Continuous-Time Derivatives Pricing

I have hardly met a mathematician
who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Toward the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation.
- The key step is recognizing that the same random process drives both securities.
- As their prices are perfectly correlated, we figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio's return to be the riskless rate.

Assumptions

- The stock price follows $dS = \mu S\, dt + \sigma S\, dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at r.
- There is unlimited riskless borrowing and lending.
- t is the current time, T is the expiration time, and $\tau \equiv T - t$.

Black-Scholes Differential Equation

- Let C be the price of a derivative on S.
- From Ito's lemma (p. 462),

$$dC = \left(\mu S \, \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \, \sigma^2 S^2 \, \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \, \frac{\partial C}{\partial S} \, dW.$$

  - The same W drives both C and S.
- Short one derivative and long $\partial C / \partial S$ shares of stock (call it Π).
- By construction,

$$\Pi = -C + S(\partial C / \partial S).$$

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is

$$d\Pi = -dC + \frac{\partial C}{\partial S} \, dS.$$

- Substitute the formulas for dC and dS into the partial differential equation to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2} \, \sigma^2 S^2 \, \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r\Pi \, dt$.

Black-Scholes Differential Equation (concluded)

- So

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2} \, \sigma^2 S^2 \, \frac{\partial^2 C}{\partial S^2} \right) dt = r \left(C - S \, \frac{\partial C}{\partial S} \right) dt.$$

- Equate the terms to finally obtain

$$\frac{\partial C}{\partial t} + rS \, \frac{\partial C}{\partial S} + \frac{1}{2} \, \sigma^2 S^2 \, \frac{\partial^2 C}{\partial S^2} = rC.$$

- When there is a dividend yield q,

$$\frac{\partial C}{\partial t} + (r - q) \, S \, \frac{\partial C}{\partial S} + \frac{1}{2} \, \sigma^2 S^2 \, \frac{\partial^2 C}{\partial S^2} = rC.$$

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2} \, \sigma^2 S^2 \Gamma = rC. \tag{53}$$

- Identity (53) leads to an alternative way of computing Θ numerically from Δ and Γ.
- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2} \, \sigma^2 S^2 \Gamma = rC.$$

  - A definite relation thus exists between Γ and Θ.

PDEs for Asian Options

- Add the new variable $A(t) \equiv \int_0^t S(u)\, du$.
- Then the value V of the Asian option satisfies this two-dimensional PDE:[a]

$$\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + S\,\frac{\partial V}{\partial A} = rV.$$

- The terminal conditions are

$$\begin{aligned} V(T,S,A) &= \max\left(\frac{A}{T} - X, 0\right) \quad \text{for call,} \\ V(T,S,A) &= \max\left(X - \frac{A}{T}, 0\right) \quad \text{for put.} \end{aligned}$$

[a] Kemna and Vorst (1990).

PDEs for Asian Options (continued)

- The two-dimensional PDE produces algorithms similar to that on pp. 320ff.
- But one-dimensional PDEs are available for Asian options.[a]
- For example, Večeř (2001) derives the following PDE for Asian calls:

$$\frac{\partial u}{\partial t} + r\left(1 - \frac{t}{T} - z\right)\frac{\partial u}{\partial z} + \frac{\left(1 - \frac{t}{T} - z\right)^2 \sigma^2}{2}\,\frac{\partial^2 u}{\partial z^2} = 0$$

with the terminal condition $u(T,z) = \max(z,0)$.

[a] Rogers and Shi (1995); Večeř (2001); Dubois and Lelièvre (2005).

PDEs for Asian Options (concluded)

- For Asian puts:

$$\frac{\partial u}{\partial t} + r\left(\frac{t}{T} - 1 - z\right)\frac{\partial u}{\partial z} + \frac{\left(\frac{t}{T} - 1 - z\right)^2 \sigma^2}{2}\,\frac{\partial^2 u}{\partial z^2} = 0$$

with the same terminal condition.

- One-dimensional PDEs lead to highly efficient numerical methods.

Hedging

Trees

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

And though the holes were rather small,
they had to count them all.
— The Beatles, *A Day in the Life* (1967)

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 228.
  - It cannot apply to American options.
- We will now apply it to price barrier options.

The Reflection Principle[a]

- Imagine a particle at position $(0, -\boldsymbol{a})$ on the integral lattice that is to reach $(n, -\boldsymbol{b})$.
- Without loss of generality, assume $\boldsymbol{a} > 0$ and $\boldsymbol{b} \geq 0$.
- This particle's movement:

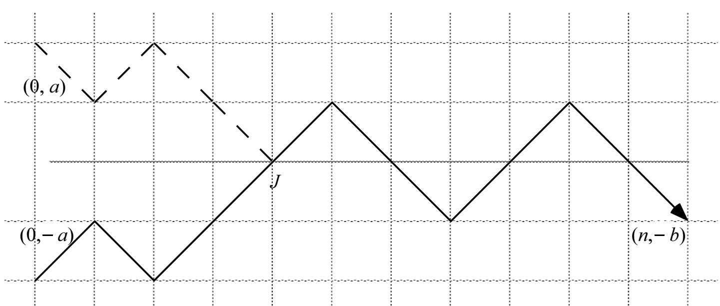
$$(i,j) \begin{array}{l} \nearrow (i+1, j+1) \quad \text{up move } S \to Su \\ \searrow (i+1, j-1) \quad \text{down move } S \to Sd \end{array}$$

- How many paths touch the x axis?

[a]André (1887).

The Reflection Principle (continued)

- For a path from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touches the x axis, let J denote the first point this happens.
- Reflect the portion of the path from $(0, -\boldsymbol{a})$ to J.
- A path from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$ is constructed.
- It also hits the x axis at J for the first time (see figure next page).
- The one-to-one mapping shows the number of paths from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touch the x axis equals the number of paths from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$.



The Reflection Principle (concluded)

- A path of this kind has $(n + \boldsymbol{b} + \boldsymbol{a})/2$ down moves and $(n - \boldsymbol{b} - \boldsymbol{a})/2$ up moves.
- Hence there are

$$\binom{n}{\frac{n+\boldsymbol{a}+\boldsymbol{b}}{2}} \tag{54}$$

such paths for even $n + \boldsymbol{a} + \boldsymbol{b}$.

  - Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

Pricing Barrier Options (Lyuu, 1998)

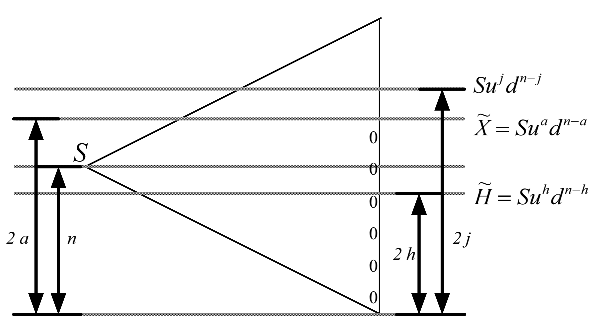
- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define

$$\begin{aligned} a &\equiv \left\lceil \frac{\ln\left(X/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil, \\ h &\equiv \left\lfloor \frac{\ln\left(H/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor. \end{aligned}$$

  - h is such that $\tilde{H} \equiv Su^h d^{n-h}$ is the terminal price that is closest to, but does not exceed H.
  - a is such that $\tilde{X} \equiv Su^a d^{n-a}$ is the terminal price that is closest to, but is not exceeded by X.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with n moves hence ends up in the money if and only if the number of up moves is at least a.
- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price Sd^n on the binomial tree.
  –
$$Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-2k}. \tag{55}$$



Pricing Barrier Options (continued)

- The number of paths from S to the terminal price $Su^j d^{n-j}$ is $\binom{n}{j}$, each with probability $p^j (1-p)^{n-j}$.
- With reference to p. 498, the reflection principle can be applied with $\boldsymbol{a} = n - 2h$ and $\boldsymbol{b} = 2j - 2h$ in Eq. (54) on p. 495 by treating the S line as the x axis.
- Therefore,
$$\binom{n}{\frac{n+(n-2h)+(2j-2h)}{2}} = \binom{n}{n-2h+j}$$
paths hit $\tilde{H}$ in the process for $h \leq n/2$.

Pricing Barrier Options (concluded)

- The terminal price $Su^j d^{n-j}$ is reached by a path that hits the effective barrier with probability
$$\binom{n}{n-2h+j} p^j (1-p)^{n-j}.$$
- The option value equals
$$\frac{\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - X \right)}{R^n}. \tag{56}$$
  – $R \equiv e^{r\tau/n}$ is the riskless return per period.
- It implies a linear-time algorithm.

Convergence of BOPM

- Equation (56) results in the sawtooth-like convergence shown on p. 303.
- The reasons are not hard to see.
- The true barrier most likely does not equal the effective barrier.
- The same holds for the strike price and the effective strike price.
- The issue of the strike price is less critical.
- But the issue of the barrier is not negligible.

Convergence of BOPM (continued)

- Convergence is actually good if we limit n to certain values—191, for example.
- These values make the true barrier coincide with or occur just above one of the stock price levels, that is, $H \approx Sd^j = Se^{-j\sigma\sqrt{\tau/n}}$ for some integer j.
- The preferred n's are thus

$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

- There is only one minor technicality left.

Convergence of BOPM (continued)

- We picked the effective barrier to be one of the $n+1$ possible terminal stock prices.
- However, the effective barrier above, Sd^j, corresponds to a terminal stock price only when $n-j$ is even by Eq. (55) on p. 497.[a]
- To close this gap, we decrement n by one, if necessary, to make $n-j$ an even number.

[a]We could have adopted the form Sd^j $(-n \leq j \leq n)$ for the effective barrier.

Convergence of BOPM (concluded)

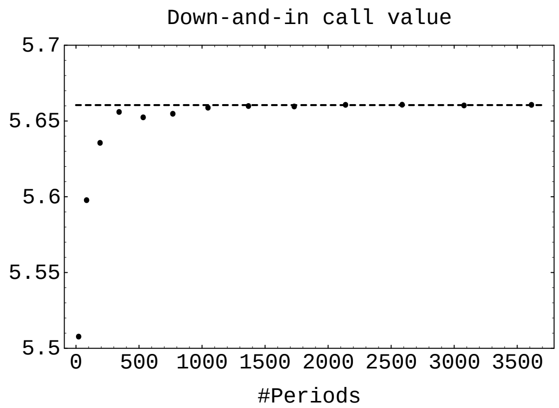
- The preferred n's are now

$$n = \begin{cases} \ell & \text{if } \ell - j \text{ is even} \\ \ell - 1 & \text{otherwise} \end{cases},$$

$j = 1, 2, 3, \ldots$, where

$$\ell \equiv \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.$$

- So evaluate pricing formula (56) on p. 500 only with the n's above.



Practical Implications

- Now that barrier options can be efficiently priced, we can afford to pick very large n's (p. 507).
- This has profound consequences.
- For example, pricing is prohibitively time consuming when $S \approx H$ because $n \sim 1/\ln^2(S/H)$.
- This observation is indeed true of standard quadratic-time binomial tree algorithms.
- But it no longer applies to linear-time algorithms (p. 508).

n	Combinatorial method	
	Value	Time (milliseconds)
21	5.507548	0.30
84	5.597597	0.90
191	5.635415	2.00
342	5.655812	3.60
533	5.652253	5.60
768	5.654609	8.00
1047	5.658622	11.10
1368	5.659711	15.00
1731	5.659416	19.40
2138	5.660511	24.70
2587	5.660592	30.20
3078	5.660099	36.70
3613	5.660498	43.70
4190	5.660388	44.10
4809	5.659955	51.60
5472	5.660122	68.70
6177	5.659981	76.70
6926	5.660263	86.90
7717	5.660272	97.20

	Barrier at 95.0			Barrier at 99.5			Barrier at 99.9	
n	Value	Time	n	Value	Time	n	Value	Time
	⋮		795	7.47761	8	19979	8.11304	253
2743	2.56095	31.1	3184	7.47626	38	79920	8.11297	1013
3040	2.56065	35.5	7163	7.47682	88	179819	8.11300	2200
3351	2.56098	40.1	12736	7.47661	166	319680	8.11299	4100
3678	2.56055	43.8	19899	7.47676	253	499499	8.11299	6300
4021	2.56152	48.1	28656	7.47667	368	719280	8.11299	8500
True	2.5615			7.4767			8.1130	

(All times in milliseconds.)

Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion $dS/S = r\,dt + \sigma\,dW$.[a]
- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Impose the matching of mean and that of variance:

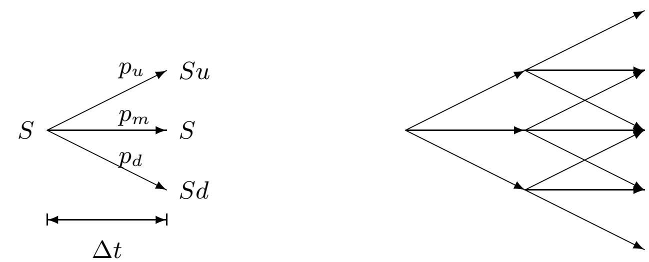
$$
\begin{aligned}
1 &= p_u + p_m + p_d,\\
SM &\equiv (p_u u + p_m + (p_d/u))\, S,\\
S^2 V &\equiv p_u (Su - SM)^2 + p_m (S - SM)^2 + p_d (Sd - SM)^2.
\end{aligned}
$$

[a] Boyle (1988).

- Above,

$$
\begin{aligned}
M &\equiv e^{r\Delta t},\\
V &\equiv M^2 (e^{\sigma^2 \Delta t} - 1),
\end{aligned}
$$

by Eqs. (17) on p. 145.



Trinomial Tree (continued)

- Use linear algebra to verify that

$$
\begin{aligned}
p_u &= \frac{u\left(V + M^2 - M\right) - (M - 1)}{(u - 1)\,(u^2 - 1)},\\
p_d &= \frac{u^2\left(V + M^2 - M\right) - u^3 (M - 1)}{(u - 1)\,(u^2 - 1)}.
\end{aligned}
$$

  - In practice, must make sure the probabilities lie between 0 and 1.
- Countless variations.

Trinomial Tree (concluded)

- Use $u = e^{\lambda \sigma \sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$\begin{aligned} p_u &\to \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}, \\ p_d &\to \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}. \end{aligned}$$

- A nice choice for λ is $\sqrt{\pi/2}$.[a]

[a] Omberg (1988).

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting λ so that the barrier is hit exactly.[a]
- It takes

$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}$$

consecutive down moves to go from S to H if h is an integer, which is easy to achieve by adjusting λ.

[a] Ritchken (1995).

Barrier Options Revisited (continued)

- Typically, we find the smallest $\lambda \geq 1$ such that h is an integer.
- That is, we find the largest integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ and then let

$$\lambda = \frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}}.$$

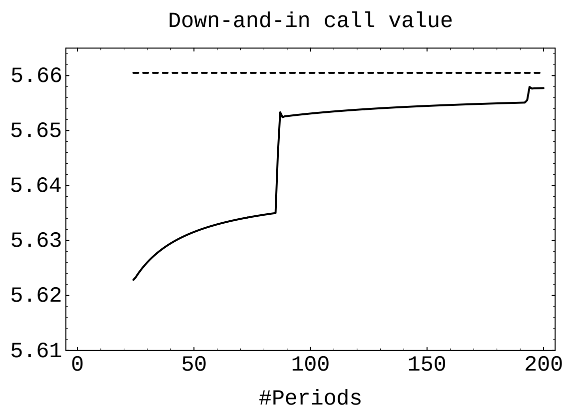
  - Such a λ may not exist for very small n's.
  - This is not hard to check.
- This done, one of the layers of the trinomial tree coincides with the barrier.

Barrier Options Revisited (concluded)

- The following probabilities may be used,

$$\begin{aligned} p_u &= \frac{1}{2\lambda^2} + \frac{\mu'\sqrt{\Delta t}}{2\lambda\sigma}, \\ p_m &= 1 - \frac{1}{\lambda^2}, \\ p_d &= \frac{1}{2\lambda^2} - \frac{\mu'\sqrt{\Delta t}}{2\lambda\sigma}. \end{aligned}$$

  - $\mu' \equiv r - \sigma^2/2$.



Algorithms Comparison[a]

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the n value at which they converge.
  - The one with the smallest n wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times.

[a] Lyuu (1998).

Algorithms Comparison (concluded)

- Pages 303 and 517 show the trinomial model converges at a smaller n than BOPM.
- It is in this sense when people say trinomial models converge faster than binomial ones.
- But is the trinomial model better then?
- The linear-time binomial tree algorithm actually performs better than the trinomial one (see next page expanded from p. 507).

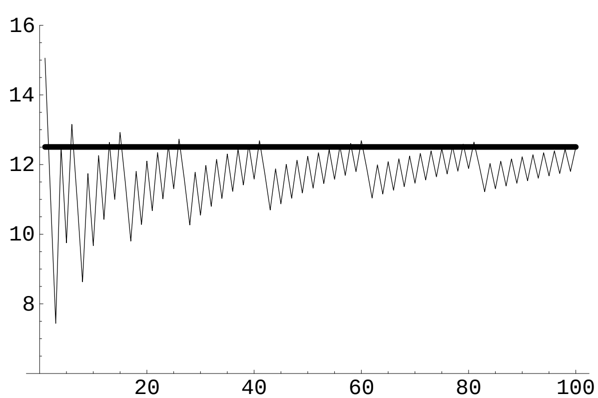
n	Combinatorial method		Trinomial tree algorithm	
	Value	Time	Value	Time
21	5.507548	0.30		
84	5.597597	0.90	5.634936	35.0
191	5.635415	2.00	5.655082	185.0
342	5.655812	3.60	5.658590	590.0
533	5.652253	5.60	5.659692	1440.0
768	5.654609	8.00	5.660137	3080.0
1047	5.658622	11.10	5.660338	5700.0
1368	5.659711	15.00	5.660432	9500.0
1731	5.659416	19.40	5.660474	15400.0
2138	5.660511	24.70	5.660491	23400.0
2587	5.660592	30.20	5.660493	34800.0
3078	5.660099	36.70	5.660488	48800.0
3613	5.660498	43.70	5.660478	67500.0
4190	5.660388	44.10	5.660466	92000.0
4809	5.659955	51.60	5.660454	130000.0
5472	5.660122	68.70		
6177	5.659981	76.70		

(All times in milliseconds.)

Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
- Assume $L < S < H$.
- The binomial model produces oscillating option values (see plot next page).[a]

[a] Chao (1999); Dai and Lyuu (2005);

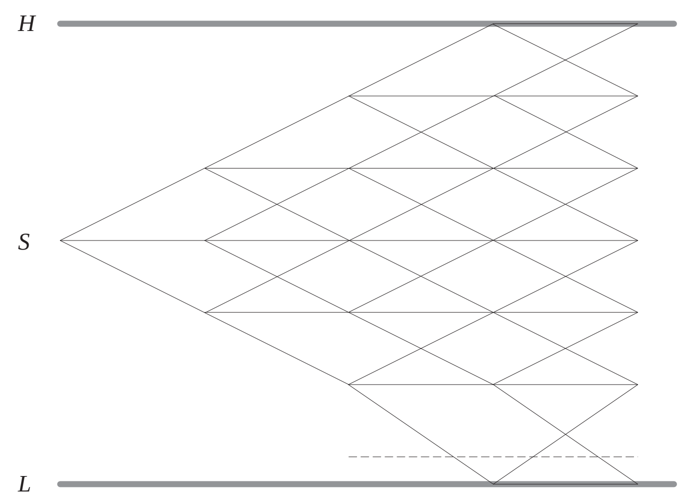


Double-Barrier Knock-Out Options

- We knew how to pick the λ so that one of the layers of the trinomial tree coincides with one barrier, say H.
- This choice, however, does not guarantee that the other barrier, L, is also hit.
- One way to handle this problem is to lower the layer of the tree just above L to coincide with L.[a]
  - More general ways to make the trinomial model hit both barriers are possible.[b]

[a] Ritchken (1995).

[b] Hsu and Lyuu (2006). Dai and Lyuu (2006) even combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options!



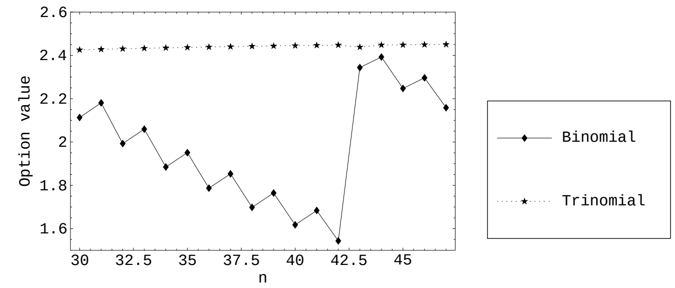
Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above L must be adjusted.
- Let ℓ be the positive integer such that
$$Sd^{\ell+1} < L < Sd^{\ell}.$$
- Hence the layer of the tree just above L has price Sd^{ℓ}.

Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying
$$L = Sd^{\ell-1} e^{-\gamma\lambda\sigma\sqrt{\Delta t}}.$$
  - The prices between the barriers are
$$L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H.$$
- The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are
$$p'_u = \frac{b + a\gamma}{1+\gamma}, \quad p'_d = \frac{b-a}{\gamma+\gamma^2}, \quad \text{and} \quad p'_m = 1 - p'_u - p'_d,$$
where $a \equiv \mu'\sqrt{\Delta t}/(\lambda\sigma)$ and $b \equiv 1/\lambda^2$.

Convergence: Binomial vs. Trinomial



Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on m assets has the terminal payoff $\max(\sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0)$, where α_i is the percentage of asset i.
- Basket options are essentially options on a portfolio of stocks or index options.
- Option on the best of two risky assets and cash has a terminal payoff of $\max(S_1(\tau), S_2(\tau), X)$.