1-q)\,d - 1. \tag{92}$$

- The variance of that return rate is

$$\widehat{\sigma}^2 \equiv q(1-q)(u-d)^2. \tag{93}$$

- The bond whose maturity is only one period away will move from a price of $1/(1+r)$ to its par value \$1.
- This is the money market account modeled by the short rate.

The Binomial Model (continued)

- The market price of risk is defined as $\lambda \equiv (\widehat{\mu} - r)/\widehat{\sigma}$.
- The same arbitrage argument as in the continuous-time case can be employed to show that λ is independent of the maturity of the bond (see text).

The Binomial Model (concluded)

- Now change the probability from q to

$$p \equiv q - \lambda\sqrt{q(1-q)} = \frac{(1+r)-d}{u-d}, \tag{94}$$

which is independent of bond maturity and q.
  - Recall the BOPM.
- The bond's expected rate of return becomes

$$\frac{pPu + (1-p)\,Pd}{P} - 1 = pu + (1-p)\,d - 1 = r.$$

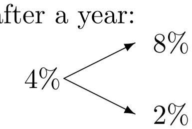
- The local expectations theory hence holds under the new probability measure p.

Numerical Examples

- Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:



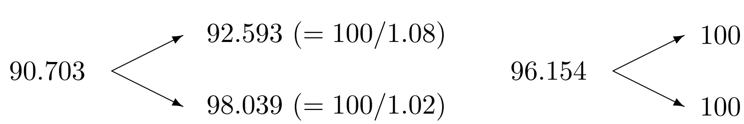
Numerical Examples (continued)

- No real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$100/1.04 = 96.154, 100/(1.05)^2 = 90.703.$$

- They follow the binomial processes on p. 779.

Numerical Examples (continued)



The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

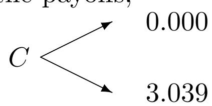
where p denotes the risk-neutral probability of an up move in rates.

Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,



- To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.

Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$\begin{aligned} x \times 100 + y \times 92.593 &= 0.000, \\ x \times 100 + y \times 98.039 &= 3.039. \end{aligned}$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

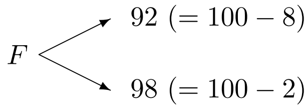
$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

the same as before.

- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity, as shown below.



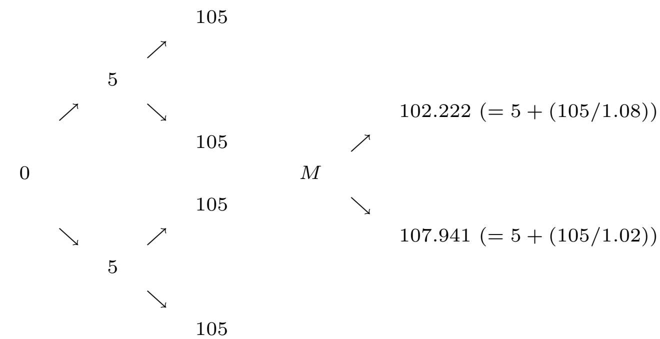
- As the futures price F is the expected future payoff (see text), $F = (1-p) \times 92 + p \times 98 = 93.914$.
- On the other hand, the forward price for a one-year forward contract on a one-year zero-coupon bond equals $90.703/96.154 = 94.331\%$.
- The forward price exceeds the futures price.

Numerical Examples: Mortgage-Backed Securities

- Consider a 5%-coupon, two-year mortgage-backed security without amortization, prepayments, and default risk.
- Its cash flow and price process are illustrated on p. 788.
- Its fair price is

$$M = \frac{(1-p) \times 102.222 + p \times 107.941}{1.04} = 100.045.$$

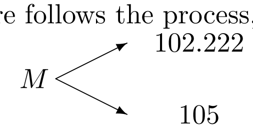
- Identical results could have been obtained via arbitrage considerations.



The left diagram depicts the cash flow; the right diagram illustrates the price process.

Numerical Examples: MBSs (continued)

- Suppose that the security can be prepaid at par.
- It will be prepaid only when its price is higher than par.
- Prepayment will hence occur only in the "down" state when the security is worth 102.941 (excluding coupon).
- The price therefore follows the process,



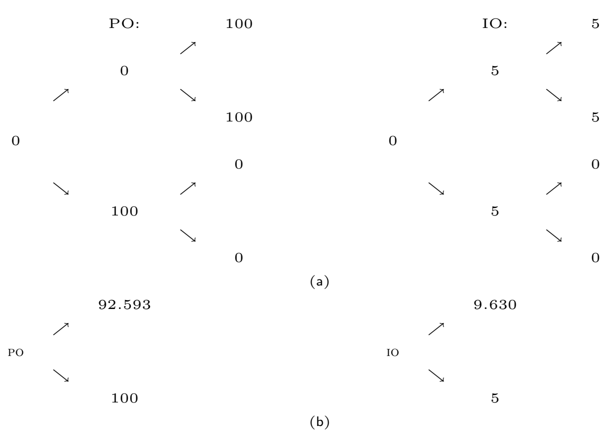
- The security is worth

$$M = \frac{(1-p) \times 102.222 + p \times 105}{1.04} = 99.142.$$

Numerical Examples: MBSs (continued)

- The cash flow of the principal-only (PO) strip comes from the mortgage's principal cash flow.
- The cash flow of the interest-only (IO) strip comes from the interest cash flow (p. 791(a)).
- Their prices hence follow the processes on p. 791(b).
- The fair prices are

$$\begin{aligned} \text{PO} &= \frac{(1-p)\times 92.593 + p\times 100}{1.04} = 91.304, \\ \text{IO} &= \frac{(1-p)\times 9.630 + p\times 5}{1.04} = 7.839. \end{aligned}$$



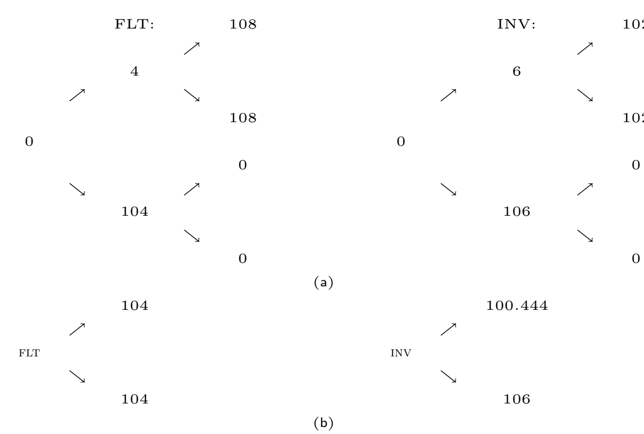
The price 9.630 is derived from $5 + (5/1.08)$.

Numerical Examples: MBSs (continued)

- Suppose the mortgage is split into half floater and half inverse floater.
- Let the floater (FLT) receive the one-year rate.
- Then the inverse floater (INV) must have a coupon rate of

$$(10\% - \text{one-year rate})$$

  to make the overall coupon rate 5%.
- Their cash flows as percentages of par and values are shown on p. 793.



Numerical Examples: MBSs (concluded)

- On p. 793, the floater's price in the up node, 104, is derived from $4 + (108/1.08)$.
- The inverse floater's price 100.444 is derived from $6 + (102/1.08)$.
- The current prices are

$$\begin{aligned} \text{FLT} &= \frac{1}{2} \times \frac{104}{1.04} = 50, \\ \text{INV} &= \frac{1}{2} \times \frac{(1-p) \times 100.444 + p \times 106}{1.04} = 49.142. \end{aligned}$$

Equilibrium Term Structure Models

8. What's your problem? Any moron can understand bond pricing models.
— *Top Ten Lies Finance Professors Tell Their Students*

Introduction

- This chapter surveys equilibrium models.
- Since the spot rates satisfy

$$r(t,T) = -\frac{\ln P(t,T)}{T-t},$$

  the discount function $P(t,T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows
$$dr = \beta(\mu - r)\,dt + \sigma\,dW.$$
- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\,dW$.
- Since the process is an Ornstein-Uhlenbeck process,
$$E[\,r(T)\,|\,r(t) = r\,] = \mu + (r - \mu)\,e^{-\beta(T-t)}$$
from Eq. (52) on p. 475.

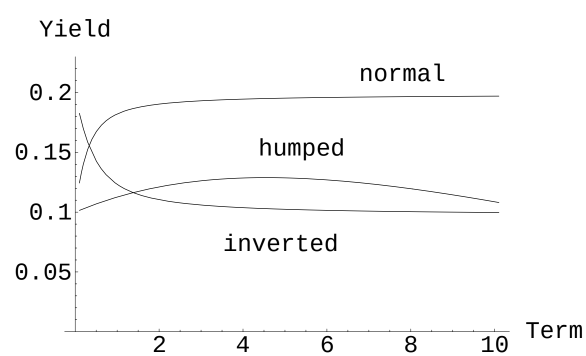
[a]Vasicek (1977).

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be
$$P(t,T) = A(t,T)\,e^{-B(t,T)\,r(t)}, \tag{95}$$
where
$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta}\right] & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2 (T-t)^3}{6}\right] & \text{if } \beta = 0. \end{cases}$$
and
$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T - t & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (concluded)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, P may exceed one for a finite T.
- The spot rate volatility structure is the curve $(\partial r(t,T)/\partial r)\,\sigma = \sigma B(t,T)/(T-t)$.
- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, β, controls the shape of the curve; indeed, higher β leads to greater attenuation of volatility with maturity.



The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t,s)\,N(x) - X\,P(t,T)\,N(x-\sigma_v).$$

[a] Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$\begin{aligned} x &\equiv \frac{1}{\sigma_v}\ln\left(\frac{P(t,s)}{P(t,T)\,X}\right)+\frac{\sigma_v}{2}, \\ \sigma_v &\equiv v(t,T)\,B(T,s), \\ v(t,T)^2 &\equiv \begin{cases} \frac{\sigma^2[1-e^{-2\beta(T-t)}]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}. \end{aligned}$$

- By the put-call parity, the price of a European put is

$$X\,P(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).$$

Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[0,T]$ divided into n identical pieces.
- Let $\Delta t \equiv T/n$ and

$$p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.$$

- The following binomial model converges to the Vasicek model,[a]

$$r(k+1) = r(k) + \sigma\sqrt{\Delta t}\;\xi(k), \quad 0 \leq k < n.$$

[a] Nelson and Ramaswamy (1990).

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\,\xi(k)=1\,] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases}.$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.
- For a general process Y with nonconstant volatility, the resulting binomial tree may not combine.

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\, dt + \sigma \sqrt{r}\, dW. \tag{96}$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.
- See text for the bond pricing formula.

[a]Cox, Ingersoll, and Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into n periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process

$$x(r) \equiv 2\sqrt{r}/\sigma.$$

- It follows

$$dx = m(x)\, dt + dW,$$

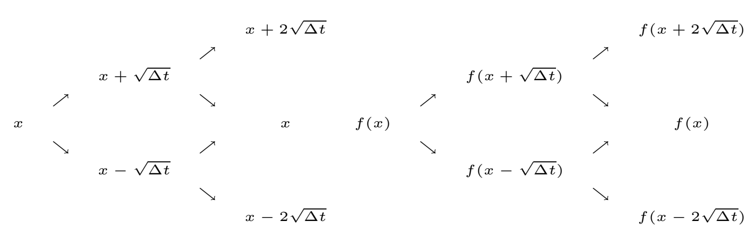
where

$$m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- Since this new process has a constant volatility, its associated binomial tree combines.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation $r = f(x) \equiv x^2\sigma^2/4$ (p. 811).



Binomial CIR (concluded)

- The probability of an up move at each node r is

$$p(r) \equiv \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}. \qquad (97)$$

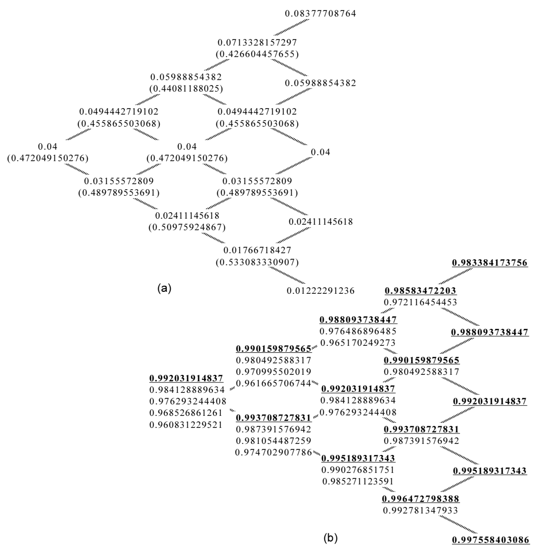
  - $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.
- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Numerical Examples

- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.
- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.
- See p. 814(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.



Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$.

Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process $dy = \alpha(y,t)\, dt + \sigma(y,t)\, dW$.
- Make sure the binomial model's drift and diffusion converge to the above process by setting the probability of an up move to

$$\frac{\alpha(y,t)\, \Delta t + y - y_{\mathrm{u}}}{y_{\mathrm{u}} - y_{\mathrm{d}}}.$$

- Here $y_{\mathrm{u}} \equiv y + \sigma(y,t)\sqrt{\Delta t}$ and $y_{\mathrm{d}} \equiv y - \sigma(y,t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.
- The displacements are identical, at $\sigma(y,t)\sqrt{\Delta t}$.

[a]Nelson and Ramaswamy (1990).

A General Method (continued)

- But the binomial tree may not combine:

$$\sigma(y,t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}},t)\sqrt{\Delta t} \neq -\sigma(y,t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}},t)\sqrt{\Delta t}$$

  in general.
- When $\sigma(y,t)$ is a constant independent of y, equality holds and the tree combines.
- To achieve this, define the transformation

$$x(y,t) \equiv \int^{y} \sigma(z,t)^{-1}\,dz.$$

- Then x follows $dx = m(y,t)\,dt + dW$ for some $m(y,t)$ (see text).

A General Method (continued)

- The key is that the diffusion term is now a constant, and the binomial tree for x combines.
- The probability of an up move remains

$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$

  where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.
- Note that $y_{\mathrm{u}}(x,t) \equiv y(x+\sqrt{\Delta t}, t+\Delta t)$ and $y_{\mathrm{d}}(x,t) \equiv y(x-\sqrt{\Delta t}, t+\Delta t)$.

A General Method (concluded)

- The transformation is

$$\int^{r} (\sigma\sqrt{z})^{-1}\,dz = 2\sqrt{r}/\sigma$$

  for the CIR model.
- The transformation is

$$\int^{S} (\sigma z)^{-1}\,dz = (1/\sigma)\ln S$$

  for the Black-Scholes model.
- The familiar binomial option pricing model in fact discretizes $\ln S$ not S.

Finis