Time Series Analysis

The historian is a prophet in reverse.
— Friedrich von Schlegel (1772–1829)

Conditional Variance Models for Price Volatility

- Although a stationary model (see text for definition) has constant variance, its *conditional* variance may vary.
- Take for example an AR(1) process $X_t = aX_{t-1} + \epsilon_t$ with $|\,a\,| < 1$.
  - Here, ϵ_t is a stationary, uncorrelated process with zero mean and constant variance σ^2.
- The conditional variance,

$$\mathrm{Var}[\,X_t \,|\, X_{t-1}, X_{t-2}, \ldots\,],$$

equals σ^2, which is smaller than the *unconditional* variance $\mathrm{Var}[\,X_t\,] = \sigma^2/(1-a^2)$.

Conditional Variance Models for Price Volatility (continued)

- Past information thus has no effects on the variance of prediction.
- To address this drawback, consider models for returns X_t consistent with a changing conditional variance:

$$X_t - \mu = V_t U_t.$$

  - U_t has zero mean and unit variance for all t.
  - $E[\,X_t\,] = \mu$ for all t.
  - $\mathrm{Var}[\,X_t \,|\, V_t = v_t\,] = v_t^2$.

Conditional Variance Models for Price Volatility (continued)

- The process $\{ V_t^2 \}$ models the conditional variance.
- Suppose $\{ U_t \}$ and $\{ V_t \}$ are independent of each other, which means $\{ U_1, U_2, \ldots, U_n \}$ and $\{ V_1, V_2, \ldots, V_n \}$ are independent for all n.
- Then $\{ X_t \}$ is uncorrelated because

$$\mathrm{Cov}[\, X_t, X_{t+\tau} \,] = 0 \tag{77}$$

for $\tau > 0$ (see text for proof).

Conditional Variance Models for Price Volatility (continued)

- If, furthermore, $\{ V_t \}$ is stationary, then $\{ X_t \}$ has constant variance because

$$\begin{aligned} & E\left[(X_t - \mu)^2 \right] \\ = \; & E\left[V_t^2 U_t^2 \right] \\ = \; & E\left[V_t^2 \right] E\left[U_t^2 \right] \\ = \; & E\left[V_t^2 \right]. \end{aligned}$$

- This makes $\{ X_t \}$ stationary.

Conditional Variance Models for Price Volatility (concluded)

- In the lognormal model, the conditional variance evolves independently of past returns.
- Suppose we assume that conditional variances are deterministic functions of past returns:

$$V_t = f(X_{t-1}, X_{t-2}, \ldots)$$

for some function f.
- Then V_t can be computed given the information set of past returns:

$$I_{t-1} \equiv \{ X_{t-1}, X_{t-2}, \ldots \}.$$

ARCH Models[a]

- An influential model in this direction is the autoregressive conditional heteroskedastic (ARCH) model.
- Assume U_t is independent of $V_t, U_{t-1}, V_{t-1}, U_{t-2}, \ldots$ for all t.
- Consequently $\{ X_t \}$ is uncorrelated by Eq. (77) on p. 712.
- Assume furthermore that $\{ U_t \}$ is a Gaussian stationary, uncorrelated process.
- Then $X_t \,|\, I_{t-1} \sim N(\mu, V_t^2)$.

[a] Engle (1982), co-winner of the 2003 Nobel Prize in Economic Sciences.

ARCH Models (continued)

- The ARCH(p) process is defined by

$$X_t - \mu = \left(a_0 + \sum_{i=1}^{p} a_i (X_{t-i} - \mu)^2 \right)^{1/2} U_t,$$

where $a_1, \dots, a_p \geq 0$ and $a_0 > 0$.

- The variance V_t^2 thus satisfies

$$V_t^2 = a_0 + \sum_{i=1}^{p} a_i (X_{t-i} - \mu)^2.$$

- The volatility at time t as estimated at time $t-1$ depends on the p most recent observations on squared returns.

ARCH Models (concluded)

- The ARCH(1) process

$$X_t - \mu = (a_0 + a_1 (X_{t-1} - \mu)^2)^{1/2} U_t$$

is the simplest.

- For it,

$$\text{Var}[\, X_t \mid X_{t-1} = x_{t-1} \,] = a_0 + a_1 (x_{t-1} - \mu)^2.$$

- The process $\{ X_t \}$ is stationary with finite variance if and only if $a_1 < 1$, in which case $\text{Var}[\, X_t \,] = a_0/(1 - a_1)$.
- The parameters can be estimated by statistical techniques.

GARCH Models[a]

- A very popular extension of the ARCH model is the generalized autoregressive conditional heteroskedastic (GARCH) process.
- The simplest GARCH$(1,1)$ process adds $a_2 V_{t-1}^2$ to the ARCH(1) process, resulting in

$$V_t^2 = a_0 + a_1 (X_{t-1} - \mu)^2 + a_2 V_{t-1}^2.$$

- The volatility at time t as estimated at time $t-1$ depends on the squared return and the estimated volatility at time $t-1$.

[a]Bollerslev (1986) and Taylor (1986).

GARCH Models (concluded)

- The estimate of volatility averages past squared returns by giving heavier weights to recent squared returns (see text).
- It is usually assumed that $a_1 + a_2 < 1$ and $a_0 > 0$, in which case the unconditional, long-run variance is given by $a_0/(1 - a_1 - a_2)$.
- A popular special case of GARCH$(1,1)$ is the exponentially weighted moving average process, which sets a_0 to zero and a_2 to $1 - a_1$.
- This model is used in J.P. Morgan's RiskMetrics™.

GARCH Option Pricing

- Options can be priced when the underlying asset's return follows a GARCH process.
- Let S_t denote the asset price at date t.
- Let h_t^2 be the conditional variance of the return over the period $[\,t, t+1\,]$ given the information at date t.
  - "One day" is merely a convenient term for any elapsed time Δt.

GARCH Option Pricing (continued)

- Adopt the following risk-neutral process for the price dynamics:[a]

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{h_t^2}{2} + h_t \epsilon_{t+1}, \tag{78}$$

where

$$\begin{aligned} h_{t+1}^2 &= \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 (\epsilon_{t+1} - c)^2, &(79)\\ \epsilon_{t+1} &\sim N(0,1) \text{ given information at date } t, \\ r &= \text{daily riskless return}, \\ c &\geq 0. \end{aligned}$$

[a]Duan (1995).

GARCH Option Pricing (continued)

- The five unknown parameters of the model are c, h_0, β_0, β_1, and β_2.
- It is postulated that $\beta_0, \beta_1, \beta_2 \geq 0$ to make the conditional variance positive.
- The above process, called the nonlinear asymmetric GARCH model, generalizes the GARCH$(1,1)$ model (see text).

GARCH Option Pricing (concluded)

- With $y_t \equiv \ln S_t$ denoting the logarithmic price, the model becomes

$$y_{t+1} = y_t + r - \frac{h_t^2}{2} + h_t \epsilon_{t+1}. \tag{80}$$

- The pair (y_t, h_t^2) completely describes the current state.
- The conditional mean and variance of y_{t+1} are clearly

$$\begin{aligned} E[\,y_{t+1} \,|\, y_t, h_t^2\,] &= y_t + r - \frac{h_t^2}{2}, &(81)\\ \text{Var}[\,y_{t+1} \,|\, y_t, h_t^2\,] &= h_t^2. &(82) \end{aligned}$$

The Ritchken-Trevor (RT) Algorithm[a]

- The GARCH model is a continuous-state model.
- To approximate it, we turn to trees with *discrete* states.
- Path dependence in GARCH makes the tree for asset prices explode exponentially.
- We need to mitigate this combinatorial explosion somewhat.

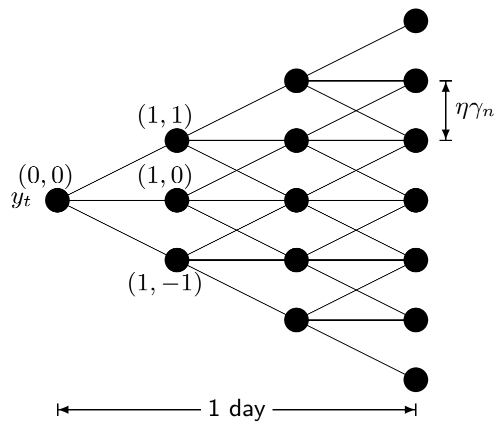
[a]Ritchken and Trevor (1999).

The Ritchken-Trevor Algorithm (continued)

- Partition a day into n periods.
- Three states follow each state (y_t, h_t^2) after a period.
- As the trinomial model combines, $2n+1$ states at date $t+1$ follow each state at date t (recall p. 550).
- These $2n+1$ values must approximate the distribution of (y_{t+1}, h_{t+1}^2).
- So the conditional moments (81)–(82) at date $t+1$ on p. 723 must be matched by the trinomial model to guarantee convergence to the continuous-state model.

The Ritchken-Trevor Algorithm (continued)

- It remains to pick the jump size and the three branching probabilities.
- The role of σ in the Black-Scholes option pricing model is played by h_t in the GARCH model.
- As a jump size proportional to $\sigma/\sqrt{n}$ is picked in the BOPM, a comparable magnitude will be chosen here.
- Define $\gamma \equiv h_0$, though other multiples of h_0 are possible, and
$$\gamma_n \equiv \frac{\gamma}{\sqrt{n}}.$$
- The jump size will be some integer multiple η of γ_n.
- We call η the jump parameter (p. 727).



The seven values on the right approximate the distribution of logarithmic price y_{t+1}.

The Ritchken-Trevor Algorithm (continued)

- The middle branch does not change the underlying asset's price.
- The probabilities for the up, middle, and down branches are

$$p_u = \frac{h_t^2}{2\eta^2\gamma^2} + \frac{r - (h_t^2/2)}{2\eta\gamma\sqrt{n}}, \tag{83}$$

$$p_m = 1 - \frac{h_t^2}{\eta^2\gamma^2}, \tag{84}$$

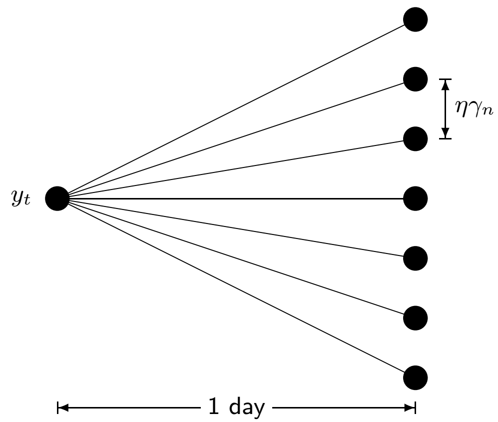
$$p_d = \frac{h_t^2}{2\eta^2\gamma^2} - \frac{r - (h_t^2/2)}{2\eta\gamma\sqrt{n}}. \tag{85}$$

The Ritchken-Trevor Algorithm (continued)

- It can be shown that:
  - The trinomial model takes on $2n+1$ values at date $t+1$ for y_{t+1}.
  - These values have a matching mean for y_{t+1}.
  - These values have an asymptotically matching variance for y_{t+1}.
- The central limit theorem thus guarantees the desired convergence as n increases.

The Ritchken-Trevor Algorithm (continued)

- We can dispense with the intermediate nodes *between* dates to create a $(2n+1)$-nomial tree (p. 731).
- The resulting model is multinomial with $2n+1$ branches from any state (y_t, h_t^2).
- There are two reasons behind this manipulation.
  - Interdate nodes are created merely to approximate the continuous-state model after one day.
  - Keeping the interdate nodes results in a tree that is n times as large.



This heptanomial tree is the outcome of the trinomial tree on p. 727 after its intermediate nodes are removed.

The Ritchken-Trevor Algorithm (continued)

- A node with logarithmic price $y_t + \ell\eta\gamma_n$ at date $t+1$ follows the current node at date t with price y_t for some $-n \leq \ell \leq n$.
- To reach that price in n periods, the number of up moves must exceed that of down moves by exactly ℓ.
- The probability that this happens is

$$P(\ell) \equiv \sum_{j_u, j_m, j_d} \frac{n!}{j_u!\, j_m!\, j_d!}\, p_u^{j_u} p_m^{j_m} p_d^{j_d},$$

with $j_u, j_m, j_d \geq 0$, $n = j_u + j_m + j_d$, and $\ell = j_u - j_d$.

The Ritchken-Trevor Algorithm (continued)

- A particularly simple way to calculate the $P(\ell)$s starts by noting that

$$(p_u x + p_m + p_d x^{-1})^n = \sum_{\ell=-n}^{n} P(\ell)\, x^{\ell}. \tag{86}$$

- So we expand $(p_u x + p_m + p_d x^{-1})^n$ and retrieve the probabilities by reading off the coefficients.
- It can be computed in $O(n^2)$ time.

The Ritchken-Trevor Algorithm (continued)

- The updating rule (79) on p. 721 must be modified to account for the adoption of the discrete-state model.
- The logarithmic price $y_t + \ell\eta\gamma_n$ at date $t+1$ following state (y_t, h_t^2) at date t has a variance equal to

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 (\epsilon'_{t+1} - c)^2, \tag{87}$$

  – Above,

$$\epsilon'_{t+1} = \frac{\ell\eta\gamma_n - (r - h_t^2/2)}{h_t}, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm n,$$

  is a discrete random variable with $2n+1$ values.

The Ritchken-Trevor Algorithm (continued)

- Different conditional variances h_t^2 may require different η so that the probabilities calculated by Eqs. (83)–(85) on p. 728 lie between 0 and 1.
- This implies varying jump sizes.
- The necessary requirement $p_m \geq 0$ implies $\eta \geq h_t/\gamma$.
- Hence we try

$$\eta = \lceil h_t/\gamma \rceil, \lceil h_t/\gamma \rceil + 1, \lceil h_t/\gamma \rceil + 2, \ldots$$

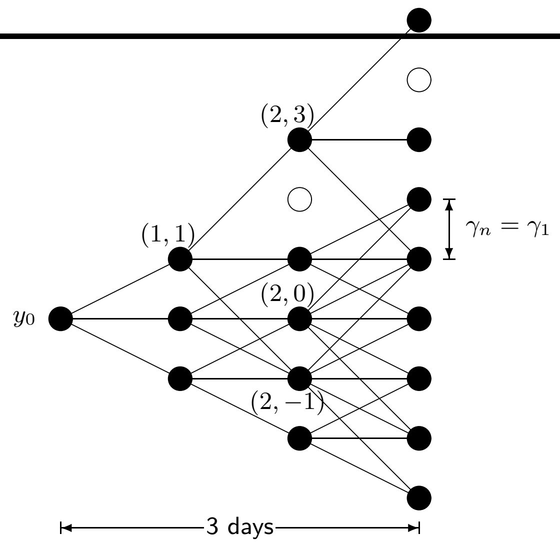
until valid probabilities are obtained or until their nonexistence is confirmed.

The Ritchken-Trevor Algorithm (continued)

- The sufficient and necessary condition for valid probabilities to exist is

$$\frac{|\, r-(h_t^2/2)\,|}{2\eta\gamma\sqrt{n}} \leq \frac{h_t^2}{2\eta^2\gamma^2} \leq \min\left(1-\frac{|\, r-(h_t^2/2)\,|}{2\eta\gamma\sqrt{n}}, \frac{1}{2}\right).$$

- Obviously, the magnitude of η tends to grow with h_t.
- The plot on p. 737 uses $n=1$ to illustrate our points for a 3-day model.
- For example, node $(1,1)$ of date 1 and node $(2,3)$ of date 2 pick $\eta=2$.



The Ritchken-Trevor Algorithm (continued)

- The topology of the tree is not a standard combining multinomial tree.
- For example, a few nodes on p. 737 such as nodes $(2,0)$ and $(2,-1)$ have multiple jump sizes.
- The reason is the path dependence of the model.
  - Two paths can reach node $(2,0)$ from the root node, each with a different variance for the node.
  - One of the variances results in $\eta=1$, whereas the other results in $\eta=2$.

The Ritchken-Trevor Algorithm (concluded)

- The possible values of h_t^2 at a node are exponential nature.
- To address this problem, we record only the maximum and minimum h_t^2 at each node.[a]
- Therefore, each node on the tree contains only two states $(y_t, h_{\max}^2)$ and $(y_t, h_{\min}^2)$.
- Each of $(y_t, h_{\max}^2)$ and $(y_t, h_{\min}^2)$ carries its own η and set of $2n+1$ branching probabilities.

[a]Cakici and Topyan (2000).

Negative Aspects of the Ritchken-Trevor Algorithm[a]

- A small n may yield inaccurate option prices.
- But the tree will grow exponentially if n is large enough.
  - Specifically, $n > (1 - \beta_1)/\beta_2$ when $r = c = 0$.
- A large n has another serious problem: The tree cannot grow beyond a certain date.
- Thus the choice of n may be limited in practice.
- The RT algorithm can be modified to be free of exponential complexity and shortened maturity.[b]

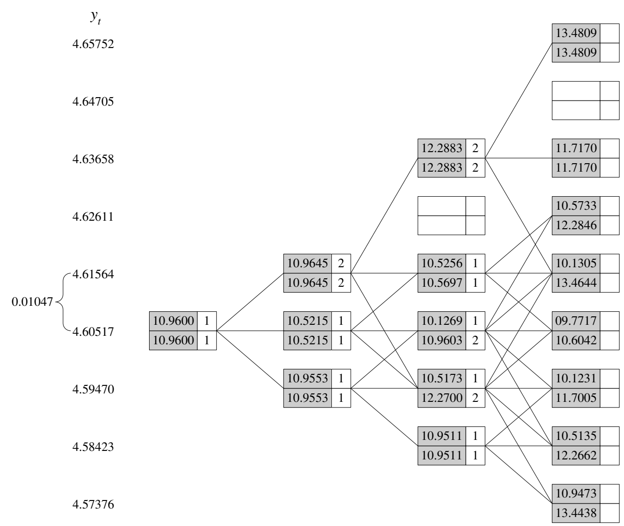
[a] Lyuu and Wu (2003).
[b] Lyuu and Wu (2005).

Numerical Examples

- Assume $S_0 = 100$, $y_0 = \ln S_0 = 4.60517$, $r = 0$, $h_0^2 = 0.0001096$, $\gamma = h_0 = 0.010469$, $n = 1$, $\gamma_n = \gamma/\sqrt{n} = 0.010469$, $\beta_0 = 0.000006575$, $\beta_1 = 0.9$, $\beta_2 = 0.04$, and $c = 0$.
- A daily variance of 0.0001096 corresponds to an annual volatility of $\sqrt{365 \times 0.0001096} \approx 20\%$.
- Let $h^2(i, j)$ denote the variance at node (i, j).
- Initially, $h^2(0, 0) = h_0^2 = 0.0001096$.

Numerical Examples (continued)

- Let $h^2_{\max}(i, j)$ denote the maximum variance at node (i, j).
- Let $h^2_{\min}(i, j)$ denote the minimum variance at node (i, j).
- Initially, $h^2_{\max}(0, 0) = h^2_{\min}(0, 0) = h_0^2$.
- The resulting three-day tree is depicted on p. 743.



A top (bottom) number inside a gray box refers to the minimum (maximum, respectively) variance $h^2_{\min}$ ($h^2_{\max}$, respectively) for the node. Variances are multiplied by 100,000 for readability. A top (bottom) number inside a white box refers to η corresponding to $h^2_{\min}$ ($h^2_{\max}$, respectively).

Numerical Examples (continued)

- Let us see how the numbers are calculated.
- Start with the root node, node $(0, 0)$.
- Try $\eta = 1$ in Eqs. (83)–(85) on p. 728 first to obtain

$$\begin{aligned} p_u &= 0.4974, \\ p_m &= 0, \\ p_d &= 0.5026. \end{aligned}$$

- As they are valid probabilities, the three branches from the root node use single jumps.

Numerical Examples (continued)

- Move on to node $(1, 1)$.
- It has one predecessor node—node $(0, 0)$—and it takes an up move to reach the current node.
- So apply updating rule (87) on p. 734 with $\ell = 1$ and $h_t^2 = h^2(0, 0)$.
- The result is $h^2(1, 1) = 0.000109645$.

Numerical Examples (continued)

- Because $\lfloor h(1, 1)/\gamma \rfloor = 2$, we try $\eta = 2$ in Eqs. (83)–(85) on p. 728 first to obtain

$$\begin{aligned} p_u &= 0.1237, \\ p_m &= 0.7499, \\ p_d &= 0.1264. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(1, 1)$ use double jumps.

Numerical Examples (continued)

- Carry out similar calculations for node $(1,0)$ with $\ell = 0$ in updating rule (87) on p. 734.
- Carry out similar calculations for node $(1,-1)$ with $\ell = -1$ in updating rule (87).
- Single jump $\eta = 1$ works in both nodes.
- The resulting variances are

$$\begin{aligned} h^2(1,0) &= 0.000105215, \\ h^2(1,-1) &= 0.000109553. \end{aligned}$$

Numerical Examples (continued)

- Node $(2,0)$ has 2 predecessor nodes, $(1,0)$ and $(1,-1)$.
- Both have to be considered in deriving the variances.
- Let us start with node $(1,0)$.
- Because it takes a middle move to reach the current node, we apply updating rule (87) on p. 734 with $\ell = 0$ and $h_t^2 = h^2(1,0)$.
- The result is $h_{t+1}^2 = 0.000101269$.

Numerical Examples (continued)

- Now move on to the other predecessor node $(1,-1)$.
- Because it takes an up move to reach the current node, apply updating rule (87) on p. 734 with $\ell = 1$ and $h_t^2 = h^2(1,-1)$.
- The result is $h_{t+1}^2 = 0.000109603$.
- We hence record

$$\begin{aligned} h^2_{\min}(2,0) &= 0.000101269, \\ h^2_{\max}(2,0) &= 0.000109603. \end{aligned}$$

Numerical Examples (continued)

- Consider state $h^2_{\max}(2,0)$ first.
- Because $\lfloor h_{\max}(2,0)/\gamma \rfloor = 2$, we first try $\eta = 2$ in Eqs. (83)–(85) on p. 728 to obtain

$$\begin{aligned} p_u &= 0.1237, \\ p_m &= 0.7500, \\ p_d &= 0.1263. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,0)$ with the maximum variance use double jumps.

Numerical Examples (continued)

- Now consider state $h^2_{\min}(2,0)$.
- Because $\lfloor h_{\min}(2,0)/\gamma \rfloor = 1$, we first try $\eta = 1$ in Eqs. (83)–(85) on p. 728 to obtain

$$\begin{aligned} p_u &= 0.4596, \\ p_m &= 0.0760, \\ p_d &= 0.4644. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,0)$ with the minimum variance use single jumps.

Numerical Examples (continued)

- Node $(2,-1)$ has 3 predecessor nodes.
- Start with node $(1,1)$.
- Because it takes a down move to reach the current node, we apply updating rule (87) on p. 734 with $\ell = -1$ and $h^2_t = h^2(1,1)$.
- The result is $h^2_{t+1} = 0.0001227$.

Numerical Examples (continued)

- Now move on to predecessor node $(1,0)$.
- Because it also takes a down move to reach the current node, we apply updating rule (87) on p. 734 with $\ell = -1$ and $h^2_t = h^2(1,0)$.
- The result is $h^2_{t+1} = 0.000105609$.

Numerical Examples (continued)

- Finally, consider predecessor node $(1,-1)$.
- Because it takes a middle move to reach the current node, we apply updating rule (87) on p. 734 with $\ell = 0$ and $h^2_t = h^2(1,-1)$.
- The result is $h^2_{t+1} = 0.000105173$.
- We hence record

$$\begin{aligned} h^2_{\min}(2,-1) &= 0.000105173, \\ h^2_{\max}(2,-1) &= 0.0001227. \end{aligned}$$

Numerical Examples (continued)

- Consider state $h^2_{\max}(2,-1)$.
- Because $\lfloor h_{\max}(2,-1)/\gamma \rfloor = 2$, we first try $\eta = 2$ in Eqs. (83)–(85) on p. 728 to obtain

$$\begin{aligned} p_u &= 0.1385, \\ p_m &= 0.7201, \\ p_d &= 0.1414. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,-1)$ with the maximum variance use double jumps.

Numerical Examples (continued)

- Next, consider state $h^2_{\min}(2,-1)$.
- Because $\lfloor h_{\min}(2,-1)/\gamma \rfloor = 1$, we first try $\eta = 1$ in Eqs. (83)–(85) on p. 728 to obtain

$$\begin{aligned} p_u &= 0.4773, \\ p_m &= 0.0404, \\ p_d &= 0.4823. \end{aligned}$$

- As they are valid probabilities, the three branches from node $(2,-1)$ with the minimum variance use single jumps.

Numerical Examples (concluded)

- Other nodes at dates 2 and 3 can be handled similarly.
- In general, if a node has k predecessor nodes, then $2k$ variances will be calculated using the updating rule.
  - This is because each predecessor node keeps two variance numbers.
- But only the maximum and minimum variances will be kept.

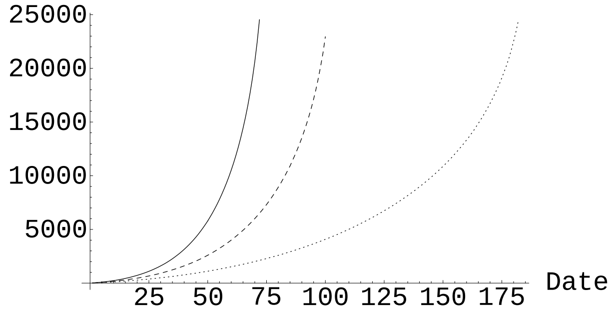
Negative Aspects of the RT Algorithm Revisited[a]

- Recall the problems mentioned on p. 740.
- In our case, combinatorial explosion occurs when

$$n > \frac{1-\beta_1}{\beta_2} = \frac{1-0.9}{0.04} = 2.5.$$

- Suppose we are willing to accept the exponential running time and pick $n = 100$ to seek accuracy.
- But the problem of shortened maturity forces the tree to stop at date 9!

[a] Lyuu and Wu (2003).



Dotted line: $n = 3$; dashed line: $n = 4$; solid line: $n = 5$.

Backward Induction on the RT Tree

- After the RT tree is constructed, it can be used to price options by backward induction.
- Recall that each node keeps two variances $h^2_{\max}$ and $h^2_{\min}$.
- We now increase that number to K equally spaced variances between $h^2_{\max}$ and $h^2_{\min}$ at each node.
- Besides the minimum and maximum variances, the other $K - 2$ variances in between are linearly interpolated.[a]

[a]In practice, log-linear interpolation works better; Lyuu and Wu (2005). Log-cubic interpolation works even better; Liu (2005).

Backward Induction on the RT Tree (continued)

- For example, if $K = 3$, then a variance of 10.5436×10^{-6} will be added between the maximum and minimum variances at node $(2, 0)$ on p. 743.
- In general, the kth variance at node (i, j) is

$$h^2_{\min}(i, j) + k \, \frac{h^2_{\max}(i, j) - h^2_{\min}(i, j)}{K - 1},$$

$k = 0, 1, \ldots, K - 1$.

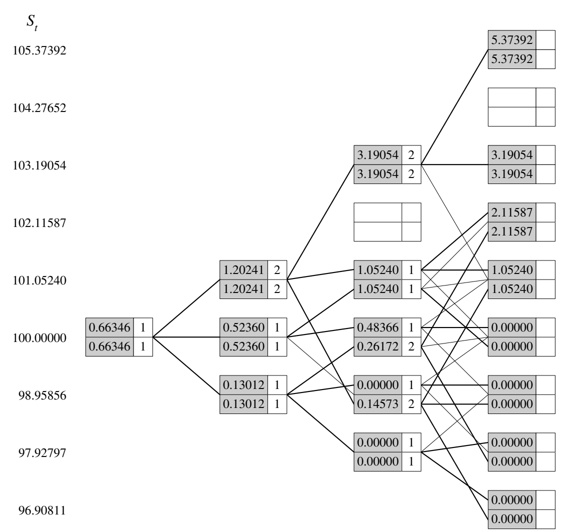
- Each interpolated variance's jump parameter and branching probabilities can be computed as before.

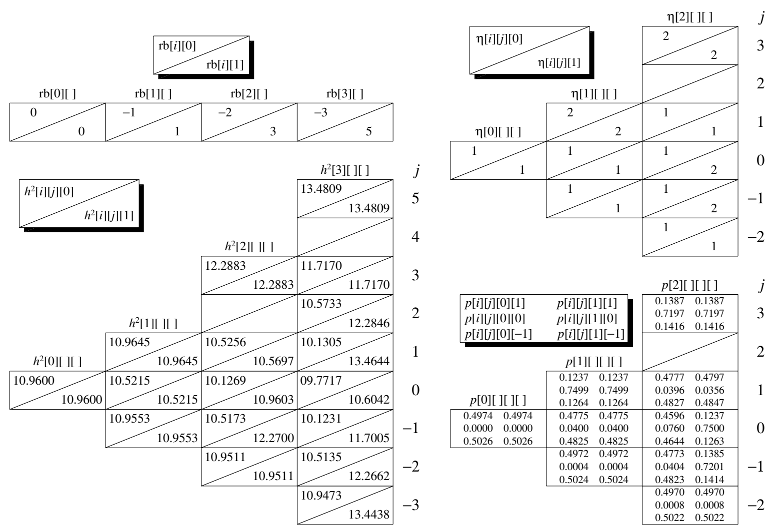
Backward Induction on the RT Tree (concluded)

- During backward induction, if a variance falls between two of the K variances, linear interpolation of the option prices corresponding to the two bracketing variances will be used as the approximate option price.
- The above ideas are reminiscent of the ones on p. 319, where we dealt with arithmetic average-rate options.

Numerical Examples

- We next use the numerical example on p. 743 to price a European call option with a strike price of 100 and expiring at date 3.
- Recall that the riskless interest rate is zero.
- Assume $K = 2$; hence there are no interpolated variances.
- The pricing tree is shown on p. 765 with a call price of 0.66346.
  - The branching probabilities needed in backward induction can be found on p. 766.





Numerical Examples (continued)

- Let us derive some of the numbers on p. 765.
- The option price for a terminal node at date 3 equals $\max(S_3 - 100, 0)$, independent of the variance level.
- Now move on to nodes at date 2.
- The option price at node $(2, 3)$ depends on those at nodes $(3, 5)$, $(3, 3)$, and $(3, 1)$.
- It therefore equals

$$0.1387 \times 5.37392 + 0.7197 \times 3.19054 + 0.1416 \times 1.05240 = 3.19054.$$

- Option prices for other nodes at date 2 can be computed similarly.

Numerical Examples (continued)

- For node $(1, 1)$, the option price for both variances is

$$0.1237 \times 3.19054 + 0.7499 \times 1.05240 + 0.1264 \times 0.14573 = 1.20241.$$

- Node $(1, 0)$ is most interesting.
- We knew that a down move from it gives a variance of 0.000105609.
- This number falls between the minimum variance 0.000105173 and the maximum variance 0.0001227 at node $(2, -1)$ on p. 743.

Numerical Examples (continued)

- The option price corresponding to the minimum variance is 0.
- The option price corresponding to the maximum variance is 0.14573.
- The equation

$$x \times 0.000105173 + (1 - x) \times 0.0001227 = 0.000105609$$

is satisfied by $x = 0.9751$.

- So the option for the down state is approximated by

$$x \times 0 + (1 - x) \times 0.14573 = 0.00362.$$

Numerical Examples (continued)

- The up move leads to the state with option price 1.05240.
- The middle move leads to the state with option price 0.48366.
- The option price at node $(1, 0)$ is finally calculated as

$$0.4775 \times 1.05240 + 0.0400 \times 0.48366 + 0.4825 \times 0.00362 = 0.52360.$$

Numerical Examples (concluded)

- It is possible for some of the three variances following an interpolated variance to exceed the maximum variance or be exceeded by the minimum variance.
- When this happens, the option price corresponding to the maximum or minimum variance will be used during backward induction.
- An interpolated variance may choose a branch that goes into a node that is not reached in the forward-induction tree-building phase.[a]

[a]Lyuu and Wu (2005).