Geometric Brownian Motion

- Consider the geometric Brownian motion process $Y(t) \equiv e^{X(t)}$
  - $X(t)$ is a (μ, σ) Brownian motion.
- As $\partial Y/\partial X = Y$ and $\partial^2 Y/\partial X^2 = Y$, Ito's formula (51) on p. 453 implies

$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right) dt + \sigma\, dW.$$

- The annualized instantaneous rate of return is $\mu + \sigma^2/2$ not μ.

Product of Geometric Brownian Motion Processes

- Let

$$\begin{aligned} dY/Y &= a\, dt + b\, dW_Y, \\ dZ/Z &= f\, dt + g\, dW_Z. \end{aligned}$$

- Consider the Ito process $U \equiv YZ$.
- Apply Ito's lemma (Theorem 18 on p. 457):

$$\begin{aligned} dU &= Z\, dY + Y\, dZ + dY\, dZ \\ &= ZY(a\, dt + b\, dW_Y) + YZ(f\, dt + g\, dW_Z) \\ &\quad + YZ(a\, dt + b\, dW_Y)(f\, dt + g\, dW_Z) \\ &= U(a + f + bg\rho)\, dt + Ub\, dW_Y + Ug\, dW_Z. \end{aligned}$$

Product of Geometric Brownian Motion Processes (continued)

- The product of two (or more) correlated geometric Brownian motion processes thus remains geometric Brownian motion.
- Note that

$$\begin{aligned} Y &= \exp\left[\left(a - b^2/2\right) dt + b\, dW_Y\right], \\ Z &= \exp\left[\left(f - g^2/2\right) dt + g\, dW_Z\right], \\ U &= \exp\left[\left(a + f - \left(b^2 + g^2\right)/2\right) dt + b\, dW_Y + g\, dW_Z\right]. \end{aligned}$$

Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.
- This holds even if Y and Z are correlated.
- Finally, $\ln Y$ and $\ln Z$ have correlation ρ.

Quotients of Geometric Brownian Motion Processes

- Suppose Y and Z are drawn from p. 460.
- Let $U \equiv Y/Z$.
- We now show that

$$\frac{dU}{U} = (a - f + g^2 - bg\rho)\, dt + b\, dW_Y - g\, dW_Z. \tag{52}$$

- Keep in mind that dW_Y and dW_Z have correlation ρ.

Quotients of Geometric Brownian Motion Processes (concluded)

- The multidimensional Ito's lemma (Theorem 18 on p. 457) can be employed to show that

$$\begin{aligned}
& dU \\
= \; & (1/Z)\, dY - (Y/Z^2)\, dZ - (1/Z^2)\, dY\, dZ + (Y/Z^3)\, (dZ)^2 \\
= \; & (1/Z)(aY\, dt + bY\, dW_Y) - (Y/Z^2)(fZ\, dt + gZ\, dW_Z) \\
& -(1/Z^2)(bgYZ\rho\, dt) + (Y/Z^3)(g^2 Z^2\, dt) \\
= \; & U(a\, dt + b\, dW_Y) - U(f\, dt + g\, dW_Z) \\
& -U(bg\rho\, dt) + U(g^2\, dt) \\
= \; & U(a - f + g^2 - bg\rho)\, dt + Ub\, dW_Y - Ug\, dW_Z.
\end{aligned}$$

Ornstein-Uhlenbeck Process

- The Ornstein-Uhlenbeck process:

$$dX = -\kappa X\, dt + \sigma\, dW,$$

  where $\kappa, \sigma \geq 0$.
- It is known that

$$\begin{aligned}
E[\, X(t)\,] &= e^{-\kappa(t-t_0)}\, E[\, x_0\,], \\
\mathrm{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-t_0)}\right) + e^{-2\kappa(t-t_0)}\, \mathrm{Var}[\, x_0\,], \\
\mathrm{Cov}[\, X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa(t-s)}\left[\, 1 - e^{-2\kappa(s-t_0)}\,\right] \\
&\quad + e^{-\kappa(t+s-2t_0)}\, \mathrm{Var}[\, x_0\,],
\end{aligned}$$

  for $t_0 \leq s \leq t$ and $X(t_0) = x_0$.

Ornstein-Uhlenbeck Process (continued)

- $X(t)$ is normally distributed if x_0 is a constant or normally distributed.
- X is said to be a normal process.
- $E[\, x_0\,] = x_0$ and $\mathrm{Var}[\, x_0\,] = 0$ if x_0 is a constant.
- The Ornstein-Uhlenbeck process has the following mean reversion property.
  - When $X > 0$, X is pulled X toward zero.
  - When $X < 0$, it is pulled toward zero again.

Ornstein-Uhlenbeck Process (continued)

- Another version:

$$dX = \kappa(\mu - X)\, dt + \sigma\, dW,$$

where $\sigma \geq 0$.

- Given $X(t_0) = x_0$, a constant, it is known that

$$\begin{aligned} E[\, X(t)\,] &= \mu + (x_0 - \mu)\, e^{-\kappa(t-t_0)}, \quad (53) \\ \text{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa} \left[1 - e^{-2\kappa(t-t_0)} \right], \end{aligned}$$

for $t_0 \leq t$.

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly μ and $\sigma/\sqrt{2\kappa}$, respectively.
- For large t, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$ (say $\mu > 4\sigma/\sqrt{2\kappa}$).
- The process is mean-reverting.
  - X tends to move toward μ.
  - Useful for modeling term structure, stock price volatility, and stock price return.

Interest Rate Models[a]

- Suppose the short rate r follows process $dr = \mu(r, t)\, dt + \sigma(r, t)\, dW$.
- Let $P(r, t, T)$ denote the price at time t of a zero-coupon bond that pays one dollar at time T.
- Write its dynamics as

$$\frac{dP}{P} = \mu_p\, dt + \sigma_p\, dW.$$

  - The expected instantaneous rate of return on a $(T - t)$-year zero-coupon bond is μ_p.
  - The instantaneous variance is σ_p^2.

[a] Merton (1970).

Interest Rate Models (continued)

- Surely $P(r, T, T) = 1$ for any T.
- By Ito's lemma (Theorem 17 on p. 455),

$$\begin{aligned} dP &= \frac{\partial P}{\partial T}\, dT + \frac{\partial P}{\partial r}\, dr + \frac{1}{2} \frac{\partial^2 P}{\partial r^2}\, (dr)^2 \\ &= -\frac{\partial P}{\partial T}\, dt + \frac{\partial P}{\partial r}\, [\, \mu(r, t)\, dt + \sigma(r, t)\, dW\,] \\ &\quad + \frac{1}{2} \frac{\partial^2 P}{\partial r^2}\, [\, \mu(r, t)\, dt + \sigma(r, t)\, dW\,]^2 \\ &= \left[-\frac{\partial P}{\partial T} + \mu(r, t)\, \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2}\, \frac{\partial^2 P}{\partial r^2} \right] dt \\ &\quad + \sigma(r, t)\, \frac{\partial P}{\partial r}\, dW. \end{aligned}$$

Interest Rate Models (concluded)

- Hence,

$$-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2} = P\mu_p, \quad (54)$$

$$\sigma(r,t)\,\frac{\partial P}{\partial r} = P\sigma_p.$$

- Models with the short rate as the only explanatory variable are called short rate models.

Continuous-Time Derivatives Pricing

I have hardly met a mathematician
who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Toward the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation.
- The key step is recognizing that the same random process drives both securities.
- As their prices are perfectly correlated, we figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio's return to be the riskless rate.

Assumptions

- The stock price follows $dS = \mu S\,dt + \sigma S\,dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at r.
- There is unlimited riskless borrowing and lending.
- t is the current time, T is the expiration time, and $\tau \equiv T - t$.

Black-Scholes Differential Equation

- Let C be the price of a derivative on S.
- From Ito's lemma (p. 455),

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\,\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S}\,dW.$$

  - The same W drives both C and S.
- Short one derivative and long $\partial C/\partial S$ shares of stock (call it Π).
- By construction,

$$\Pi = -C + S(\partial C/\partial S).$$

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is

$$d\Pi = -dC + \frac{\partial C}{\partial S}\,dS.$$

- Substitute the formulas for dC and dS into the partial differential equation to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\,\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r\Pi\,dt$.

Black-Scholes Differential Equation (concluded)

- So

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\,\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt = r \left(C - S\,\frac{\partial C}{\partial S} \right) dt.$$

- Equate the terms to finally obtain

$$\frac{\partial C}{\partial t} + rS\,\frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$

- When there is a dividend yield q,

$$\frac{\partial C}{\partial t} + (r - q)\,S\,\frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rC. \tag{55}$$

- Identity (55) leads to an alternative way of computing Θ numerically from Δ and Γ.
- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rC.$$

  - A definite relation thus exists between Γ and Θ.

PDEs for Asian Options

- Add the new variable $A(t) \equiv \int_0^t S(u)\, du$.
- Then the value V of the Asian option satisfies this two-dimensional PDE:[a]

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial A} = rV.$$

- The terminal conditions are

$$\begin{aligned} V(T,S,A) &= \max\left(\frac{A}{T} - X, 0\right) \quad \text{for call,} \\ V(T,S,A) &= \max\left(X - \frac{A}{T}, 0\right) \quad \text{for put.} \end{aligned}$$

[a] Kemna and Vorst (1990).

PDEs for Asian Options (continued)

- The two-dimensional PDE produces algorithms similar to that on pp. 316ff.
- But one-dimensional PDEs are available for Asian options.[a]
- For example, Večeř (2001) derives the following PDE for Asian calls:

$$\frac{\partial u}{\partial t} + r\left(1 - \frac{t}{T} - z\right)\frac{\partial u}{\partial z} + \frac{\left(1 - \frac{t}{T} - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

with the terminal condition $u(T, z) = \max(z, 0)$.

[a] Rogers and Shi (1995), Večeř (2001), and Dubois and Lelièvre (2005).

PDEs for Asian Options (concluded)

- For Asian puts:

$$\frac{\partial u}{\partial t} + r\left(\frac{t}{T} - 1 - z\right)\frac{\partial u}{\partial z} + \frac{\left(\frac{t}{T} - 1 - z\right)^2 \sigma^2}{2}\frac{\partial^2 u}{\partial z^2} = 0$$

with the same terminal condition.

- One-dimensional PDEs lead to highly efficient numerical methods.

Exchange Options[a]

- A correlation option has value dependent on multiple assets.
- An exchange option is a correlation option.
- It gives the holder the right to exchange one asset for another.
- Its value at expiration is thus

$$\max(S_2(T) - S_1(T), 0),$$

where $S_1(T)$ and $S_2(T)$ denote the prices of the two assets at expiration.

[a] Margrabe (1978).

Exchange Options (concluded)

- The payoff implies two ways of looking at the option.
  - It is a call on asset 2 with a strike price equal to the future price of asset 1.
  - It is a put on asset 1 with a strike price equal to the future value of asset 2.

Pricing of Exchange Options

- Assume that the two underlying assets do not pay dividends and that their prices follow

$$\frac{dS_1}{S_1} = \mu_1\, dt + \sigma_1\, dW_1,$$
$$\frac{dS_2}{S_2} = \mu_2\, dt + \sigma_2\, dW_2,$$

where ρ is the correlation between dW_1 and dW_2.

Pricing of Exchange Options (concluded)

- The option value at time t is

$$V(S_1, S_2, t) = S_2 N(x) - S_1 N(x - \sigma\sqrt{T-t}),$$

where

$$x \equiv \frac{\ln(S_2/S_1) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$\sigma^2 \equiv \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2. \quad (56)$$

- This is called Margrabe's formula.

Derivation of Margrabe's Formula

- Observe first that $V(x, y, t)$ is homogeneous of degree one in x and y.
  - That is, $V(\lambda S_1, \lambda S_2, t) = \lambda V(S_1, S_2, t)$.
  - An exchange option based on λ times the prices of the two assets is thus equal in value to λ original exchange options.
  - Intuitively, this is true because of

$$\max(\lambda S_2(T) - \lambda S_1(T), 0) = \lambda \times \max(S_2(T) - S_1(T), 0)$$

    and the perfect market assumption.

Derivation of Margrabe's Formula (continued)

- The price of asset 2 relative to asset 1 is $S \equiv S_2/S_1$.
- The diffusion of dS/S is $\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$ by Eq. (52) on p. 463 (proving Eq. (56) on p. 486).
- Hence, the option sells for

$$V(S_1, S_2, t)/S_1 = V(1, S_2/S_1, t)$$

  with asset 1 as the numeraire.

Derivation of Margrabe's Formula (continued)

- The interest rate on a riskless loan denominated in asset 1 is zero in a perfect market.
  - A lender of one unit of asset 1 demands one unit of asset 1 back as repayment of principal.
- The option to exchange asset 1 for asset 2 is a call on asset 2 with a strike price equal to unity and the interest rate equal to zero.

Derivation of Margrabe's Formula (concluded)

- So the Black-Scholes formula applies:

$$\begin{aligned} \frac{V(S_1, S_2, t)}{S_1} &= V(1, S, t) \\ &= SN(x) - 1 \times e^{-0 \times (T-t)} N(x - \sigma\sqrt{T-t}), \end{aligned}$$

  where

$$x \equiv \frac{\ln(S/1) + (0 + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln(S_2/S_1) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Margrabe's Formula with Dividends

- Margrabe's formula is not much more complicated if S_i pays out a continuous dividend yield of q_i, $i = 1, 2$.
- Simply replace each occurrence of S_i with $S_i e^{-q_i(T-t)}$ to obtain

$$\begin{aligned} V(S_1, S_2, t) &= S_2 e^{-q_2(T-t)} N(x) \\ &\quad - S_1 e^{-q_1(T-t)} N(x - \sigma\sqrt{T-t}), \\ x &\equiv \frac{\ln(S_2/S_1) + (q_1 - q_2 + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ \sigma^2 &\equiv \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2. \end{aligned} \tag{57}$$

Options on Foreign Currencies and Assets

- Correlation options involving foreign currencies and assets can be analyzed either take place in the domestic market or the foreign market before being converted back into the domestic currency.
- In the following, $S(t)$ denotes the spot exchange rate in terms of the domestic value of one unit of foreign currency.
- We knew from p. 305 that foreign currency is analogous to a stock paying a continuous dividend yield equal to the foreign riskless interest rate $r_{\rm f}$ in foreign currency.

Options on Foreign Currencies and Assets (concluded)

- So $S(t)$ follows the geometric Brownian motion process,

$$\frac{dS}{S} = (r - r_{\rm f})\, dt + \sigma_{\rm s}\, dW_{\rm s}(t),$$

in a risk-neutral economy.

- The foreign asset will be assumed to pay a continuous dividend yield of $q_{\rm f}$, and its price follows

$$\frac{dG_{\rm f}}{G_{\rm f}} = (\mu_{\rm f} - q_{\rm f})\, dt + \sigma_{\rm f}\, dW_{\rm f}(t)$$

in foreign currency.

- ρ is the correlation between $dW_{\rm s}$ and $dW_{\rm f}$.

Inverse Exchange Rates

- Suppose we have to work with the inverse of the exchange rate, $Y \equiv 1/S$, instead of S.
  - Because the option payoff is a function of Y; or
  - Because the parameters for Y are quoted in the markets but not S.
- Y follows

$$\frac{dY}{Y} = -(r - r_{\rm f} - \sigma_{\rm s}^2)\, dt - \sigma_{\rm s}\, dW_{\rm s}(t)$$

by Eq. (52) on p. 463.

Inverse Exchange Rates (concluded)

- Hence the volatility of Y equals that of S.
  - If a simulation of S gives wildly different sample volatilities for S and Y, you probably forgot to take logarithms before calculating the standard deviations.
- The correlation between Y and G_{f} equals

$$\frac{E[\,(-Y\sigma_{\mathrm{s}}\,dW_{\mathrm{s}})(G_{\mathrm{f}}\sigma_{\mathrm{f}}\,dW_{\mathrm{f}})\,]}{\sqrt{E[\,(-Y\sigma_{\mathrm{s}}\,dW_{\mathrm{s}})^2\,]E[\,(G_{\mathrm{f}}\sigma_{\mathrm{f}}\,dW_{\mathrm{f}})^2\,]}}$$

$$= \quad -\frac{E[\,dW_{\mathrm{s}}\,dW_{\mathrm{f}}\,]}{\sqrt{E[\,dW_{\mathrm{s}}^2\,]E[\,dW_{\mathrm{f}}^2\,]}} = -\rho$$

as the correlation between S and G_{f} is ρ.

Foreign Equity Options

- From Eq. (26) on p. 255, a European option on the foreign asset G_{f} with the terminal payoff $S(T) \times \max(G_{\mathrm{f}}(T) - X_{\mathrm{f}}, 0)$ is worth

$$C_{\mathrm{f}} = G_{\mathrm{f}} e^{-q_{\mathrm{f}}\tau} N(x) - X_{\mathrm{f}} e^{-r_{\mathrm{f}}\tau} N(x - \sigma_{\mathrm{f}}\sqrt{\tau})$$

in foreign currency.

  - Above,

$$x \equiv \frac{\ln(G_{\mathrm{f}}/X_{\mathrm{f}}) + (r_{\mathrm{f}} - q_{\mathrm{f}} + \sigma_{\mathrm{f}}^2/2)\,\tau}{\sigma_{\mathrm{f}}\sqrt{\tau}}.$$

  - X_{f} is the strike price in foreign currency.

Foreign Equity Options (concluded)

- Similarly, a European option on the foreign asset G_{f} with the terminal payoff $S(T) \times \max(X_{\mathrm{f}} - G_{\mathrm{f}}(T), 0)$ is worth

$$P_{\mathrm{f}} = X_{\mathrm{f}} e^{-r_{\mathrm{f}}\tau} N(-x + \sigma_{\mathrm{f}}\sqrt{\tau}) - G_{\mathrm{f}} e^{-q_{\mathrm{f}}\tau} N(-x)$$

in foreign currency.

- They will fetch SC_{f} and SP_{f}, respectively, in domestic currency.
- These options are called foreign equity options struck in foreign currency.

Foreign Domestic Options

- Foreign equity options fundamentally involve values in the foreign currency.
- A foreign equity call may allow the holder to participate in a foreign market rally.
- But the profits can be wiped out if the foreign currency depreciates against the domestic currency.
- What is really needed is a call in *domestic* currency with a payoff of $\max(S(T)\,G_{\mathrm{f}}(T) - X, 0)$.
  - For foreign equity options, the strike price in domestic currency is the uncertain $S(T)\,X_{\mathrm{f}}$.
- This is called a foreign domestic option.

Pricing of Foreign Domestic Options

- To foreign investors, this call is an option to exchange X units of domestic currency (foreign currency to them) for one share of foreign asset (domestic asset to them).
- It is an exchange option, that is.
- By Eq. (57) on p. 491, its price in foreign currency equals

$$G_{\rm f} e^{-q_{\rm f}\tau} N(x) - \frac{X}{S}\, e^{-r\tau} N(x - \sigma\sqrt{\tau}),$$

$$x \equiv \frac{\ln(G_{\rm f}S/X) + (r - q_{\rm f} + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}},$$

$$\sigma^2 \equiv \sigma_{\rm s}^2 + 2\rho\sigma_{\rm s}\sigma_{\rm f} + \sigma_{\rm f}^2.$$

Pricing of Foreign Domestic Options (concluded)

- The domestic price is therefore

$$C = SG_{\rm f} e^{-q_{\rm f}\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}).$$

- Similarly, a put has a price of

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SG_{\rm f} e^{-q_{\rm f}\tau} N(-x).$$

Cross-Currency Options

- A cross-currency option is an option in which the currency of the strike price is different from the currency in which the underlying asset is denominated.
  - An option to buy 100 yen at a strike price of 1.18 Canadian dollars provides one example.
- Usually, a third currency, the U.S. dollar, is involved because of the lack of relevant exchange-traded options for the two currencies in question (yen and Canadian dollars in the above example).
- So the notations below will be slightly different.

Cross-Currency Options (continued)

- Let $S_{\rm A}$ denote the price of the foreign asset and $S_{\rm C}$ the price of currency C that the strike price X is based on.
- Both $S_{\rm A}$ and $S_{\rm C}$ are in U.S. dollars, say.
- If S is the price of the foreign asset as measured in currency C, then we have the triangular arbitrage $S = S_{\rm A}/S_{\rm C}$.[a]

[a]Triangular arbitrage had been known for centuries. See Montesquieu's *The Spirit of Laws.*

Cross-Currency Options (concluded)

- Assume $S_{\rm A}$ and $S_{\rm C}$ follow the geometric Brownian motion processes $dS_{\rm A}/S_{\rm A} = \mu_{\rm A}\, dt + \sigma_{\rm A}\, dW_{\rm A}$ and $dS_{\rm C}/S_{\rm C} = \mu_{\rm C}\, dt + \sigma_{\rm C}\, dW_{\rm C}$, respectively.
  - Parameters $\sigma_{\rm A}$, $\sigma_{\rm C}$, and ρ can be inferred from exchange-traded options.
- By an exercise in the text,

$$\frac{dS}{S} = (\mu_{\rm A} - \mu_{\rm C} + \sigma_{\rm C}^2 - \rho\sigma_{\rm A}\sigma_{\rm C})\, dt + \sigma_{\rm A}\, dW_{\rm A} - \sigma_{\rm C}\, dW_{\rm C},$$

  where ρ is the correlation between $dW_{\rm A}$ and $dW_{\rm C}$.
- The volatility of dS/S is hence $(\sigma_{\rm A}^2 - 2\rho\sigma_{\rm A}\sigma_{\rm C} + \sigma_{\rm C}^2)^{1/2}$.

Quanto Options

- Consider a call with a terminal payoff $\widehat{S} \times \max(G_{\rm f}(T) - X_{\rm f}, 0)$ in domestic currency, where $\widehat{S}$ is a constant.
- This amounts to fixing the exchange rate to $\widehat{S}$.
  - For instance, a call on the Nikkei 225 futures, if it existed, fits this framework with $\widehat{S} = 5$ and $G_{\rm f}$ denoting the futures price.
- A guaranteed exchange rate option is called a quanto option or simply a quanto.

Quanto Options (continued)

- The process $U \equiv \widehat{S} G_{\rm f}$ in a risk-neutral economy follows

$$\frac{dU}{U} = (r_{\rm f} - q_{\rm f} - \rho\sigma_{\rm s}\sigma_{\rm f})\, dt + \sigma_{\rm f}\, dW \qquad (58)$$

  in domestic currency.
- Hence, it can be treated as a stock paying a continuous dividend yield of $q \equiv r - r_{\rm f} + q_{\rm f} + \rho\sigma_{\rm s}\sigma_{\rm f}$.
- Apply Eq. (26) on p. 255 to obtain

$$C = \widehat{S}\, (G_{\rm f}\, e^{-q\tau} N(x) - X_{\rm f} e^{-r\tau} N(x - \sigma_{\rm f}\sqrt{\tau}))$$
$$P = \widehat{S}\, (X_{\rm f}\, e^{-r\tau} N(-x + \sigma_{\rm f}\sqrt{\tau}) - G_{\rm f} e^{-q\tau} N(-x))$$

  where $x \equiv \frac{\ln(G_{\rm f}/X_{\rm f}) + (r - q + \sigma_{\rm f}^2/2)\,\tau}{\sigma_{\rm f}\sqrt{\tau}}$.

Quanto Options (concluded)

- In general, a quanto derivative has nominal payments in the foreign currency which are converted into the domestic currency at a fixed exchange rate.
- A cross-rate swap, for example, is like a currency swap except that the foreign currency payments are converted into the domestic currency at a fixed exchange rate.
- Quanto derivatives form a rapidly growing segment of international financial markets.