Finesse

- Equations (7) on p. 77 and (8) on p. 78 hold only if the coupon C, the par value F, and the maturity n are all independent of the yield y.
  - That is, if the cash flow is independent of yields.
- To see this point, suppose the market yield declines.
- The MD will be lengthened.
- But for securities whose maturity actually decreases as a result, the MD may actually decrease.

How Not To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But you use it that way at your peril.
- The MD should be seen mainly as measuring price volatility.
- Many, if not most, duration-related terminology cannot be comprehended otherwise.

Conversion

- For the MD to be year-based, modify Eq. (8) on p. 78 to
$$\frac{1}{P}\left[\sum_{i=1}^{n}\frac{i}{k}\frac{C}{\left(1+\frac{y}{k}\right)^{i}}+\frac{n}{k}\frac{F}{\left(1+\frac{y}{k}\right)^{n}}\right],$$
where y is the *annual* yield and k is the compounding frequency per annum.
- Equation (7) on p. 77 also becomes
$$\text{MD}=-\left(1+\frac{y}{k}\right)\frac{\partial P}{\partial y}\frac{1}{P}.$$
- By definition, MD (in years) $= \frac{\text{MD (in periods)}}{k}$.

Modified Duration

- Modified duration is defined as
$$\text{modified duration} \equiv -\frac{\partial P}{\partial y}\frac{1}{P}=\frac{\text{MD}}{(1+y)}. \tag{9}$$
- By Taylor expansion,
percent price change $\approx$ $-$modified duration $\times$ yield change.

Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

Modified Duration of a Portfolio

- The modified duration of a portfolio equals

$$\sum_i \omega_i D_i.$$

  - D_i is the modified duration of the ith asset.
  - ω_i is the market value of that asset expressed as a percentage of the market value of the portfolio.

Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

$$\frac{P_- - P_+}{P_0(y_+ - y_-)}.$$

  - P_- is the price if the yield is decreased by Δy.
  - P_+ is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.

Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use

$$\frac{P_0 - P_+}{P_0 \, \Delta y}.$$

  - More economical but less accurate.

Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as

$$\text{modified duration} \times \text{price } (\% \text{ of par}) = -\frac{\partial P}{\partial y}.$$

- The approximate dollar price change per \$100 of par value is

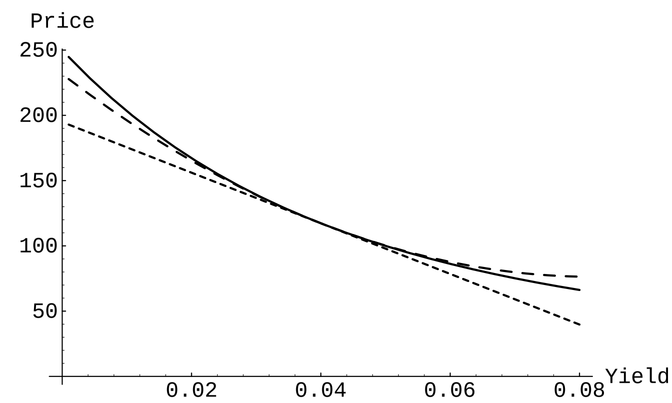
$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$

Convexity

- Convexity is defined as

$$\text{convexity (in periods)} \equiv \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.$$

- The convexity of a coupon bond is positive (prove it!).
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot on the next page).
- Hence, between two bonds with the same duration, the one with a higher convexity is more valuable.



Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}$$

when there are k periods per annum.

Use of Convexity

- The approximation $\Delta P/P \approx -$ duration $\times$ yield change works for small yield changes.
- To improve upon it for larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y}\frac{1}{P}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\frac{1}{P}(\Delta y)^2$$
$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$

- Recall the figure on p. 89.

Effective Convexity

- The effective convexity is defined as

$$\frac{P_+ + P_- - 2P_0}{P_0\,(0.5 \times (y_+ - y_-))^2},$$

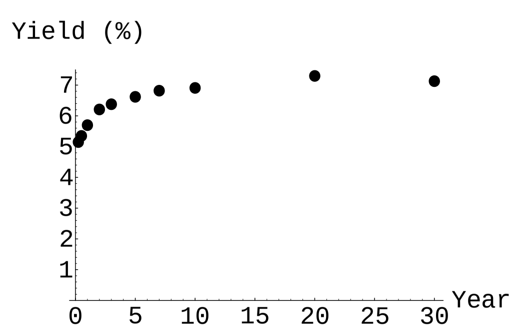
  - P_- is the price if the yield is decreased by Δy.
  - P_+ is the price if the yield is increased by Δy.
  - P_0 is the initial price, y is the initial yield.
  - Δy is small.
- Effective convexity is most relevant when a bond's cash flow is interest rate sensitive.
- Numerically, choosing the right Δy is a delicate matter.

Term Structure of Interest Rates

Why is it that the interest of money is lower,
when money is plentiful?
— Samuel Johnson (1709–1784)

Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds forms the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.



Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots yields to maturity against maturity.
- A par yield curve is constructed from bonds trading near par.

Four Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

Spot Rates

- The i-period spot rate $S(i)$ is the yield to maturity of an i-period zero-coupon bond.
- The PV of one dollar i periods from now is

$$[\,1 + S(i)\,]^{-i}.$$

- The one-period spot rate is called the short rate.
- A spot rate curve is a plot of spot rates against maturity.

Problems with the PV Formula

- In the bond price formula,

$$\sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n},$$

every cash flow is discounted at the same yield y.

- Consider two riskless bonds with different yields to maturity because of their different cash flow streams.
- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.
- But shouldn't they be discounted at the same rate?

Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the ith bond paying C_i dollars at time i.
- So a level-coupon bond has the price

$$P = \sum_{i=1}^{n} \frac{C}{[\,1+S(i)\,]^i} + \frac{F}{[\,1+S(n)\,]^n}. \tag{10}$$

- This pricing method incorporates information from the term structure.
- Discount each cash flow at the corresponding spot rate.

Discount Factors

- In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

  - Above, $d(i) \equiv [\,1 + S(i)\,]^{-i}$, $i = 1, 2, \ldots, n$, are called discount factors.
  - $d(i)$ is the PV of one dollar i periods from now.
- The discount factors are often interpolated to form a continuous function called the discount function.

Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.
- Compute $S(2)$ from the two-period coupon bond price P by solving

$$P = \frac{C}{1+S(1)} + \frac{C+100}{[\,1+S(2)\,]^2}.$$

Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price P of the n-period coupon bond and $S(1), S(2), \ldots, S(n-1)$.
- Then $S(n)$ can be computed from Eq. (10), repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[\,1+S(i)\,]^i} + \frac{F}{[\,1+S(n)\,]^n}.$$

- The running time is $O(n)$.
- The procedure is called bootstrapping.

Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.
  - Lack economic justifications.

Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for P.
- Were this bond riskless, it would fetch

$$P^* = \sum_{t=1}^{n} \frac{C_t}{[\,1+S(t)\,]^t}.$$

- Since riskiness must be compensated, $P < P^*$.
- Yield spread is the difference between the IRR of the risky bond and that of a riskless bond with comparable maturity.

Static Spread

- The static spread is the amount s by which the spot rate curve has to shift in parallel in order to price the risky bond correctly,

$$P = \sum_{t=1}^{n} \frac{C_t}{[\,1 + s + S(t)\,]^t}.$$

- Unlike the yield spread, the static spread incorporates information from the term structure.

Of Spot Rate Curve and Yield Curve

- y_k: yield to maturity for the k-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

Coupon Effect on the Yield to Maturity

- Under a normal spot rate curve, a coupon bond has a lower yield than a zero-coupon bond of equal maturity.
- Picking a zero-coupon bond over a coupon bond based purely on the zero's higher yield to maturity is flawed.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But this is only a trend not a mathematical truth.
- See a counterexample in the textbook.

Forward Rates

- The yield curve contains information regarding future interest rates currently "expected" by the market.
- Invest \$1 for j periods to end up with $[1+S(j)]^j$ dollars at time j.
  - The maturity strategy.
- Invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j-i$ periods where $j>i$.
- Will have $[1+S(i)]^i[1+S(i,j)]^{j-i}$ dollars at time j.
  - $S(i,j)$: $(j-i)$-period spot rate i periods from now.
  - The rollover strategy.

Forward Rates (concluded)

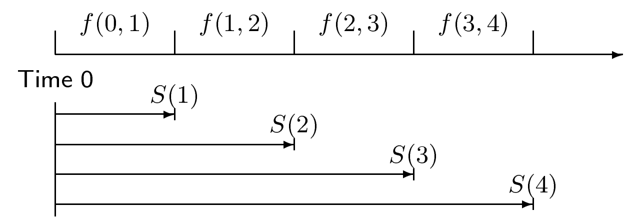
- When $S(i,j)$ equals

$$f(i,j) \equiv \left[\frac{(1+S(j))^j}{(1+S(i))^i}\right]^{1/(j-i)} - 1, \qquad (11)$$

we will end up with $[1+S(j)]^j$ dollars again.

- By definition, $f(0,j)=S(j)$.
- $f(i,j)$ is called the (implied) forward rates.
  - More precisely, the $(j-i)$-period forward rate i periods from now.

Time Line



Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i,j)$ and future spot rate $S(i,j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate two investment strategies.
- $f(i,i+1)$ are instantaneous forward rates or one-period forward rates.

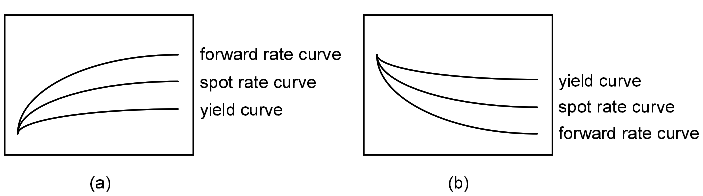
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i,j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i,j) < S(j) < \cdots < S(i).$$



Forward Rates=Spot Rates=Yield Curve

- The FV of \$1 at time n can be derived in two ways.
- Buy n-period zero-coupon bonds and receive $[\,1+S(n)\,]^n$.
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is $[\,1+S(1)\,][\,1+f(1,2)\,]\cdots[\,1+f(n-1,n)\,]$.

Forward Rates=Spot Rates=Yield Curve (concluded)

- Since they are identical,

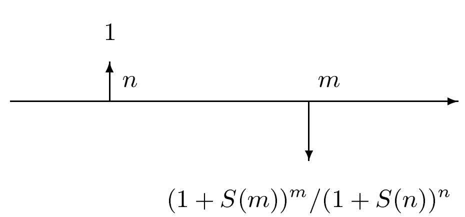
$$S(n) = ((1+S(1))(1+f(1,2)) \cdots (1+f(n-1,n)))^{1/n} - 1. \quad (12)$$

- Hence, the forward rates, specifically the one-period forward rates, determine the spot rate curve.
- Other equivalency can be derived similarly, such as

$$f(T,T+1) = d(T)/d(T+1) - 1.$$

Locking in the Forward Rate $f(n, m)$

- Buy one n-period zero-coupon bond for $1/(1+S(n))^n$.
- Sell $(1+S(m))^m/(1+S(n))^n$ m-period zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow $1/(1+S(n))^n$.
- At time n there will be a cash inflow of $1.
- At time m there will be a cash outflow of $(1+S(m))^m/(1+S(n))^n$ dollars.
- This implies the rate $f(n, m)$ between times n and m.



Forward Contracts

- We generated the cash flow of a financial instrument called forward contract.
- Agreed upon today, it enables one to borrow money at time n in the future and repay the loan at time $m > n$ with an interest rate equal to the forward rate $f(n, m)$.
- Can the spot rate curve be an arbitrary curve?[a]

[a] Contributed by Mr. Dai, Tian-Shyr (R86526008, D8852600) in 1998.

Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} C e^{-iS(i)} + F e^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

Spot and Forward Rates under Continuous Compounding (concluded)

- The formula for the forward rate:

$$f(i,j) = \frac{jS(j) - iS(i)}{j - i}.$$

- The one-period forward rate:

$$f(j, j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

- $$f(T) \equiv \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T\,\frac{\partial S}{\partial T}.$$

- $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$.