Algorithms Comparison[a]

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the n value at which they converge.
  - The one with the smallest n wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times.

[a]Lyuu (1998).

Algorithms Comparison (concluded)

- Pages 299 and 556 show the trinomial model converges at a smaller n than BOPM.
- It is in this sense when people say trinomial models converge faster than binomial ones.
- But is the trinomial model better then?
- The linear-time binomial tree algorithm actually performs better than the trinomial one (see next page expanded from p. 546).

n	Combinatorial method		Trinomial tree algorithm	
	Value	Time	Value	Time
21	5.507548	0.30		
84	5.597597	0.90	5.634936	35.0
191	5.635415	2.00	5.655082	185.0
342	5.655812	3.60	5.658590	590.0
533	5.652253	5.60	5.659692	1440.0
768	5.654609	8.00	5.660137	3080.0
1047	5.658622	11.10	5.660338	5700.0
1368	5.659711	15.00	5.660432	9500.0
1731	5.659416	19.40	5.660474	15400.0
2138	5.660511	24.70	5.660491	23400.0
2587	5.660592	30.20	5.660493	34800.0
3078	5.660099	36.70	5.660488	48800.0
3613	5.660498	43.70	5.660478	67500.0
4190	5.660388	44.10	5.660466	92000.0
4809	5.659955	51.60	5.660454	130000.0
5472	5.660122	68.70		
6177	5.659981	76.70		

(All times in milliseconds.)

Double-Barrier Options

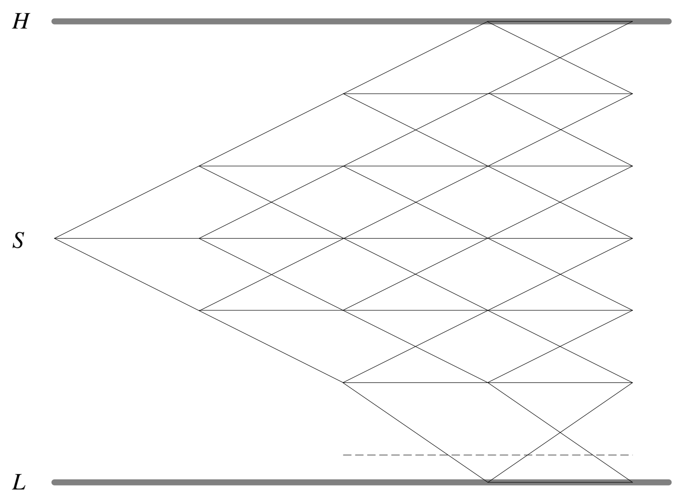
- Double-barrier options are barrier options with two barriers $L < H$.
- Assume $L < S < H$.
- The binomial model produces oscillating option values (see plot next page).[a]
- The trinomial model can be modified so that both barriers coincide with a layer of the tree.[b]

[a]Chao (1999); Dai and Lyuu (2005);
[b]Ritchken (1995); Hull (1999); Hsu and Lyuu (2004).

Double-Barrier Knock-Out Options

- We knew how to pick the λ so that one of the layers of the trinomial tree coincides with one of the barriers, say H.
- This choice, however, does not guarantee that the other barrier, L, is also hit.
- One way to handle this problem is to lower the layer of the tree just above L to coincide with L (see the plot next page).[a]

[a]Ritchken (1995).



Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above L must be adjusted.
- Let ℓ be the positive integer such that

$$Sd^{\ell+1} < L < Sd^{\ell}.$$

- Hence the layer of the tree just above L has price Sd^{ℓ}.

Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying

$$L = Sd^{\ell-1}e^{-\gamma\lambda\sigma\sqrt{\Delta t}}.$$

  – The prices between the barriers are

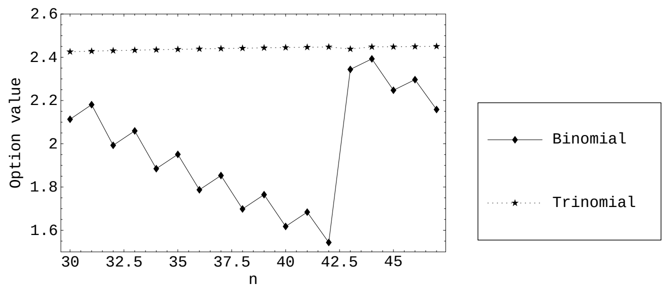
$$L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H.$$

- The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are

$$p'_u = \frac{b + a\gamma}{1 + \gamma}, \quad p'_d = \frac{b - a}{\gamma + \gamma^2}, \quad \text{and} \quad p'_m = 1 - p'_u - p'_d,$$

where $a \equiv \mu'\sqrt{\Delta t}/(\lambda\sigma)$ and $b \equiv 1/\lambda^2$.

Convergence: Binomial vs. Trinomial



Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on m assets has the terminal payoff $\max(\sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0)$, where α_i is the percentage of asset i.
- Basket options are essentially options on a portfolio of stocks or index options.
- Option on the best of two risky assets and cash has a terminal payoff of $\max(S_1(\tau), S_2(\tau), X)$.

Correlated Trinomial Model

- Two risky assets S_1 and S_2 follow $dS_i/S_i = r\,dt + \sigma_i\,dW_i$ in a risk-neutral economy, $i = 1, 2$.
- Let

$$\begin{aligned} M_i &\equiv e^{r\Delta t}, \\ V_i &\equiv M_i^2(e^{\sigma_i^2 \Delta t} - 1). \end{aligned}$$

  – $S_i M_i$ is the mean of S_i at time Δt.
  – $S_i^2 V_i$ the variance of S_i at time Δt.

Correlated Trinomial Model (continued)

- The value of $S_1 S_2$ at time Δt has a joint lognormal distribution with mean $S_1 S_2 M_1 M_2 e^{\rho\sigma_1\sigma_2\Delta t}$, where ρ is the correlation between dW_1 and dW_2.
- Next match the 1st and 2nd moments of the approximating discrete distribution to those of the continuous counterpart.
- At time Δt from now, there are five distinct outcomes.

Correlated Trinomial Model (continued)

- The five-point probability distribution of the asset prices is (as usual, we impose $u_i d_i = 1$)

Probability	Asset 1	Asset 2
p_1	$S_1 u_1$	$S_2 u_2$
p_2	$S_1 u_1$	$S_2 d_2$
p_3	$S_1 d_1$	$S_2 d_2$
p_4	$S_1 d_1$	$S_2 u_2$
p_5	S_1	S_2

Correlated Trinomial Model (continued)

- The probabilities must sum to one, and the means must be matched:

$$\begin{aligned} 1 &= p_1 + p_2 + p_3 + p_4 + p_5, \\ S_1 M_1 &= (p_1 + p_2)\, S_1 u_1 + p_5 S_1 + (p_3 + p_4)\, S_1 d_1, \\ S_2 M_2 &= (p_1 + p_4)\, S_2 u_2 + p_5 S_2 + (p_2 + p_3)\, S_2 d_2. \end{aligned}$$

Correlated Trinomial Model (continued)

- Let $R \equiv M_1 M_2 e^{\rho\sigma_1\sigma_2\Delta t}$.
- Match the variances and covariance:

$$\begin{aligned} S_1^2 V_1 &= (p_1 + p_2)((S_1 u_1)^2 - (S_1 M_1)^2) + p_5 (S_1^2 - (S_1 M_1)^2) \\ &\quad +(p_3 + p_4)((S_1 d_1)^2 - (S_1 M_1)^2), \\ S_2^2 V_2 &= (p_1 + p_4)((S_2 u_2)^2 - (S_2 M_2)^2) + p_5 (S_2^2 - (S_2 M_2)^2) \\ &\quad +(p_2 + p_3)((S_2 d_2)^2 - (S_2 M_2)^2), \\ S_1 S_2 R &= (p_1 u_1 u_2 + p_2 u_1 d_2 + p_3 d_1 d_2 + p_4 d_1 u_2 + p_5)\, S_1 S_2. \end{aligned}$$

Correlated Trinomial Model (continued)

- The solutions are

$$
\begin{aligned}
p_1 &= \frac{u_1 u_2 (R-1) - f_1 (u_1^2 - 1) - f_2 (u_2^2 - 1) + (f_2 + g_2)(u_1 u_2 - 1)}{(u_1^2 - 1)(u_2^2 - 1)},\\
p_2 &= \frac{-u_1 u_2 (R-1) + f_1 (u_1^2 - 1) u_2^2 + f_2 (u_2^2 - 1) - (f_2 + g_2)(u_1 u_2 - 1)}{(u_1^2 - 1)(u_2^2 - 1)},\\
p_3 &= \frac{u_1 u_2 (R-1) - f_1 (u_1^2 - 1) u_2^2 + g_2 (u_2^2 - 1) u_1^2 + (f_2 + g_2)(u_1 u_2 - u_2^2)}{(u_1^2 - 1)(u_2^2 - 1)},\\
p_4 &= \frac{-u_1 u_2 (R-1) + f_1 (u_1^2 - 1) + f_2 (u_2^2 - 1) u_1^2 - (f_2 + g_2)(u_1 u_2 - 1)}{(u_1^2 - 1)(u_2^2 - 1)}.
\end{aligned}
$$

Correlated Trinomial Model (concluded)

- In the above,

$$
\begin{aligned}
f_1 &= p_1 + p_2 = \frac{u_1 \left(V_1 + M_1^2 - M_1\right) - (M_1 - 1)}{(u_1 - 1)\,(u_1^2 - 1)},\\
f_2 &= p_1 + p_4 = \frac{u_2 \left(V_2 + M_2^2 - M_2\right) - (M_2 - 1)}{(u_2 - 1)\,(u_2^2 - 1)},\\
g_1 &= p_3 + p_4 = \frac{u_1^2 \left(V_1 + M_1^2 - M_1\right) - u_1^3 (M_1 - 1)}{(u_1 - 1)\,(u_1^2 - 1)},\\
g_2 &= p_2 + p_3 = \frac{u_2^2 \left(V_2 + M_2^2 - M_2\right) - u_2^3 (M_2 - 1)}{(u_2 - 1)\,(u_2^2 - 1)}.
\end{aligned}
$$

- As $f_1 + g_1 = f_2 + g_2$, we can solve for u_2 given $u_1 = e^{\lambda \sigma_1 \sqrt{\Delta t}}$ for an appropriate $\lambda > 1$.

Extrapolation

- It is a method to speed up numerical convergence.
- Say $f(n)$ converges to an unknown limit f at rate of $1/n$:

$$f(n) = f + \frac{c}{n} + o\left(\frac{1}{n}\right). \tag{64}$$

- Assume c is an unknown constant independent of n.
  - Convergence is basically monotonic and smooth.

Extrapolation (concluded)

- From two approximations $f(n_1)$ and $f(n_2)$ and by ignoring the smaller terms,

$$
\begin{aligned}
f(n_1) &= f + \frac{c}{n_1},\\
f(n_2) &= f + \frac{c}{n_2}.
\end{aligned}
$$

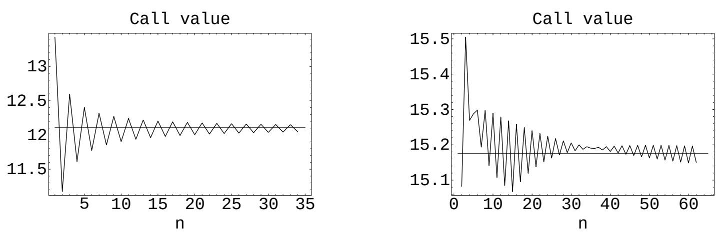
- A better approximation to the desired f is

$$f = \frac{n_1 f(n_1) - n_2 f(n_2)}{n_1 - n_2}. \tag{65}$$

- This estimate should converge faster than $1/n$.
- The Richardson extrapolation uses $n_2 = 2n_1$.

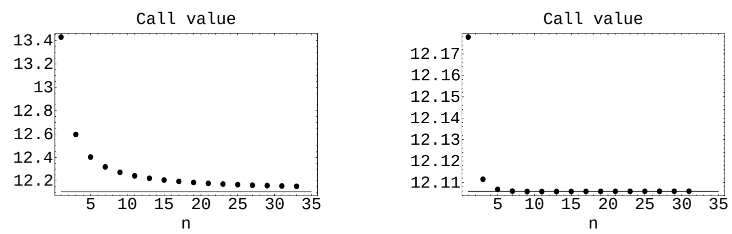
Improving BOPM with Extrapolation

- Consider standard European options.
- Denote the option value under BOPM using n time periods by $f(n)$.
- It is known that BOPM convergences at the rate of $1/n$, consistent with Eq. (64) on p. 575.
- But the plots on p. 241 (redrawn on next page) demonstrate that convergence to the true option value oscillates with n.
- Extrapolation is inapplicable at this stage.



Improving BOPM with Extrapolation (concluded)

- Take the at-the-money option in the left plot on p. 578.
- The sequence with odd n turns out to be monotonic and smooth (see the left plot on p. 580).
- Apply extrapolation (65) on p. 576 with $n_2 = n_1 + 2$, where n_1 is odd.
- Result is shown in the right plot on p. 580.
- The convergence rate is amazing.
- See Exercise 9.3.8 of the textbook (p. 111) for ideas in the general case.

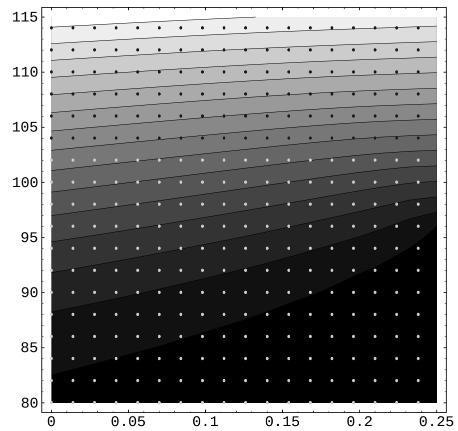


Numerical Methods

All science is dominated
by the idea of approximation.
— Bertrand Russell

Finite-Difference Methods

- Place a grid of points on the space over which the desired function takes value.
- Then approximate the function value at each of these points (p. 584).
- Solve the equation numerically by introducing difference equations in place of derivatives.



Example: Poisson's Equation

- It is $\partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 = -\rho(x, y)$.
- Replace second derivatives with finite differences through central difference.
- Introduce evenly spaced grid points with distance of Δx along the x axis and Δy along the y axis.
- The finite difference form is

$$
\begin{aligned}
-\rho(x_i, y_j) = {} & \frac{\theta(x_{i+1}, y_j) - 2\theta(x_i, y_j) + \theta(x_{i-1}, y_j)}{(\Delta x)^2} \\
& + \frac{\theta(x_i, y_{j+1}) - 2\theta(x_i, y_j) + \theta(x_i, y_{j-1})}{(\Delta y)^2}.
\end{aligned}
$$

Example: Poisson's Equation (concluded)

- In the above, $\Delta x \equiv x_i - x_{i-1}$ and $\Delta y \equiv y_j - y_{j-1}$ for $i, j = 1, 2, \ldots$.
- When the grid points are evenly spaced in both axes so that $\Delta x = \Delta y = h$, the difference equation becomes

$$
\begin{aligned}
-h^2 \rho(x_i, y_j) = {} & \theta(x_{i+1}, y_j) + \theta(x_{i-1}, y_j) \\
& + \theta(x_i, y_{j+1}) + \theta(x_i, y_{j-1}) - 4\theta(x_i, y_j).
\end{aligned}
$$

- Given boundary values, we can solve for the x_is and the y_js within the square $[\pm L, \pm L]$.
- From now on, $\theta_{i,j}$ will denote the finite-difference approximation to the exact $\theta(x_i, y_j)$.

Explicit Methods

- Consider the diffusion equation $D(\partial^2\theta/\partial x^2) - (\partial\theta/\partial t) = 0$.
- Use evenly spaced grid points (x_i, t_j) with distances Δx and Δt, where $\Delta x \equiv x_{i+1} - x_i$ and $\Delta t \equiv t_{j+1} - t_j$.
- Employ central difference for the second derivative and forward difference for the time derivative to obtain

$$
\begin{aligned}
\left.\frac{\partial\theta(x, t)}{\partial t}\right|_{t=t_j} &= \frac{\theta(x, t_{j+1}) - \theta(x, t_j)}{\Delta t} + \cdots, & (66) \\
\left.\frac{\partial^2\theta(x, t)}{\partial x^2}\right|_{x=x_i} &= \frac{\theta(x_{i+1}, t) - 2\theta(x_i, t) + \theta(x_{i-1}, t)}{(\Delta x)^2} + \cdots. & (67)
\end{aligned}
$$

Explicit Methods (continued)

- To assemble Eqs. (66) and (67) into a single equation at (x_i, t_j), need to decide how to evaluate x in the first equation and t in the second.
- Since central difference around x_i is used in Eq. (67), we might as well use x_i for x in Eq. (66).
- Two choices are possible for t in Eq. (67).
- The first choice uses $t = t_j$ to yield the following finite-difference equation,

$$
\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D\, \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2}. \tag{68}
$$

Explicit Methods (concluded)

- The stencil of grid points involves four values, $\theta_{i,j+1}$, $\theta_{i,j}$, $\theta_{i+1,j}$, and $\theta_{i-1,j}$.
- We can calculate $\theta_{i,j+1}$ from $\theta_{i,j}, \theta_{i+1,j}, \theta_{i-1,j}$, at the previous time t_j (see figure (a) on next page).
- Starting from the initial conditions at t_0, that is, $\theta_{i,0} = \theta(x_i, t_0)$, $i = 1, 2, \ldots$, we calculate

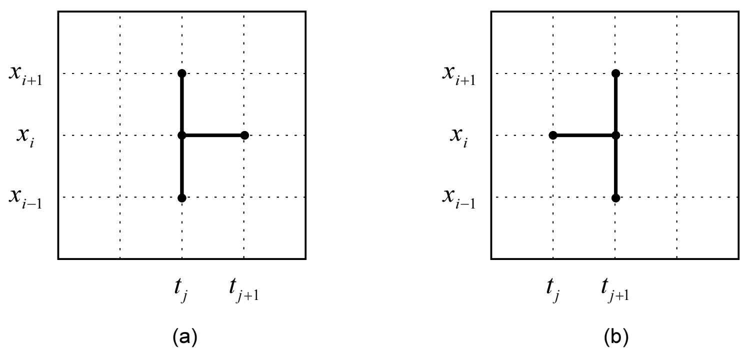
  $$\theta_{i,1}, \quad i = 1, 2, \ldots ,$$

  and then

  $$\theta_{i,2}, \quad i = 1, 2, \ldots ,$$

  and so on.

Stencils



Stability

- The explicit method is numerically unstable unless $\Delta t \leq (\Delta x)^2/(2D)$.
  - A numerical method is unstable if the solution is highly sensitive to changes in initial conditions.
- The stability condition may lead to high running times and memory requirements.
- For instance, halving Δx would imply quadrupling $(\Delta t)^{-1}$, resulting in a running time eight times as much.

Explicit Method and Trinomial Tree

- Rearrange Eq. (68) on p. 588 as

  $$\theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right)\theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2}\,\theta_{i-1,j}.$$

- When the stability condition is satisfied, the three coefficients for $\theta_{i+1,j}$, $\theta_{i,j}$, and $\theta_{i-1,j}$ all lie between zero and one and sum to one.
- They can therefore be interpreted as probabilities.
- So the finite-difference equation becomes identical to backward induction on trinomial trees.
- The freedom in choosing Δx corresponds to similar freedom in the construction of the trinomial trees.

Implicit Methods

- If we use $t = t_{j+1}$ in Eq. (67) on p. 587 instead, the finite-difference equation becomes

$$\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D \, \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2}. \tag{69}$$

- The stencil involves $\theta_{i,j}$, $\theta_{i,j+1}$, $\theta_{i+1,j+1}$, and $\theta_{i-1,j+1}$.
- This method is implicit because the value of any one of the three quantities at t_{j+1} cannot be calculated unless the other two are known (see figure (b) on p. 590).

Implicit Methods (continued)

- Equation (69) can be rearranged as

$$\theta_{i-1,j+1} - (2+\gamma)\,\theta_{i,j+1} + \theta_{i+1,j+1} = -\gamma\theta_{i,j},$$

  where $\gamma \equiv (\Delta x)^2/(D\Delta t)$.
- This equation is unconditionally stable.
- Suppose the boundary conditions are given at $x = x_0$ and $x = x_{N+1}$.
- After $\theta_{i,j}$ has been calculated for $i = 1, 2, \ldots, N$, the values of $\theta_{i,j+1}$ at time t_{j+1} can be computed as the solution to the following tridiagonal linear system,

Implicit Methods (continued)

$$\begin{bmatrix} a & 1 & 0 & \cdots & \cdots & 0 \\ 1 & a & 1 & 0 & \cdots & 0 \\ 0 & 1 & a & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & a & 1 \\ 0 & \cdots & \cdots & \cdots & 1 & a \end{bmatrix} \begin{bmatrix} \theta_{1,j+1} \\ \theta_{2,j+1} \\ \theta_{3,j+1} \\ \vdots \\ \vdots \\ \vdots \\ \theta_{N,j+1} \end{bmatrix} = \begin{bmatrix} -\gamma\theta_{1,j} - \theta_{0,j+1} \\ -\gamma\theta_{2,j} \\ -\gamma\theta_{3,j} \\ \vdots \\ \vdots \\ -\gamma\theta_{N-1,j} \\ -\gamma\theta_{N,j} - \theta_{N+1,j+1} \end{bmatrix},$$

where $a \equiv -2 - \gamma$.

Implicit Methods (concluded)

- Tridiagonal systems can be solved in $O(N)$ time and $O(N)$ space.
- The matrix above is nonsingular when $\gamma \geq 0$.
  - A square matrix is nonsingular if its inverse exists.

Crank-Nicolson Method

- Take the average of explicit method (68) on p. 588 and implicit method (69) on p. 593:

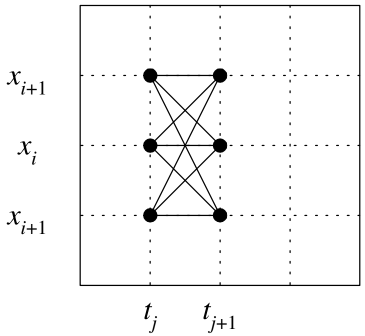
$$\begin{aligned}&\frac{\theta_{i,j+1}-\theta_{i,j}}{\Delta t}\\ =\;& \frac{1}{2}\left(D\,\frac{\theta_{i+1,j}-2\theta_{i,j}+\theta_{i-1,j}}{(\Delta x)^2}+D\,\frac{\theta_{i+1,j+1}-2\theta_{i,j+1}+\theta_{i-1,j+1}}{(\Delta x)^2}\right).\end{aligned}$$

- After rearrangement,

$$\gamma\theta_{i,j+1}-\frac{\theta_{i+1,j+1}-2\theta_{i,j+1}+\theta_{i-1,j+1}}{2}=\gamma\theta_{i,j}+\frac{\theta_{i+1,j}-2\theta_{i,j}+\theta_{i-1,j}}{2}.$$

- This is an unconditionally stable implicit method with excellent rates of convergence.

Stencil



Numerically Solving the Black-Scholes PDE

- We focus on American puts.
- The technique can be applied to any derivative satisfying the Black-Scholes PDE as only the initial and the boundary conditions need to be changed.
- The Black-Scholes PDE for American puts is

$$\frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 P}{\partial S^2}+(r-q)\,S\,\frac{\partial P}{\partial S}-rP+\frac{\partial P}{\partial t}=0$$

with $P(S,T)=\max(X-S,0)$ and $P(S,t)=\max(\overline{P}(S,t),X-S)$ for $t<T$.

- $\overline{P}$ denotes the option value at time t if it is not exercised for the next instant of time.

Numerically Solving the Black-Scholes PDE (continued)

- After the change of variable $V\equiv\ln S$, the option value becomes $U(V,t)\equiv P(e^V,t)$ and

$$\frac{\partial P}{\partial t}=\frac{\partial U}{\partial t},\quad \frac{\partial P}{\partial S}=\frac{1}{S}\,\frac{\partial U}{\partial V},\,\frac{\partial^2 P}{\partial^2 S}=\frac{1}{S^2}\,\frac{\partial^2 U}{\partial V^2}-\frac{1}{S^2}\,\frac{\partial U}{\partial V}.$$

- The Black-Scholes PDE is now transformed into

$$\frac{1}{2}\,\sigma^2\,\frac{\partial^2 U}{\partial V^2}+\left(r-q-\frac{\sigma^2}{2}\right)\frac{\partial U}{\partial V}-rU+\frac{\partial U}{\partial t}=0$$

subject to $U(V,T)=\max(X-e^V,0)$ and $U(V,t)=\max(\overline{U}(V,t),X-e^V)$, $t<T$.

Numerically Solving the Black-Scholes PDE (concluded)

- Along the V axis, the grid will span from $V_{\min}$ to $V_{\min} + N \times \Delta V$ at ΔV apart for some suitably small $V_{\min}$.
- So boundary conditions at the lower ($V = V_{\min}$) and upper ($V = V_{\min} + N \times \Delta V$) boundaries will have to be specified.
- S_0 as usual denotes the current stock price.

Explicit Method

- The explicit scheme for the Black-Scholes differential equation is

$$\begin{aligned} 0 &= \frac{1}{2}\sigma^2 \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta V)^2} \\ &+ \left(r - q - \frac{\sigma^2}{2}\right) \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta V} - rU_{i,j} + \frac{U_{i,j} - U_{i,j-1}}{\Delta t} \end{aligned}$$

for $1 \leq i \leq N - 1$.

- The computation moves backward in time.
- There are $N - 1$ difference equations.

Explicit Method (continued)

- Regroup the terms to obtain

$$U_{i,j-1} = aU_{i-1,j} + bU_{i,j} + cU_{i+1,j},$$

where

$$\begin{aligned} a &\equiv \left(\left(\frac{\sigma}{\Delta V}\right)^2 - \frac{r - q - \sigma^2/2}{\Delta V}\right)\frac{\Delta t}{2}, \\ b &\equiv 1 - r\Delta t - \left(\frac{\sigma}{\Delta V}\right)^2 \Delta t, \\ c &\equiv \left(\left(\frac{\sigma}{\Delta V}\right)^2 + \frac{r - q - \sigma^2/2}{\Delta V}\right)\frac{\Delta t}{2}. \end{aligned}$$

Explicit Method (continued)

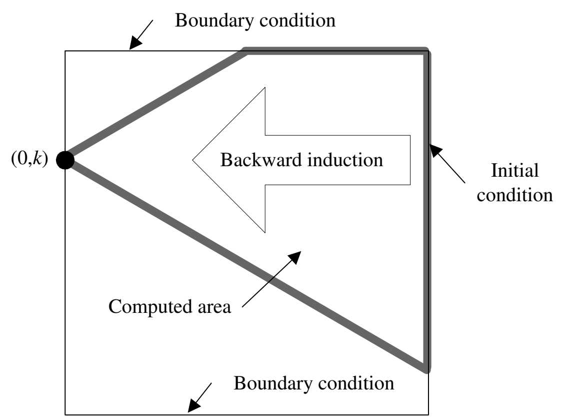
- These $N - 1$ equations express option values at time step $j - 1$ in terms of those at time step j.
- For American puts, we assume for U's lower boundary that the first derivative at grid point $(0, j)$ for every time step j equals $-e^{V_{\min}}$.
- This essentially makes the put value $X - S = X - e^V$.
- So $U_{0,j-1} = U_{1,j-1} + (e^{V_{\min}+\Delta V} - e^{V_{\min}})$.
- For the upper boundary, we set $U_{N,j-1} = 0$.
- The put's value at any grid point at time step $j - 1$ is therefore an explicit function of its values at time step j.

Explicit Method (concluded)

- $U_{i,j}$ is set to the greater of the value derived above and $X - e^{V_{\min}+i\times\Delta V}$ for early-exercise considerations.
- Repeating this process as we move backward in time, we will eventually arrive at the solution at time zero, $U_{k,0}$.
  - k is the integer so that $V_{\min} + k \times \Delta V$ is closest to $\ln S_0$.
  - Interpolation is another alternative.
- By the stability condition, given ΔV, the value of Δt must be small enough for the method to converge.
  - The conditions to satisfy are $a > 0$, $b > 0$, and $c > 0$.

Region of Influence

- The explicit method evaluates all the grid points in a rectangle.
- But we are only interested in the single grid point at time zero, $(0, k)$, that corresponds to the current stock price.
- The grid points that may influence the desired value form a triangular subset of the rectangle.
- This triangle could be truncated further by the two boundary conditions (see figure on next page).
- Only those points within the truncated triangle need be evaluated.



Implicit Method

The partial differential equation now becomes the following $N - 1$ difference equations,

$$\begin{aligned} 0 &= \frac{1}{2}\sigma^2 \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta V)^2} \\ &\quad + \left(r - q - \frac{\sigma^2}{2}\right)\frac{U_{i+1,j} - U_{i-1,j}}{2\Delta V} - rU_{i,j} + \frac{U_{i,j+1} - U_{i,j}}{\Delta t} \end{aligned}$$

for $1 \le i \le N - 1$.

Implicit Method (continued)

Regroup the terms to obtain

$$aU_{i-1,j} + bU_{i,j} + cU_{i+1,j} = U_{i,j+1},$$

where

$$\begin{aligned}
a &\equiv \left(-\left(\frac{\sigma}{\Delta V} \right)^2 + \frac{r - q - \sigma^2/2}{\Delta V} \right) \frac{\Delta t}{2}, \\
b &\equiv 1 + r\Delta t + \left(\frac{\sigma}{\Delta V} \right)^2 \Delta t, \\
c &\equiv -\left(\left(\frac{\sigma}{\Delta V} \right)^2 + \frac{r - q - \sigma^2/2}{\Delta V} \right) \frac{\Delta t}{2}.
\end{aligned}$$

Implicit Method (continued)

The system of equations can be written in matrix form,

$$\begin{bmatrix}
b^* & c & 0 & \cdots & \cdots & \cdots & 0 \\
a & b & c & 0 & \cdots & \cdots & 0 \\
0 & a & b & c & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & a & b & c \\
0 & \cdots & \cdots & \cdots & 0 & a & b
\end{bmatrix}
\begin{bmatrix}
U_{1,j} \\ U_{2,j} \\ U_{3,j} \\ \vdots \\ \vdots \\ \vdots \\ U_{N-1,j}
\end{bmatrix}
=
\begin{bmatrix}
U_{1,j+1} - K \\ U_{2,j+1} \\ U_{3,j+1} \\ \vdots \\ \vdots \\ U_{N-2,j+1} \\ U_{N-1,j+1}
\end{bmatrix},$$

where $b^* \equiv a + b$ and $K \equiv a(e^{V_{\min} + \Delta V} - e^{V_{\min}})$.

Implicit Method (concluded)

- The values of

$$U_{1,j}, U_{2,j}, \ldots, U_{N-1,j}$$

  can be obtained by inverting the tridiagonal matrix.
  - But *never* literally invert a matrix numerically.
- As before, at every time step and before going to the next, we should set the option value to the intrinsic value of the option if the latter is larger.

Monte Carlo Simulation[a]

- Monte Carlo simulation is a sampling scheme.
- In many important applications within finance and without, Monte Carlo is one of the few feasible tools.
- When the time evolution of a stochastic process is not easy to describe analytically, Monte Carlo may very well be the only strategy that succeeds consistently.

[a] A top 10 algorithm according to Dongarra and Sullivan (2000).

The Big Idea

- Assume $X_1, X_2, \ldots, X_n$ have a joint distribution.
- $\theta \equiv E[\, g(X_1, X_2, \ldots, X_n)\,]$ for some function g is desired.
- We generate

$$\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)} \right), \quad 1 \leq i \leq N$$

independently with the same joint distribution as $(X_1, X_2, \ldots, X_n)$ and set

$$Y_i \equiv g\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)} \right).$$

The Big Idea (concluded)

- $Y_1, Y_2, \ldots, Y_N$ are independent and identically distributed random variables.
- Each Y_i has the same distribution as $Y \equiv g(X_1, X_2, \ldots, X_n)$.
- Since the average of these N random variables, $\overline{Y}$, satisfies $E[\,\overline{Y}\,] = \theta$, it can be used to estimate θ.
- The strong law of large numbers says that this procedure converges almost surely.
- The number of replications (or independent trials), N, is called the sample size.

Example

- Suppose we want to evaluate the definite integral $\int_a^b g(x)\, dx$ numerically.
- Consider the random variable $Y \equiv (b - a)\, g(X)$.
  - X is uniformly distributed over $[\, a, b\,]$.
  - Note that $\text{Prob}[\, X \leq x\,] = (x - a)/(b - a)$ for $a \leq x \leq b$.

Example (concluded)

- Note that

$$\begin{aligned} E[\, Y\,] &= (b - a)\, E[\, g(X)\,] \\ &= (b - a) \int_a^b \frac{g(x)}{b - a}\, dx \\ &= \int_a^b g(x)\, dx. \end{aligned}$$

- So any unbiased estimator of $E[\, Y\,]$ can be used to evaluate the integral.

Accuracy

- The Monte Carlo estimate and true value may differ owing to two reasons:
  1. Sampling variation.
  2. The discreteness of the sample paths.[a]
- The first can be controlled by the number of replications.
- The second can be controlled by the number of observations along the sample path.

[a]This may not be an issue if the derivative only requires discrete sampling along the time dimension.

Accuracy and Number of Replications

- The statistical error of the sample mean $\overline{Y}$ of the random variable Y grows as $1/\sqrt{N}$.
  - Because $\text{Var}[\overline{Y}\,] = \text{Var}[Y\,]/N$.
- In fact, this convergence rate is asymptotically optimal by the Berry-Esseen theorem.
- So the variance of the estimator $\overline{Y}$ can be reduced by a factor of $1/N$ by doing N times as much work.
- This is amazing because the same order of convergence holds independently of the dimension n.

Accuracy and Number of Replications (concluded)

- In contrast, classic numerical integration schemes have an error bound of $O(N^{-c/n})$ for some constant $c > 0$.
  - n is the dimension.
- The required number of evaluations thus grows exponentially in n to achieve a given level of accuracy.
  - The familiar curse of dimensionality.
- The Monte Carlo method, for example, is more efficient than alternative procedures for securities depending on more than one asset, the multivariate derivatives.

Brownian Bridge

- A Brownian bridge is one more alternative to generating the Wiener process.
- Let the time interval $[\,0, T\,]$ be partitioned at time points $t_0 = 0, t_1, t_2, \ldots, t_n = T$.
- We used to use

$$W(t_j) = W(t_{j-1}) + \sqrt{t_j - t_{j-1}}\,\xi, \quad \xi \sim N(0,1)$$

  to generate a sample path for the Wiener process.
- It is sequential.

Brownian Bridge (continued)

- The new method uses

$$W(t_j) = \frac{t_k - t_j}{t_k - t_i} W(t_i) + \frac{t_j - t_i}{t_k - t_i} W(t_k) + \sqrt{\frac{(t_k - t_j)(t_j - t_i)}{t_k - t_i}}\, \xi, \qquad (70)$$

  where $t_i < t_j < t_k$.

- The sample path is not generated sequentially: $W(t_i)$ is a past value and $W(t_k)$ a future value.
- It is critical to make sure that $W(t_i)$ and $W(t_k)$ are produced before $W(t_j)$.

Brownian Bridge (continued)

- Assume n is a power of two, 2^m.
- First set $W(0) = 0$ and $W(T) = \sqrt{T}\,\xi$.
- Then set the midpoint $W(T/2)$ according to Eq. (70) on p. 621.
- From here, we find the midpoints for intervals $(W(0), W(T/2))$ and $(W(T/2), W(T))$, that is, $W(T/4)$ and $W(3T/4)$, respectively.
- Iterate for $m - 2$ more times.

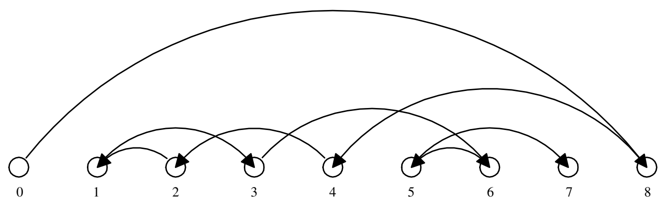
Brownian Bridge (continued)

- Consider 9 time points $t_0, t_1, \ldots, t_8$.
- The desired sample path $(W(t_0), W(t_1), \ldots, W(t_8))$ is generated in the sequence

  $W(t_0), W(t_8), W(t_4), W(t_2), W(t_1), W(t_3), W(t_6), W(t_5), W(t_7)$

  (see figure next page).

- In particular, $W(t_4)$ depends on $W(t_0)$ and $W(t_8)$, $W(t_2)$ depends on $W(t_0)$ and $W(t_4)$, $W(t_1)$ depends on $W(t_0)$ and $W(t_2)$, and so on.



Brownian Bridge (continued)

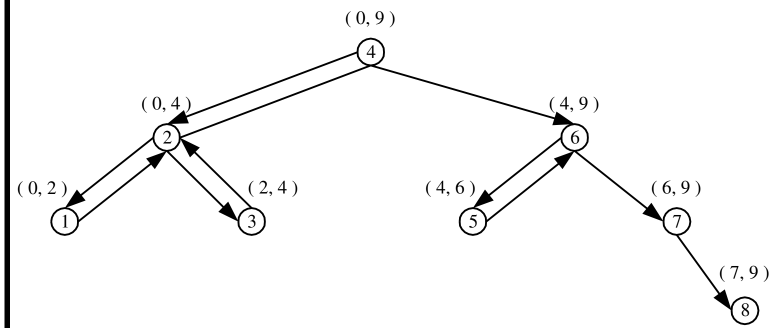
- Now we tackle the general case.
- As before, set $W(0) = 0$ and $W(t_n) = \sqrt{t_n}\ \xi$ first.
- Next, determine the midpoint for the interval (t_0, t_n) as $t = t_{t_0 + \lfloor (t_n - t_0)/2 \rfloor}$.
- Set $W(t)$ from $W(t_0)$ and $W(t_n)$ according to Eq. (70) on p. 621.
- Do the same thing for the problem for each of the smaller intervals (t_0, t) and (t, t_n).

Brownian Bridge (continued)

- A top-down tree best demonstrates the working of the procedure.
- Each tree node contains an integer j that signifies $W(t_j)$ is to be generated.
- The interval (t_i, t_k) over the node records the fact that $W(t_j)$ depends on $W(t_i)$ and $W(t_k)$.

Brownian Bridge (continued)

- Consider the case with 10 time points $t_0, t_1, \ldots, t_9$.
- The generation sequence starts with $W(t_0)$ and $W(t_9)$.
- The algorithm then proceeds to tackle the interval (t_0, t_9).
- The midpoint being t_4, the tree records 4 for the root node.
- There are now two smaller problems (t_0, t_4) and (t_4, t_9) to solve.
- Each of the two problems is again solved in the same fashion, starting with the left one first.



The generation sequence is

$W(t_0), W(t_9), W(t_4), W(t_2), W(t_1), W(t_3), W(t_6), W(t_5), W(t_7), W(t_8)$.

Brownian Bridge (concluded)

- The tree makes the data dependency clear.
  - $W(t_5)$ uses $W(t_4)$ and $W(t_6)$ in the application of Eq. (70) on p. 621.