General Derivatives Pricing

- In general the underlying asset S may not be traded.
  - Interest rate, for instance, is not a traded security.
- Let S follow the Ito process $dS/S = \boldsymbol{\mu}\, dt + \boldsymbol{\sigma}\, dW$, where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ may depend only on S and t.
- Let $f_1(S, t)$ and $f_2(S, t)$ be the prices of two derivatives with dynamics $df_i / f_i = \mu_i\, dt + \sigma_i\, dW$, $i = 1, 2$.
  - They share the same Wiener process as S.

General Derivatives Pricing (continued)

- A portfolio consisting of $\sigma_2 f_2$ units of the first derivative and $-\sigma_1 f_1$ units of the second derivative is instantaneously riskless:

$$\begin{aligned} & \sigma_2 f_2\, df_1 - \sigma_1 f_1\, df_2 \\ = \; & \sigma_2 f_2 f_1 (\mu_1\, dt + \sigma_1\, dW) - \sigma_1 f_1 f_2 (\mu_2\, dt + \sigma_2\, dW) \\ = \; & (\sigma_2 f_2 f_1 \mu_1 - \sigma_1 f_1 f_2 \mu_2)\, dt. \end{aligned}$$

- Therefore,

$$(\sigma_2 f_2 f_1 \mu_1 - \sigma_1 f_1 f_2 \mu_2)\, dt = r(\sigma_2 f_2 f_1 - \sigma_1 f_1 f_2)\, dt,$$

or $\sigma_2 \mu_1 - \sigma_1 \mu_2 = r(\sigma_2 - \sigma_1)$.

General Derivatives Pricing (continued)

- After rearranging the terms,

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \equiv \lambda \;\text{ for some }\; \lambda.$$

- A derivative whose value depends only on S and t and which follows the Ito process $df / f = \mu\, dt + \sigma\, dW$ must thus satisfy

$$\frac{\mu - r}{\sigma} = \lambda \;\text{ or }\; \mu = r + \lambda\sigma. \tag{59}$$

- We call λ the market price of risk, which is independent of the specifics of the derivative.

General Derivatives Pricing (continued)

- Ito's lemma can be used to derive the formulas for μ and σ:

$$\begin{aligned} \mu &= \frac{1}{f} \left(\frac{\partial f}{\partial t} + \boldsymbol{\mu} S \frac{\partial f}{\partial S} + \frac{1}{2} \boldsymbol{\sigma}^2 S^2 \frac{\partial^2 f}{\partial S^2} \right), \\ \sigma &= \frac{\boldsymbol{\sigma} S}{f} \frac{\partial f}{\partial S}. \end{aligned}$$

- Substitute the above into Eq. (59) on p. 509 to obtain

$$\frac{\partial f}{\partial t} + (\boldsymbol{\mu} - \lambda \boldsymbol{\sigma})\, S \frac{\partial f}{\partial S} + \frac{1}{2} \boldsymbol{\sigma}^2 S^2 \frac{\partial^2 f}{\partial S^2} = r f. \tag{60}$$

General Derivatives Pricing (concluded)

- The presence of $\boldsymbol{\mu}$ shows that the investor's risk preference is relevant.
- The derivative may be dependent on the underlying asset's growth rate and the market price of risk.
- Only when the underlying variable is the price of a traded security can we assume $\boldsymbol{\mu} = r$ in pricing.
  - Note that in such a case, $\lambda = 0$.

Hedging

When Professors Scholes and Merton and I
invested in warrants,
Professor Merton lost the most money.
And I lost the least.
— Fischer Black (1938–1995)

Delta Hedge

- The delta (hedge ratio) of a derivative f is defined as $\Delta \equiv \partial f / \partial S$.
- Thus $\Delta f \approx \Delta \times \Delta S$ for relatively small changes in the stock price, ΔS.
- A delta-neutral portfolio is hedged in the sense that it is immunized against small changes in the stock price.
- A trading strategy that dynamically maintains a delta-neutral portfolio is called delta hedge.

Delta Hedge (concluded)

- Delta changes with the stock price.
- A delta hedge needs to be rebalanced periodically in order to maintain delta neutrality.
- In the limit where the portfolio is adjusted continuously, perfect hedge is achieved and the strategy becomes self-financing.
- This was the gist of the Black-Scholes-Merton argument.

Implementing Delta Hedge

- We want to hedge N short derivatives.
- Assume the stock pays no dividends.
- The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus B borrowed dollars such that

$$-N \times f + N \times \Delta \times S - B = 0.$$

- At next rebalancing point when the delta is Δ', buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares with a total borrowing of $B' = N \times \Delta' \times S' - N \times f'$.
- Delta hedge is the discrete-time analog of the continuous-time limit and will rarely be self-financing.

Example

- A hedger is short 10,000 European calls.
- $\sigma = 30\%$ and $r = 6\%$.
- This call's expiration is four weeks away, its strike price is \$50, and each call has a current value of $f = 1.76791$.
- As an option covers 100 shares of stock, $N = 1{,}000{,}000$.
- The trader adjusts the portfolio weekly.
- The calls are replicated well if the cumulative cost of trading *stock* is close to the call premium's FV.

Example (continued)

- As $\Delta = 0.538560$, $N \times \Delta = 538,560$ shares are purchased for a total cost of $538{,}560 \times 50 = 26{,}928{,}000$ dollars to make the portfolio delta-neutral.
- The trader finances the purchase by borrowing

$$B = N \times \Delta \times S - N \times f = 25{,}160{,}090$$

  dollars net.
- The portfolio has zero net value now.

Example (continued)

- At 3 weeks to expiration, the stock price rises to \$51.
- The new call value is $f' = 2.10580$.
- So the portfolio is worth
$$-N \times f' + 538{,}560 \times 51 - Be^{0.06/52} = 171,622$$
before rebalancing.

Example (continued)

- A delta hedge does not replicate the calls perfectly; it is not self-financing as \$171,622 can be withdrawn.
- The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.
- In fact, the tracking error is positive about 68% of the time even though its expected value is essentially zero.[a]
- It is furthermore proportional to vega.

[a] Boyle and Emanuel (1980).

Example (continued)

- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.
- With a higher delta $\Delta' = 0.640355$, the trader buys $N \times (\Delta' - \Delta) = 101,795$ shares for \$5,191,545.
- The number of shares is increased to $N \times \Delta' = 640,355$.

Example (continued)

- The cumulative cost is
$$26{,}928{,}000 \times e^{0.06/52} + 5{,}191{,}545 = 32{,}150{,}634.$$
- The total borrowed amount is
$$B' = 640{,}355 \times 51 - N \times f' = 30{,}552{,}305.$$
- The portfolio is again delta-neutral with zero value.

		Option value	Delta	Change in delta	No. shares bought	Cost of shares	Cumulative cost
τ	S	f	Δ		$N\times(5)$	$(1)\times(6)$	FV(8')+(7)
	(1)	(2)	(3)	(5)	(6)	(7)	(8)
4	50	1.7679	0.53856	—	538,560	26,928,000	26,928,000
3	51	2.1058	0.64036	0.10180	101,795	5,191,545	32,150,634
2	53	3.3509	0.85578	0.21542	215,425	11,417,525	43,605,277
1	52	2.2427	0.83983	−0.01595	−15,955	−829,660	42,825,960
0	54	4.0000	1.00000	0.16017	160,175	8,649,450	51,524,853

The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).

Example (concluded)

- At expiration, the trader has 1,000,000 shares.
- They are exercised against by the in-the-money calls for \$50,000,000.
- The trader is left with an obligation of

$$51{,}524{,}853 - 50{,}000{,}000 = 1{,}524{,}853,$$

which represents the replication cost.

- Compared with the FV of the call premium,

$$1{,}767{,}910 \times e^{0.06\times 4/52} = 1{,}776{,}088,$$

the net gain is $1{,}776{,}088 - 1{,}524{,}853 = 251{,}235$.

Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, Δf, due to changes in the stock price, ΔS.
- When ΔS is not small, the second-order term, gamma $\Gamma \equiv \partial^2 f/\partial S^2$, helps (theoretically).
- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.
- To meet this extra condition, one more security needs to be brought in.

Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call f_2 is brought in.
- To set up a delta-gamma hedge, we solve

$$\begin{aligned} -N \times f + n_1 \times S + n_2 \times f_2 - B &= 0 \quad \text{(self-financing)}, \\ -N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 &= 0 \quad \text{(delta neutrality)}, \\ -N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 &= 0 \quad \text{(gamma neutrality)}, \end{aligned}$$

for n_1, n_2, and B.

  - The gammas of the stock and bond are 0.

Other Hedges

- If volatility changes, delta-gamma hedge may not work well.
- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.
- To accomplish this, one more security has to be brought into the process.
- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.

Trees

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

And though the holes were rather small,
they had to count them all.
— The Beatles, *A Day in the Life* (1967)

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 227.
  - It cannot apply to American options.
- We will now apply it to price barrier options.

The Reflection Principle[a]

- Imagine a particle at position $(0, -\boldsymbol{a})$ on the integral lattice that is to reach $(n, -\boldsymbol{b})$.
- Without loss of generality, assume $\boldsymbol{a} > 0$ and $\boldsymbol{b} \geq 0$.
- This particle's movement:

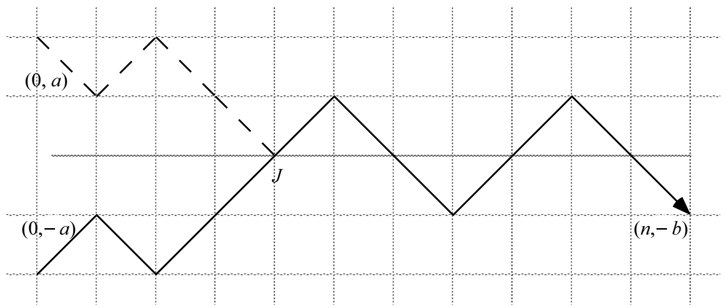
$$(i, j) \nearrow (i+1, j+1) \quad \text{up move } S \to Su$$
$$(i, j) \searrow (i+1, j-1) \quad \text{down move } S \to Sd$$

- How many paths touch the x axis?

[a] André (1887).

The Reflection Principle (continued)

- For a path from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touches the x axis, let J denote the first point this happens.
- Reflect the portion of the path from $(0, -\boldsymbol{a})$ to J.
- A path from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$ is constructed.
- It also hits the x axis at J for the first time (see figure next page).
- The one-to-one mapping shows the number of paths from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touch the x axis equals the number of paths from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$.



The Reflection Principle (concluded)

- A path of this kind has $(n + \boldsymbol{b} + \boldsymbol{a})/2$ down moves and $(n - \boldsymbol{b} - \boldsymbol{a})/2$ up moves.
- Hence there are

$$\binom{n}{\frac{n+\boldsymbol{a}+\boldsymbol{b}}{2}} \tag{61}$$

such paths for even $n + \boldsymbol{a} + \boldsymbol{b}$.

– Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define

$$a \equiv \left\lceil \frac{\ln\left(X/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil,$$

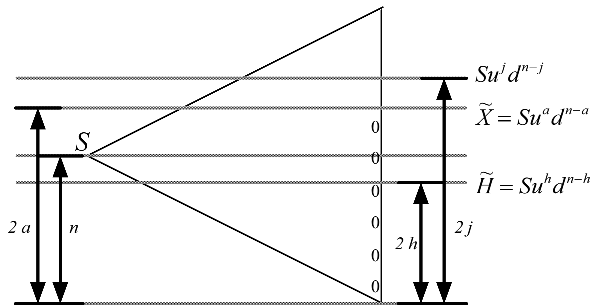
$$h \equiv \left\lfloor \frac{\ln\left(H/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor.$$

  - h is such that $\tilde{H} \equiv Su^h d^{n-h}$ is the terminal price that is closest to, but does not exceed H.
  - a is such that $\tilde{X} \equiv Su^a d^{n-a}$ is the terminal price that is closest to, but is not exceeded by X.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with n moves hence ends up in the money if and only if the number of up moves is at least a.
- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price Sd^n on the binomial tree.
  -

$$Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-2k}. \qquad (62)$$



Pricing Barrier Options (continued)

- The number of paths from S to the terminal price $Su^j d^{n-j}$ is $\binom{n}{j}$, each with probability $p^j (1-p)^{n-j}$.
- With reference to p. 537, the reflection principle can be applied with $\boldsymbol{a} = n - 2h$ and $\boldsymbol{b} = 2j - 2h$ in Eq. (61) on p. 534 by treating the S line as the x axis.
- Therefore,

$$\binom{n}{\frac{n+(n-2h)+(2j-2h)}{2}} = \binom{n}{n-2h+j}$$

paths hit $\tilde{H}$ in the process for $h \leq n/2$.

Pricing Barrier Options (concluded)

- The terminal price $Su^j d^{n-j}$ is reached by a path that hits the effective barrier with probability

$$\binom{n}{n-2h+j} p^j (1-p)^{n-j}.$$

- The option value equals

$$R^{-n} \sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - X \right). \tag{63}$$

  - $R \equiv e^{r\tau/n}$ is the riskless return per period.
- It implies a linear-time algorithm.

Convergence of BOPM

- Equation (63) results in the sawtooth-like convergence shown on p. 299.
- The reasons are not hard to see.
- The true barrier most likely does not equal the effective barrier.
- The same holds for the strike price and the effective strike price.
- The issue of the strike price is less critical.
- But the issue of the barrier is not negligible.

Convergence of BOPM (continued)

- Convergence is actually good if we limit n to certain values—191, for example.
- These values make the true barrier coincide with or occur just above one of the stock price levels, that is, $H \approx Sd^j = Se^{-j\sigma\sqrt{\tau/n}}$ for some integer j.
- The preferred n's are thus

$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

- There is only one minor technicality left.

Convergence of BOPM (continued)

- We picked the effective barrier to be one of the $n+1$ possible terminal stock prices.
- However, the effective barrier above, Sd^j, corresponds to a terminal stock price only when $n-j$ is even by Eq. (62) on p. 536.[a]
- To close this gap, we decrement n by one, if necessary, to make $n-j$ an even number.

[a] We could have adopted the form Sd^j $(-n \leq j \leq n)$ for the effective barrier.

Convergence of BOPM (concluded)

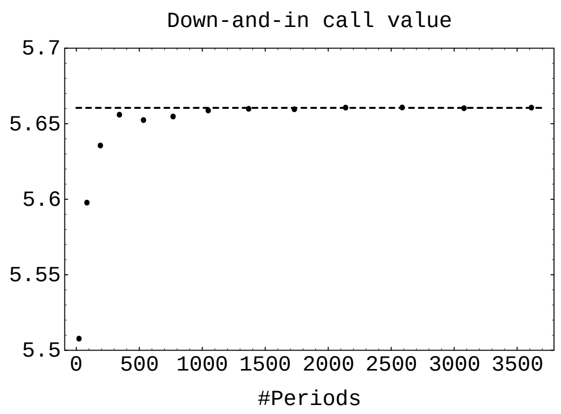
- The preferred n's are now

$$n = \begin{cases} \ell & \text{if } \ell - j \text{ is even} \\ \ell - 1 & \text{otherwise} \end{cases},$$

$j = 1, 2, 3, \ldots$, where

$$\ell \equiv \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.$$

- So evaluate pricing formula (63) on p. 539 only with the n's above.



Practical Implications

- Now that barrier options can be efficiently priced, we can afford to pick very large n's (p. 546).
- This has profound consequences.
- For example, pricing is prohibitively time consuming when $S \approx H$ because $n \sim 1/\ln^2(S/H)$.
- This observation is indeed true of standard quadratic-time binomial tree algorithms.
- But it no longer applies to linear-time algorithms (p. 547).

n	Combinatorial method	
	Value	Time (milliseconds)
21	5.507548	0.30
84	5.597597	0.90
191	5.635415	2.00
342	5.655812	3.60
533	5.652253	5.60
768	5.654609	8.00
1047	5.658622	11.10
1368	5.659711	15.00
1731	5.659416	19.40
2138	5.660511	24.70
2587	5.660592	30.20
3078	5.660099	36.70
3613	5.660498	43.70
4190	5.660388	44.10
4809	5.659955	51.60
5472	5.660122	68.70
6177	5.659981	76.70
6926	5.660263	86.90
7717	5.660272	97.20

	Barrier at 95.0			Barrier at 99.5			Barrier at 99.9	
n	Value	Time	n	Value	Time	n	Value	Time
	⋮		795	7.47761	8	19979	8.11304	253
2743	2.56095	31.1	3184	7.47626	38	79920	8.11297	1013
3040	2.56065	35.5	7163	7.47682	88	179819	8.11300	2200
3351	2.56098	40.1	12736	7.47661	166	319680	8.11299	4100
3678	2.56055	43.8	19899	7.47676	253	499499	8.11299	6300
4021	2.56152	48.1	28656	7.47667	368	719280	8.11299	8500
True	2.5615			7.4767			8.1130	

(All times in milliseconds.)

Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion $dS/S = r\,dt + \sigma\,dW$.[a]
- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Impose the matching of mean and that of variance:

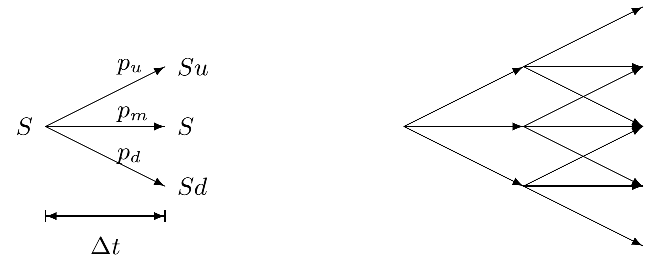
$$\begin{aligned} 1 &= p_u + p_m + p_d, \\ SM &\equiv (p_u u + p_m + (p_d/u))\, S, \\ S^2 V &\equiv p_u (Su - SM)^2 + p_m (S - SM)^2 + p_d (Sd - SM)^2. \end{aligned}$$

[a] Boyle (1988).

- Above,

$$\begin{aligned} M &\equiv e^{r \Delta t}, \\ V &\equiv M^2 (e^{\sigma^2 \Delta t} - 1), \end{aligned}$$

by Eqs. (18) on p. 144.



Trinomial Tree (continued)

- Use linear algebra to verify that

$$p_u = \frac{u\left(V + M^2 - M\right) - (M-1)}{(u-1)\left(u^2-1\right)},$$

$$p_d = \frac{u^2\left(V + M^2 - M\right) - u^3(M-1)}{(u-1)\left(u^2-1\right)}.$$

  - In practice, must make sure the probabilities lie between 0 and 1.
- Countless variations.

Trinomial Tree (concluded)

- Use $u = e^{\lambda\sigma\sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$p_u \rightarrow \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma},$$

$$p_d \rightarrow \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}.$$

- A nice choice for λ is $\sqrt{\pi/2}$.[a]

[a]Omberg (1988).

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting λ so that the barrier is hit exactly.[a]
- It takes

$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}$$

  consecutive down moves to go from S to H if h is an integer, which is easy to achieve by adjusting λ.

[a]Ritchken (1995).

Barrier Options Revisited (continued)

- Typically, we find the smallest $\lambda \geq 1$ such that h is an integer.
- That is, we find the largest integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ and then let

$$\lambda = \frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}}.$$

  - Such a λ may not exist for very small n's.
  - This is not hard to check.
- This done, one of the layers of the trinomial tree coincides with the barrier.

Barrier Options Revisited (concluded)

- The following probabilities may be used,

$$p_u = \frac{1}{2\lambda^2} + \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma},$$
$$p_m = 1 - \frac{1}{\lambda^2},$$
$$p_d = \frac{1}{2\lambda^2} - \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}.$$

  – $\mu' \equiv r - \sigma^2/2$.