Foundations of Term Structure Modeling

[Meriwether] scoring especially high marks
in mathematics — an indispensable subject
for a bond trader.
— Roger Lowenstein,
When Genius Failed

Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time t, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

t**:** a point in time.

$r(t)$**:** the one-period riskless rate prevailing at time t for repayment one period later (the instantaneous spot rate, or short rate, at time t).

$P(t, T)$**:** the present value at time t of one dollar at time T.

Standard Notations (continued)

$r(t,T)$**:** the $(T-t)$-period interest rate prevailing at time t stated on a per-period basis and compounded once per period—in other words, the $(T-t)$-period spot rate at time t.

- The long rate is defined as $r(t,\infty)$.

$F(t,T,M)$**:** the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \geq T$.

Standard Notations (concluded)

$f(t,T,L)$**:** the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.

$f(t,T)$**:** the one-period or instantaneous forward rate at time T as seen at time t stated on a per period basis and compounded once per period.

- It is $f(t,T,1)$ in the discrete-time model and $f(t,T,dt)$ in the continuous-time model.
- Note that $f(t,t)$ equals the short rate $r(t)$.

Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t,T) = \begin{cases} (1+r(t,T))^{-(T-t)} & \text{in discrete time,} \\ e^{-r(t,T)(T-t)} & \text{in continuous time.} \end{cases}$$

- $r(t,T)$ as a function of T defines the spot rate curve at time t.
- By definition,

$$f(t,t) = \begin{cases} r(t,t+1) & \text{in discrete time,} \\ r(t,t) & \text{in continuous time.} \end{cases}$$

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t,T,M) = \frac{P(t,M)}{P(t,T)}, \quad T \leq M. \tag{93}$$

  – The forward price equals the future value at time T of the underlying asset (see text for proof).

- Equation (93) holds whether the model is discrete-time or continuous-time, and it implies

$$F(t,T,M) = F(t,T,S)\,F(t,S,M), \quad T \leq S \leq M.$$

Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t,T,L) = \left(\frac{1}{F(t,T,T+L)}\right)^{1/L} - 1 = \left(\frac{P(t,T)}{P(t,T+L)}\right)^{1/L} - 1 \tag{94}$$

  in discrete time.
  - $f(t,T,L) = \frac{1}{L}\left(\frac{P(t,T)}{P(t,T+L)} - 1\right)$ is the analog to Eq. (94) under simple compounding.

Fundamental Relations (continued)

- In continuous time,

$$f(t,T,L) = -\frac{\ln F(t,T,T+L)}{L} = \frac{\ln(P(t,T)/P(t,T+L))}{L} \tag{95}$$

  by Eq. (93) on p. 845.
- Furthermore,

$$\begin{aligned} f(t,T,\Delta t) &= \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t,T)}{\partial T} \\ &= -\frac{\partial P(t,T)/\partial T}{P(t,T)}. \end{aligned}$$

Fundamental Relations (continued)

- So

$$f(t,T) \equiv \lim_{\Delta t \to 0} f(t,T,\Delta t) = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \leq T. \tag{96}$$

- Because Eq. (96) is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}, \tag{97}$$

  the spot rate curve is

$$r(t,T) = \frac{1}{T-t}\int_t^T f(t,s)\,ds.$$

Fundamental Relations (concluded)

- The discrete analog to Eq. (97) is

$$P(t,T) = \frac{1}{(1+r(t))(1+f(t,t+1))\cdots(1+f(t,T-1))}. \tag{98}$$

- The short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}.$$

  - This can be verified with Eq. (96) on p. 848 and the observation that $P(t,t) = 1$ and $r(t) = f(t,t)$.

Risk-Neutral Pricing

- Under the local expectations theory, the expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t+1 < T$,

$$\frac{E_t[\,P(t+1,T)\,]}{P(t,T)} = 1 + r(t). \tag{99}$$

  - Relation (99) in fact follows from the risk-neutral valuation principle, Theorem 14 (p. 412).

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π.
- Rewrite Eq. (99) as

$$\frac{E_t^{\pi}[\,P(t+1,T)\,]}{1+r(t)} = P(t,T).$$

  - It says the current spot rate curve equals the expected spot rate curve one period from now discounted by the short rate.

Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

$$\begin{aligned} & P(t,T) \\ = \ & E_t^{\pi}\left[\frac{P(t+1,T)}{1+r(t)}\right] \\ = \ & E_t^{\pi}\left[\frac{E_{t+1}^{\pi}[\,P(t+2,T)\,]}{(1+r(t))(1+r(t+1))}\right] = \cdots \\ = \ & E_t^{\pi}\left[\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))}\right]. \end{aligned} \tag{100}$$

Risk-Neutral Pricing (concluded)

- Equation (99) on p. 850 can also be expressed as

$$E_t[\,P(t+1,T)\,] = F(t,t+1,T).$$

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

$$P(t,T) = E_t\left[e^{-\int_t^T r(s)\,ds}\right], \quad t < T. \tag{101}$$

- Note that $e^{\int_t^T r(s)\,ds}$ is the bank account process, which denotes the rolled-over money market account.
- When the local expectations theory holds, riskless arbitrage opportunities are impossible.

Interest Rate Swaps

- Consider an interest rate swap made at time t with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is c per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant Δt for all i, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$.

Interest Rate Swaps (continued)

- The amount to be paid out at time t_{i+1} is $(f_i - c)\,\Delta t$ for the floating-rate payer.
  - Simple rates are adopted here.
- Hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

Interest Rate Swaps (continued)

- The value of the swap at time t is thus

$$\begin{aligned}
& \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds}(f_{i-1} - c)\,\Delta t\right] \\
= & \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds}\left(\frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t)\right)\right] \\
= & \sum_{i=1}^{n}(P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i)) \\
= & \; P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}$$

Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.

Swap Rate

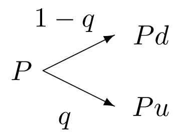
- The swap rate, which gives the swap zero value, equals

$$S_n(t) \equiv \frac{P(t,t_0) - P(t,t_n)}{\sum_{i=1}^{n} P(t,t_i)\,\Delta t}. \tag{102}$$

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, $P(t,t_0) = 1$.

The Binomial Model

- The analytical framework can be nicely illustrated with the binomial model.
- Suppose the bond price P can move with probability q to Pu and probability $1-q$ to Pd, where $u > d$:



The Binomial Model (continued)

- Over the period, the bond's expected rate of return is

$$\widehat{\mu} \equiv \frac{qPu + (1-q)\,Pd}{P} - 1 = qu + (1-q)\,d - 1. \tag{103}$$

- The variance of that return rate is

$$\widehat{\sigma}^2 \equiv q(1-q)(u-d)^2. \tag{104}$$

- The bond whose maturity is only one period away will move from a price of $1/(1+r)$ to its par value \$1.
- This is the money market account modeled by the short rate.

The Binomial Model (continued)

- The market price of risk is defined as $\lambda \equiv (\widehat{\mu} - r)/\widehat{\sigma}$.
- The same arbitrage argument as in the continuous-time case can be employed to show that λ is independent of the maturity of the bond (see text).

The Binomial Model (concluded)

- Now change the probability from q to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1+r)-d}{u-d}, \qquad (105)$$

  which is independent of bond maturity and q.
  - Recall the BOPM.
- The bond's expected rate of return becomes

$$\frac{pPu + (1-p)\,Pd}{P} - 1 = pu + (1-p)\,d - 1 = r.$$

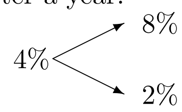
- The local expectations theory hence holds under the new probability measure p.

Numerical Examples

- Assume this spot rate curve:

Year	1	2
Spot rate	4%	5%

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:



Numerical Examples (continued)

- No real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,

$$100/1.04 = 96.154, 100/(1.05)^2 = 90.703.$$

- They follow the binomial processes on p. 866.

Numerical Examples (continued)

$$90.703 \begin{array}{l} \nearrow \; 92.593 \; (= 100/1.08) \\ \searrow \; 98.039 \; (= 100/1.02) \end{array} \qquad 96.154 \begin{array}{l} \nearrow \; 100 \\ \searrow \; 100 \end{array}$$

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

  where p denotes the risk-neutral probability of an up move in rates.

Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.

Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a \$95 strike price has the payoffs,

$$C \begin{array}{l} \nearrow \; 0.000 \\ \searrow \; 3.039 \end{array}$$

- To solve for the option value C, we replicate the call by a portfolio of x one-year and y two-year zeros.

Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

$$\begin{aligned} x \times 100 + y \times 92.593 &= 0.000, \\ x \times 100 + y \times 98.039 &= 3.039. \end{aligned}$$

- They give $x = -0.5167$ and $y = 0.5580$.
- Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.

Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

the same as before.

- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Futures and Forward Prices

- A one-year futures contract on the one-year rate has a payoff of $100 - r$, where r is the one-year rate at maturity, as shown below.

$$\begin{tikzcd} & 92\ (= 100 - 8) \\ F \arrow[ur] \arrow[dr] & \\ & 98\ (= 100 - 2) \end{tikzcd}$$

- As the futures price F is the expected future payoff (see text), $F = (1-p) \times 92 + p \times 98 = 93.914$.
- On the other hand, the forward price for a one-year forward contract on a one-year zero-coupon bond equals $90.703/96.154 = 94.331\%$.
- The forward price exceeds the futures price.

Numerical Examples: Mortgage-Backed Securities

- Consider a 5%-coupon, two-year mortgage-backed security without amortization, prepayments, and default risk.
- Its cash flow and price process are illustrated on p. 875.
- Its fair price is

$$M = \frac{(1-p) \times 102.222 + p \times 107.941}{1.04} = 100.045.$$

- Identical results could have been obtained via arbitrage considerations.

$$\begin{array}{ccccc} & & & & 105 \\ & & & \nearrow & \\ & & 5 & & \\ & \nearrow & & \searrow & \\ & & & & 105 \\ 0 & & & & \\ & & & & 105 \\ & \searrow & & \nearrow & \\ & & 5 & & \\ & & & \searrow & \\ & & & & 105 \end{array} \qquad \begin{array}{ccl} & & 102.222\ (= 5 + (105/1.08)) \\ & \nearrow & \\ M & & \\ & \searrow & \\ & & 107.941\ (= 5 + (105/1.02)) \end{array}$$

The left diagram depicts the cash flow; the right diagram illustrates the price process.

Numerical Examples: MBSs (continued)

- Suppose that the security can be prepaid at par.
- It will be prepaid only when its price is higher than par.
- Prepayment will hence occur only in the "down" state when the security is worth 102.941 (excluding coupon).
- The price therefore follows the process,

$$\begin{array}{ccl} & \nearrow & 102.222 \\ M & & \\ & \searrow & 105 \end{array}$$

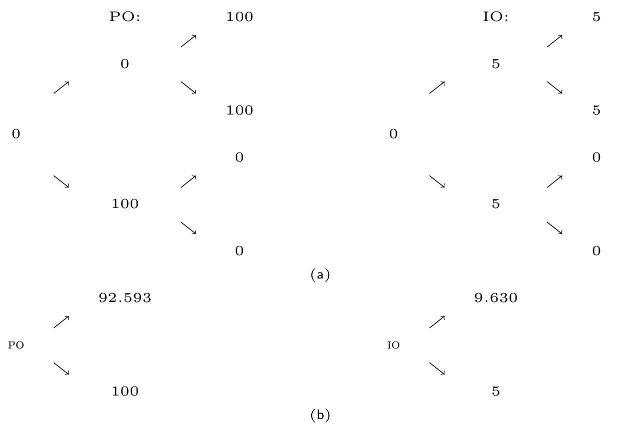
- The security is worth

$$M = \frac{(1-p) \times 102.222 + p \times 105}{1.04} = 99.142.$$

Numerical Examples: MBSs (continued)

- The cash flow of the principal-only (PO) strip comes from the mortgage's principal cash flow.
- The cash flow of the interest-only (IO) strip comes from the interest cash flow (p. 878(a)).
- Their prices hence follow the processes on p. 878(b).
- The fair prices are

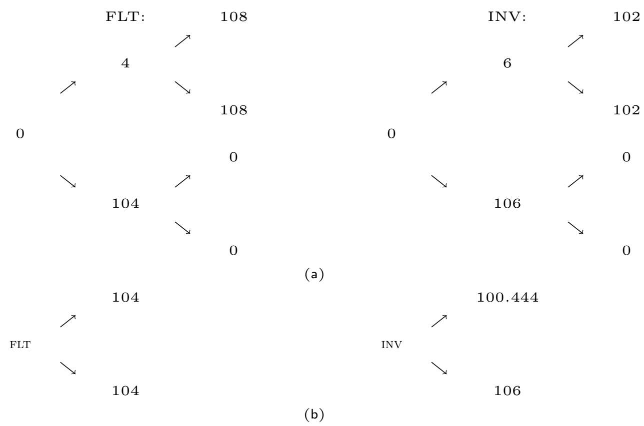
$$\begin{aligned} \text{PO} &= \frac{(1-p) \times 92.593 + p \times 100}{1.04} = 91.304, \\ \text{IO} &= \frac{(1-p) \times 9.630 + p \times 5}{1.04} = 7.839. \end{aligned}$$



The price 9.630 is derived from $5 + (5/1.08)$.

Numerical Examples: MBSs (continued)

- Suppose the mortgage is split into half floater and half inverse floater.
- Let the floater (FLT) receive the one-year rate.
- Then the inverse floater (INV) must have a coupon rate of
$$(10\% - \textsf{one-year rate})$$
to make the overall coupon rate 5%.
- Their cash flows as percentages of par and values are shown on p. 880.



Numerical Examples: MBSs (concluded)

- On p. 880, the floater's price in the up node, 104, is derived from $4 + (108/1.08)$.
- The inverse floater's price 100.444 is derived from $6 + (102/1.08)$.
- The current prices are

$$\begin{aligned} \text{FLT} &= \frac{1}{2} \times \frac{104}{1.04} = 50, \\ \text{INV} &= \frac{1}{2} \times \frac{(1-p) \times 100.444 + p \times 106}{1.04} = 49.142. \end{aligned}$$

Equilibrium Term Structure Models

8. What's your problem? Any moron can understand bond pricing models.
— *Top Ten Lies Finance Professors Tell Their Students*

Introduction

- This chapter surveys equilibrium models.
- Since the spot rates satisfy
$$r(t,T) = -\frac{\ln P(t,T)}{T-t},$$
the discount function $P(t,T)$ suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

The Vasicek Model[a]

- The short rate follows
$$dr = \beta(\mu - r)\,dt + \sigma\,dW.$$
- The short rate is pulled to the long-term mean level μ at rate β.
- Superimposed on this "pull" is a normally distributed stochastic term $\sigma\,dW$.
- Since the process is an Ornstein-Uhlenbeck process,
$$E[\,r(T)\,|\,r(t) = r\,] = \mu + (r - \mu)\,e^{-\beta(T-t)}$$
from Eq. (53) on p. 467.

[a] Vasicek (1977).

The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

$$P(t,T) = A(t,T)\, e^{-B(t,T)\, r(t)}, \tag{106}$$

where

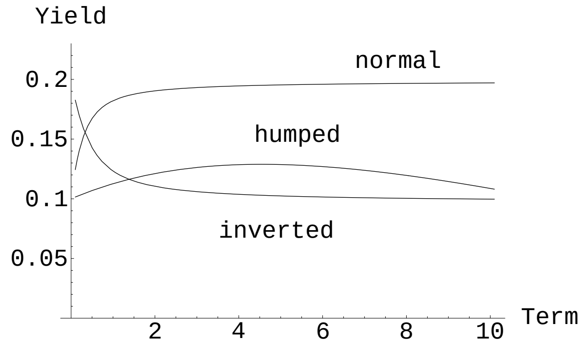
$$A(t,T) = \begin{cases} \exp\left[\frac{(B(t,T)-T+t)(\beta^2\mu-\sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\ \exp\left[\frac{\sigma^2 (T-t)^3}{6} \right] & \text{if } \beta = 0. \end{cases}$$

and

$$B(t,T) = \begin{cases} \frac{1-e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T-t & \text{if } \beta = 0. \end{cases}$$

The Vasicek Model (concluded)

- If $\beta = 0$, then P goes to infinity as $T \to \infty$.
- Sensibly, P goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, P may exceed one for a finite T.
- The spot rate volatility structure is the curve $(\partial r(t,T)/\partial r)\, \sigma = \sigma B(t,T)/(T-t)$.
- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, β, controls the shape of the curve; indeed, higher β leads to greater attenuation of volatility with maturity.



The Vasicek Model: Options on Zeros[a]

- Consider a European call with strike price X expiring at time T on a zero-coupon bond with par value \$1 and maturing at time $s > T$.
- Its price is given by

$$P(t,s)\, N(x) - XP(t,T)\, N(x-\sigma_v).$$

[a] Jamshidian (1989).

The Vasicek Model: Options on Zeros (concluded)

- Above

$$
\begin{aligned}
x &\equiv \frac{1}{\sigma_v} \ln\left(\frac{P(t,s)}{P(t,T)\,X}\right) + \frac{\sigma_v}{2}, \\
\sigma_v &\equiv v(t,T)\,B(T,s), \\
v(t,T)^2 &\equiv \begin{cases} \frac{\sigma^2\left[1-e^{-2\beta(T-t)}\right]}{2\beta}, & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0 \end{cases}.
\end{aligned}
$$

- By the put-call parity, the price of a European put is

$$
X P(t,T)\,N(-x+\sigma_v) - P(t,s)\,N(-x).
$$

Binomial Vasicek

- Consider a binomial model for the short rate in the time interval $[\,0,T\,]$ divided into n identical pieces.
- Let $\Delta t \equiv T/n$ and

$$
p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r)\sqrt{\Delta t}}{2\sigma}.
$$

- The following binomial model converges to the Vasicek model,[a]

$$
r(k+1) = r(k) + \sigma\sqrt{\Delta t}\;\xi(k), \quad 0 \leq k < n.
$$

[a] Nelson and Ramaswamy (1990).

Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$
\text{Prob}[\,\xi(k) = 1\,] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases}.
$$

- Observe that the probability of an up move, p, is a decreasing function of the interest rate r.
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its *constant* volatility, σ.
- For a general process Y with nonconstant volatility, the resulting binomial tree may not combine.

The Cox-Ingersoll-Ross Model[a]

- It is the following square-root short rate model:

$$dr = \beta(\mu - r)\, dt + \sigma\sqrt{r}\, dW. \qquad (107)$$

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.
- The parameter β determines the speed of adjustment.
- The short rate can reach zero only if $2\beta\mu < \sigma^2$.
- See text for the bond pricing formula.

[a] Cox, Ingersoll, and Ross (1985).

Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into n periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will *not* combine.

Binomial CIR (continued)

- Instead, consider the transformed process

$$x(r) \equiv 2\sqrt{r}/\sigma.$$

- It follows

$$dx = m(x)\, dt + dW,$$

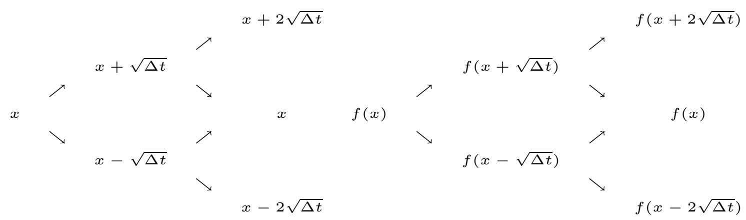
where

$$m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x).$$

- Since this new process has a constant volatility, its associated binomial tree combines.

Binomial CIR (continued)

- Construct the combining tree for r as follows.
- First, construct a tree for x.
- Then transform each node of the tree into one for r via the inverse transformation $r = f(x) \equiv x^2\sigma^2/4$ (p. 898).



Binomial CIR (concluded)

- The probability of an up move at each node r is

$$p(r) \equiv \frac{\beta(\mu - r)\,\Delta t + r - r^-}{r^+ - r^-}. \tag{108}$$

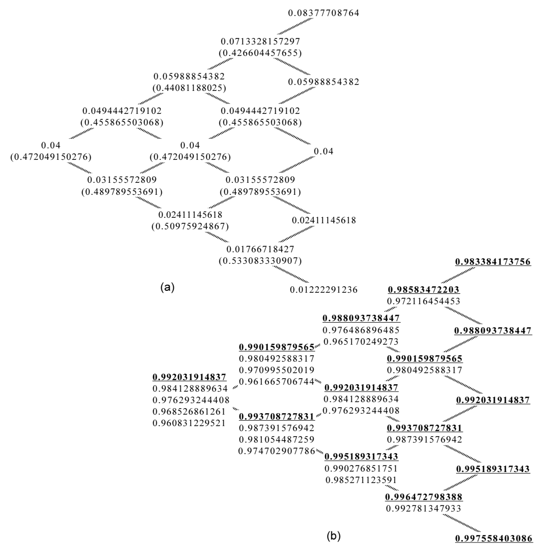
  - $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from r.
  - $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.
- Finally, set the probability $p(r)$ to one as r goes to zero to make the probability stay between zero and one.

Numerical Examples

- Consider the process,

$$0.2\,(0.04 - r)\,dt + 0.1\sqrt{r}\,dW,$$

  for the time interval $[\,0, 1\,]$ given the initial rate $r(0) = 0.04$.
- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.
- See p. 901(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.



Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has $x = 2\sqrt{r(0)}/\sigma = 4$, this particular node's x value equals $4 + \sqrt{\Delta t} = 4.4472135955$.
- Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$.

Numerical Examples (concluded)

- Once the short rates are in place, computing the probabilities is easy.
- Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
- This phenomenon agrees with mean reversion.
- Convergence is quite good (see text).

A General Method for Constructing Binomial Models[a]

- We are given a continuous-time process $dy = \alpha(y,t)\,dt + \sigma(y,t)\,dW$.
- Make sure the binomial model's drift and diffusion converge to the above process by setting the probability of an up move to

$$\frac{\alpha(y,t)\,\Delta t + y - y_{\mathrm{u}}}{y_{\mathrm{u}} - y_{\mathrm{d}}}.$$

- Here $y_{\mathrm{u}} \equiv y + \sigma(y,t)\sqrt{\Delta t}$ and $y_{\mathrm{d}} \equiv y - \sigma(y,t)\sqrt{\Delta t}$ represent the two rates that follow the current rate y.
- The displacements are identical, at $\sigma(y,t)\sqrt{\Delta t}$.

[a] Nelson and Ramaswamy (1990).

A General Method (continued)

- But the binomial tree may not combine:

$$\sigma(y,t)\sqrt{\Delta t} - \sigma(y_{\mathrm{u}},t)\sqrt{\Delta t} \neq -\sigma(y,t)\sqrt{\Delta t} + \sigma(y_{\mathrm{d}},t)\sqrt{\Delta t}$$

  in general.
- When $\sigma(y,t)$ is a constant independent of y, equality holds and the tree combines.
- To achieve this, define the transformation

$$x(y,t) \equiv \int^{y} \sigma(z,t)^{-1}\,dz.$$

- Then x follows $dx = m(y,t)\,dt + dW$ for some $m(y,t)$ (see text).

A General Method (continued)

- The key is that the diffusion term is now a constant, and the binomial tree for x combines.
- The probability of an up move remains
$$\frac{\alpha(y(x,t),t)\,\Delta t + y(x,t) - y_{\mathrm{d}}(x,t)}{y_{\mathrm{u}}(x,t) - y_{\mathrm{d}}(x,t)},$$
where $y(x,t)$ is the inverse transformation of $x(y,t)$ from x back to y.
- Note that $y_{\mathrm{u}}(x,t) \equiv y(x+\sqrt{\Delta t}, t+\Delta t)$ and $y_{\mathrm{d}}(x,t) \equiv y(x-\sqrt{\Delta t}, t+\Delta t)$.

A General Method (concluded)

- The transformation is
$$\int^{r} (\sigma\sqrt{z})^{-1}\, dz = 2\sqrt{r}/\sigma$$
for the CIR model.
- The transformation is
$$\int^{S} (\sigma z)^{-1}\, dz = (1/\sigma)\ln S$$
for the Black-Scholes model.
- The familiar binomial option pricing model in fact discretizes $\ln S$ not S.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate *levels* only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.

On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two-factor ones.

Options on Coupon Bonds[a]

- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time T on a bond with par value \$1.
- Let X denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all i.
- The payoff for the option is

$$\max\left(\sum_{i=1}^{n} c_i P(r(T), T, t_i) - X, 0\right).$$

[a]Jamshidian (1989).

Options on Coupon Bonds (continued)

- At time T, there is a unique value r^* for $r(T)$ that renders the coupon bond's price equal the strike price X.
- This r^* can be obtained by solving $X = \sum_{i=1}^{n} c_i P(r, T, t_i)$ numerically for r.
- The solution is also unique for one-factor models whose bond price is a monotonically decreasing function of r.
- Let $X_i \equiv P(r^*, T, t_i)$, the value at time T of a zero-coupon bond with par value \$1 and maturing at time t_i if $r(T) = r^*$.

Options on Coupon Bonds (concluded)

- Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$.
- As $X = \sum_i c_i X_i$, the option's payoff equals

$$\sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

- Thus the call is a package of n options on the underlying zero-coupon bond.