Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As the number of periods increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.
- Any proper calibration of the model parameters makes the BOPM converge to the continuous-time model.
- We now skim through the proof.

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as n goes to infinity.
- First, $\hat{r} = r\tau/n$.
  - The period gross return $R = e^{\hat{r}}$.

Toward the Black-Scholes Formula (continued)

- Use
$$\widehat{\mu} \equiv \frac{1}{n} E\left[\ln \frac{S_\tau}{S}\right] \quad \text{and} \quad \widehat{\sigma}^2 \equiv \frac{1}{n} \mathrm{Var}\left[\ln \frac{S_\tau}{S}\right]$$
to denote, resp., the expected value and variance of the period continuously compounded rate of return.
- Under the BOPM, it is not hard to show that
$$\begin{aligned} \widehat{\mu} &= q\ln(u/d) + \ln d, \\ \widehat{\sigma}^2 &= q(1-q)\ln^2(u/d). \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- Assume the stock's true continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
  - Call σ the stock's (annualized) volatility.
- The BOPM converges to the distribution only if
$$\begin{aligned} n\widehat{\mu} &= n(q\ln(u/d) + \ln d) \to \mu\tau, \\ n\widehat{\sigma}^2 &= nq(1-q)\ln^2(u/d) \to \sigma^2\tau. \end{aligned}$$
- Impose $ud = 1$ to make nodes at the same horizontal level of the tree have identical price (review p. 227).
  - Other choices are possible (see text).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}. \tag{23}$$

- With Eqs. (23),

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right] \sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $u > R > d$ may not hold under Eqs. (23) on p. 235.
- If this happens, the risk-neutral probability may lie outside $[0, 1]$.
- The problem disappears when n satisfies

$$e^{\sigma\sqrt{\tau/n}} > e^{r\tau/n},$$

in other words, when $n > r^2\tau/\sigma^2$ (check it).
  - So it goes away if n is large enough.
  - Other solutions will be presented later.

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$?
- The central limit theorem says $\ln(S_\tau/S)$ converge to the normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$.
- So $\ln S_\tau$ approaches the normal distribution with mean $\mu\tau + \ln S$ and variance $\sigma^2\tau$.
- S_τ has a lognormal distribution in the limit.

Toward the Black-Scholes Formula (continued)

Lemma 7 *The continuously compounded rate of return $\ln(S_\tau/S)$ approaches the normal distribution with mean $(r - \sigma^2/2)\,\tau$ and variance $\sigma^2\tau$ in a risk-neutral economy.*

- Let q equal the risk-neutral probability $p \equiv (e^{r\tau/n} - d)/(u - d)$.
- Let $n \to \infty$.

Toward the Black-Scholes Formula (continued)

- By Lemma 7 (p. 238) and Eq. (17) on p. 145, the expected stock price at expiration in a risk-neutral economy is $Se^{r\tau}$.
- The stock's expected annual rate of return[a] is thus the riskless rate r.

[a] In the sense of $(1/\tau) \ln E[\, S_\tau / S\,]$ not $(1/\tau) E[\ln(S_{\tau/S})\,]$.

Toward the Black-Scholes Formula (concluded)

Theorem 8 (The Black-Scholes Formula)

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x), \end{aligned}$$

where

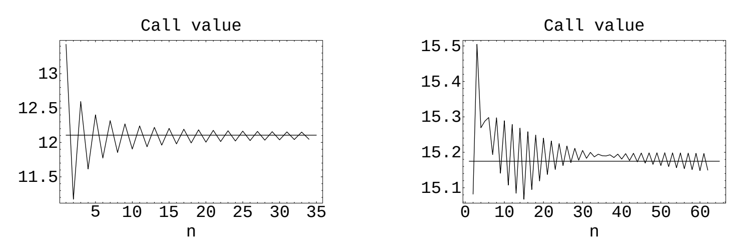
$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

BOPM and Black-Scholes Model

- The Black-Scholes formula needs five parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take six inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$u = e^{\sigma\sqrt{\tau/n}},\ d = e^{-\sigma\sqrt{\tau/n}},\ \hat{r} = r\tau/n.$$

- The binomial tree algorithms converge reasonably fast.
- Oscillations can be dealt with by the judicious choices of u and d (see text).



$S = 100$, $X = 100$ (left), and $X = 95$ (right).

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?
- This volatility is called the implied volatility.
- Implied volatility is often preferred to historical volatility in practice.[a]

[a] It is like driving a car with your eyes on the rearview mirror?

Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.

Problems; the Smile (concluded)

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But σ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.[a]
- How to incorporate these two different ways of day count into the Black-Scholes formula and binomial tree algorithms?

[a] Fama (1965); French (1980); French and Roll (1986).

Trading Days and Calendar Days (concluded)

- Suppose a year has 260 trading days.
- A quick and dirty way is to replace σ with[a]

$$\sigma \sqrt{\frac{365}{260} \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}.$$

- How about binomial tree algorithms?

[a] French (1984).

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs

$$\max(0, X - Su^j d^{n-j})$$

and applies backward induction.

- At each intermediate node, it checks for early exercise by comparing the payoff if exercised with the continuation value.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the prevailing stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\, u$, $(Su - D)\, d$, $(Sd - D)\, u$, $(Sd - D)\, d$.
  - The binomial tree no longer combines.

$$
\begin{array}{ccccc}
 & & & & (Su-D)\,u \\
 & & & \nearrow & \\
 & & Su-D & & \\
 & \nearrow & & \searrow & \\
 & & & & (Su-D)\,d \\
S & & & & \\
 & & & & (Sd-D)\,u \\
 & \searrow & & \nearrow & \\
 & & Sd-D & & \\
 & & & \searrow & \\
 & & & & (Sd-D)\,d
\end{array}
$$

An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends (Roll, 1977).
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - σ equal to the volatility of the process followed by the risky component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all future dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

An Uncompromising Approach[a]

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases.

[a] Dai and Lyuu (2004).

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.

Continuous Dividend Yields (continued)

- The Black-Scholes formulas hold with S replaced by $Se^{-q\tau}$ (Merton, 1973):

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{24}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{24'}$$

where

$$x \equiv \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (24) and (24') remain valid as long as the dividend yield is predictable.
- Replace q with the average annualized dividend yield.

Continuous Dividend Yields (concluded)

- To run binomial tree algorithms, pick the risk-neutral probability as

$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \tag{25}$$

where $\Delta t \equiv \tau/n$.
  - Because the stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- The u and d remain unchanged.
- Other than the change in Eq. (25), binomial tree algorithms stay the same.

Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
— Blaise Pascal (1623–1662)

Sensitivity Measures ("The Greeks")

- Understanding how the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- Let $x \equiv \frac{\ln(S/X)+(r+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$ (recall p. 240).
- Note that

$$N'(y) = (1/\sqrt{2\pi}\,)\, e^{-y^2/2} > 0,$$

the density function of standard normal distribution.

Delta

- Defined as $\Delta \equiv \partial f / \partial S$.
  - f is the price of the derivative.
  - S is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.
- The delta used in the BOPM is the discrete analog.

Delta (concluded)

- The delta of a European call on a non-dividend-paying stock equals

$$\frac{\partial C}{\partial S} = N(x) > 0.$$

- The delta of a European put equals

$$\frac{\partial P}{\partial S} = N(x) - 1 < 0.$$

- The delta of a long stock is 1.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
- A delta-neutral portfolio is immune to small price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  - Short Δ shares of stock to hedge a long call.
- In general, hedge a position in a security with a delta of Δ_1 by shorting Δ_1/Δ_2 units of a security with a delta of Δ_2.

Theta (Time Decay)

- Defined as the rate of change of a security's value with respect to time, or $\Theta \equiv -\partial f/\partial\tau$.
- For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x-\sigma\sqrt{\tau}) < 0.$$

  - The call loses value with the passage of time.
- For a European put,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x+\sigma\sqrt{\tau}).$$

  - Can be negative or positive.

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \equiv \partial^2\Pi/\partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- A portfolio with a high gamma needs in practice be rebalanced more often to maintain delta neutrality.
- Delta $\sim$ duration; gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$

Vega[a] (Lambda, Kappa, Sigma)

- Defined as the rate of change of its value with respect to the volatility of the underlying asset $\Lambda \equiv \partial\Pi/\partial\sigma$.
- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\,N'(x) > 0$.
  - Higher volatility increases option value.

[a] Vega is not Greek.

Rho

- Defined as the rate of change in its value with respect to interest rates $\rho \equiv \partial \Pi / \partial r$.
- The rho of a European call on a non-dividend-paying stock is
$$X \tau e^{-r\tau} N(x - \sigma\sqrt{\tau}) > 0.$$
- The rho of a European put on a non-dividend-paying stock is
$$-X \tau e^{-r\tau} N(-x + \sigma\sqrt{\tau}) < 0.$$

Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,
$$\frac{f(S+\Delta S) - f(S-\Delta S)}{2\Delta S}.$$
- The computation time roughly doubles that for evaluating the derivative security itself.

An Alternative Numerical Delta[a]

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, f_u and f_d are computed.
- These values correspond to derivative values at stock prices Su and Sd, respectively.
- Delta is approximated by
$$\frac{f_u - f_d}{Su - Sd}.$$
- Almost zero extra computational effort.

[a]Pelsser and Vorst (1994).

Numerical Gamma

- At the stock price $(Suu + Sud)/2$, delta is approximately $(f_{uu} - f_{ud})/(Suu - Sud)$.
- At the stock price $(Sud + Sdd)/2$, delta is approximately $(f_{ud} - f_{dd})/(Sud - Sdd)$.
- Gamma is the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,
$$\frac{\frac{f_{uu}-f_{ud}}{Suu-Sud} - \frac{f_{ud}-f_{dd}}{Sud-Sdd}}{(Suu - Sdd)/2}.$$

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

$$\frac{f(S+\Delta S)-2f(S)+f(S-\Delta S)}{(\Delta S)^2}.$$

- It does not work (see text).
- Why did the binomial tree version work?

Other Numerical Greeks

- The theta can be computed as

$$\frac{f_{ud}-f}{2(\tau/n)}.$$

  – In fact, the theta of a European option will be shown to be computable from delta and gamma (see p. 485).
- For vega and rho, there is no alternative but to run the binomial tree algorithm twice.