Example

- Consider the stochastic process

$$\{ Z_n \equiv \sum_{i=1}^{n} X_i, n \geq 1 \},$$

where X_i are independent random variables with zero mean.

- This process is a martingale because

$$\begin{aligned} & E[\, Z_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \; & E[\, Z_n + X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \; & E[\, Z_n \,|\, Z_1, Z_2, \ldots, Z_n \,] + E[\, X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n \,] \\ = \; & Z_n + E[\, X_{n+1} \,] = Z_n. \end{aligned}$$

Probability Measure

- A martingale is defined with respect to a probability measure, under which the expectation is taken.
  - A probability measure assigns probabilities to states of the world.
- A martingale is also defined with respect to an information set.
  - In the characterizations (41)–(42) on p. 397, the information set contains the current and past values of X by default.
  - But it needs not be so.

Probability Measure (continued)

- A stochastic process $\{ X(t), t \geq 0 \}$ is a martingale with respect to information sets $\{ I_t \}$ if, for all $t \geq 0$, $E[\,|\, X(t) \,|\,] < \infty$ and

$$E[\, X(u) \,|\, I_t \,] = X(t)$$

for all $u > t$.

- The discrete-time version: For all $n > 0$,

$$E[\, X_{n+1} \,|\, I_n \,] = X_n,$$

given the information sets $\{ I_n \}$.

Probability Measure (concluded)

- The above implies $E[\, X_{n+m} \,|\, I_n \,] = X_n$ for any $m > 0$ by Eq. (16) on p. 136.
  - A typical I_n is the price information up to time n.
  - Then the above identity says the FVs of X will not deviate systematically from today's value given the price history.

Example

- Consider the stochastic process $\{ Z_n - n\mu, n \geq 1 \}$.
  - $Z_n \equiv \sum_{i=1}^{n} X_i$.
  - $X_1, X_2, \ldots$ are independent random variables with mean μ.
- Now,

$$\begin{aligned} & E[\, Z_{n+1} - (n+1)\,\mu \,|\, X_1, X_2, \ldots, X_n \,] \\ = \; & E[\, Z_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\ = \; & Z_n + \mu - (n+1)\,\mu \\ = \; & Z_n - n\mu. \end{aligned}$$

Example (concluded)

- Define
$$I_n \equiv \{ X_1, X_2, \ldots, X_n \}.$$
- Then
$$\{ Z_n - n\mu, n \geq 1 \}$$
is a martingale with respect to $\{ I_n \}$.

Martingale Pricing

- Recall that the price of a European option is the expected discounted future payoff at expiration in a risk-neutral economy.
- This principle can be generalized using the concept of martingale.
- Recall the recursive valuation of European option via
$$C = [\, pC_u + (1-p)\, C_d \,]/R.$$
  - p is the risk-neutral probability.
  - \$1 grows to \$$R$ in a period.

Martingale Pricing (continued)

- Let $C(i)$ denote the value of the option at time i.
- Consider the discount process
$$\{ C(i)/R^i, i = 0, 1, \ldots, n \}.$$
- Then,
$$E\left[\left. \frac{C(i+1)}{R^{i+1}} \,\right|\, C(i) = C \right] = \frac{pC_u + (1-p)\, C_d}{R^{i+1}} = \frac{C}{R^i}.$$

Martingale Pricing (continued)

- In general,

$$E\left[\left.\frac{C(k)}{R^k}\,\right|\,C(i)=C\right]=\frac{C}{R^i},\quad i\le k. \tag{44}$$

- The discount process is a martingale:

$$\frac{C(i)}{R^i}=E_i^\pi\left[\frac{C(k)}{R^k}\right],\quad i\le k. \tag{45}$$

  - E_i^π is taken under the risk-neutral probability conditional on the price information up to time i.
- This risk-neutral probability is also called the EMM, or the equivalent martingale (probability) measure.

Martingale Pricing (continued)

- In general, Eq. (45) holds for all assets, not just options.
- When interest rates are stochastic, the equation becomes

$$\frac{C(i)}{M(i)}=E_i^\pi\left[\frac{C(k)}{M(k)}\right],\quad i\le k. \tag{46}$$

  - $M(j)$ is the balance in the money market account at time j using the rollover strategy with an initial investment of $1.
  - So it is called the bank account process.
- It says the discount process is a martingale under π.

Martingale Pricing (concluded)

- If interest rates are stochastic, then $M(j)$ is a random variable.
  - $M(0)=1$.
  - $M(j)$ is known at time $j-1$.
- Identity (46) on p. 411 is the general formulation of risk-neutral valuation.

Theorem 14 *A discrete-time model is arbitrage-free if and only if there exists a probability measure such that the discount process is a martingale. This probability measure is called the risk-neutral probability measure.*

Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability.
  - The expected futures price in the next period is

$$p_{\rm f}Fu+(1-p_{\rm f})\,Fd=F\left(\frac{1-d}{u-d}\,u+\frac{u-1}{u-d}\,d\right)=F$$

    (p. 374).
- Can be generalized to

$$F_i=E_i^\pi[\,F_k\,],\quad i\le k,$$

  where F_i is the futures price at time i.
- It holds under stochastic interest rates.

Martingale Pricing and Numeraire

- The martingale pricing formula (46) on p. 411 uses the money market account as numeraire.[a]
  - It expresses the price of any asset *relative to* the money market account.
- The money market account is not the only choice for numeraire.
- Suppose asset S's value is positive at all times.

[a]Leon Walras (1834–1910).

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Martingale Pricing and Numeraire (concluded)

- Choose S as numeraire.
- Martingale pricing says there exists a risk-neutral probability π under which the relative price of any asset C is a martingale:

$$\frac{C(i)}{S(i)} = E_i^{\pi}\left[\frac{C(k)}{S(k)}\right], \quad i \leq k.$$

  - $S(j)$ denotes the price of S at time j.
- So the discount process remains a martingale.

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Example

- Take the binomial model with two assets.
- In a period, asset one's price can go from S to S_1 or S_2.
- In a period, asset two's price can go from P to P_1 or P_2.
- Assume

$$\frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2}$$

to rule out arbitrage opportunities.

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Example (continued)

- For any derivative security, let C_1 be its price at time one if asset one's price moves to S_1.
- Let C_2 be its price at time one if asset one's price moves to S_2.
- Replicate the derivative by solving

$$\begin{aligned} \alpha S_1 + \beta P_1 &= C_1, \\ \alpha S_2 + \beta P_2 &= C_2, \end{aligned}$$

using α units of asset one and β units of asset two.

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Example (continued)

- This yields

$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \text{ and } \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}.$$

- The derivative costs

$$\begin{aligned} C &= \alpha S + \beta P \\ &= \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2. \end{aligned}$$

Example (concluded)

- It is easy to verify that

$$\frac{C}{P} = p \frac{C_1}{P_1} + (1-p) \frac{C_2}{P_2}.$$

  – Above,

$$p \equiv \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}.$$

- The derivative's price using asset two as numeraire is thus a martingale under the risk-neutral probability p.
- The expected returns of the two assets are irrelevant.

Brownian Motion[a]

- Brownian motion is a stochastic process $\{ X(t), t \geq 0 \}$ with the following properties.
  1. $X(0) = 0$, unless stated otherwise.
  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables

$$X(t_k) - X(t_{k-1})$$

  for $1 \leq k \leq n$ are independent.[b]
  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2 (t-s)$, where μ and $\sigma \neq 0$ are real numbers.

[a] Robert Brown (1773–1858).
[b] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

Brownian Motion (concluded)

- Such a process will be called a (μ, σ) Brownian motion with drift μ and variance σ^2.
- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.[a]
- Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is also called the Wiener process.

[a] Norbert Wiener (1894–1964).

Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then $X(t) - X(s) \sim N(0, t-s)$.
- A (μ, σ) Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t).$$

- As $Y(t+s) - Y(t) \sim N(\mu s, \sigma^2 s)$, uncertainty about the future value of Y grows as the square root of how far we look into the future.

Brownian Motion Is a Random Walk in Continuous Time

Claim 1 *A (μ, σ) Brownian motion is the limiting case of random walk.*

- A particle moves Δx to the left with probability $1-p$.
- It moves to the right with probability p after Δt time.
- Assume $n \equiv t/\Delta t$ is an integer.
- Its position at time t is

$$Y(t) \equiv \Delta x \,(X_1 + X_2 + \cdots + X_n)\,.$$

Brownian Motion as Limit of Random Walk (continued)

- (continued)
  - Here

$$X_i \equiv \begin{cases} +1 & \text{if the } i\text{th move is to the right,} \\ -1 & \text{if the } i\text{th move is to the left.} \end{cases}$$

  - X_i are independent with $\text{Prob}[\, X_i = 1\,] = p = 1 - \text{Prob}[\, X_i = -1\,]$.
- Recall $E[\, X_i\,] = 2p - 1$ and $\text{Var}[\, X_i\,] = 1 - (2p-1)^2$.

Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\, Y(t)\,] = n(\Delta x)(2p-1),$$
$$\text{Var}[\, Y(t)\,] = n(\Delta x)^2 \left[\, 1 - (2p-1)^2 \,\right].$$

- With $\Delta x \equiv \sigma \sqrt{\Delta t}$ and $p \equiv [\, 1 + (\mu/\sigma)\sqrt{\Delta t}\,]/2$,

$$\begin{aligned} E[\, Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t, \\ \text{Var}[\, Y(t)\,] &= n\sigma^2 \Delta t \left[\, 1 - (\mu/\sigma)^2 \Delta t \,\right] \to \sigma^2 t, \end{aligned}$$

as $\Delta t \to 0$.

Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{ Y(t), t \geq 0 \}$ converges to a (μ, σ) Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
- Note that

$$\begin{aligned} & \text{Var}[\, Y(t + \Delta t) - Y(t) \,] \\ =& \text{Var}[\, \Delta x \, X_{n+1} \,] = (\Delta x)^2 \times \text{Var}[\, X_{n+1} \,] \to \sigma^2 \Delta t. \end{aligned}$$

- Similarity to the the BOPM: The p is identical to the probability in Eq. (25) on p. 234 and $\Delta x = \ln u$.

Geometric Brownian Motion

- Let $X \equiv \{ X(t), t \geq 0 \}$ be a Brownian motion process.
- The process

$$\{ Y(t) \equiv e^{X(t)}, t \geq 0 \},$$

is called geometric Brownian motion.

- Suppose further that X is a (μ, σ) Brownian motion.
- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function

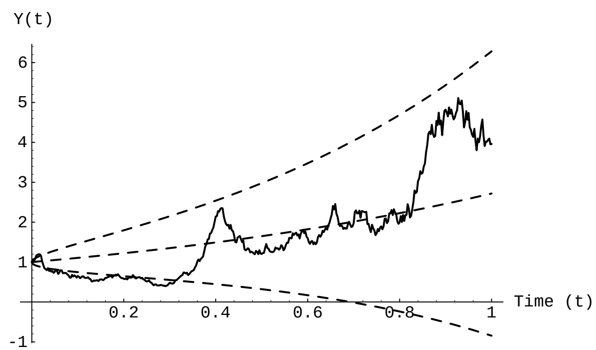
$$E \left[e^{s X(t)} \right] = E \left[\, Y(t)^s \,\right] = e^{\mu t s + (\sigma^2 t s^2 / 2)}$$

from Eq. (17) on p 138.

Geometric Brownian Motion (continued)

- In particular,

$$\begin{aligned} E[\, Y(t) \,] &= e^{\mu t + (\sigma^2 t / 2)}, \\ \text{Var}[\, Y(t) \,] &= E \left[\, Y(t)^2 \,\right] - E[\, Y(t) \,]^2 \\ &= e^{2\mu t + \sigma^2 t} \left(e^{\sigma^2 t} - 1 \right). \end{aligned}$$



Geometric Brownian Motion (continued)

- It is useful for situations in which percentage changes are independent and identically distributed.
- Let Y_n denote the stock price at time n and $Y_0 = 1$.
- Assume relative returns

$$X_i \equiv \frac{Y_i}{Y_{i-1}}$$

are independent and identically distributed.

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Geometric Brownian Motion (concluded)

- Then

$$\ln Y_n = \sum_{i=1}^{n} \ln X_i$$

is a sum of independent, identically distributed random variables.
- Thus $\{ \ln Y_n, n \geq 0 \}$ is approximately Brownian motion.
  - And $\{ Y_n, n \geq 0 \}$ is approximately geometric Brownian motion.

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Continuous-Time Financial Mathematics

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A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an
unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a
divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
Science and the Modern World

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Stochastic Integrals

- Use $W \equiv \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of X from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \equiv \int_0^t X \, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of X with respect to W.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X \, dW$.

[a] Ito (1915–).

Stochastic Integrals (concluded)

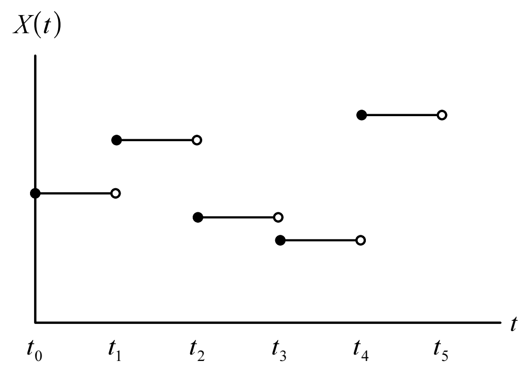
- Typical requirements for X in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s) \, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)] \, ds < \infty$.
  - The information set at time t includes the history of X and W up to that point in time.
  - But it contains nothing about the evolution of X or W after t (nonanticipating, so to speak).
  - The future cannot influence the present.
- $\{ X(s), 0 \leq s \leq t \}$ is independent of $\{ W(t+u) - W(t), u > 0 \}$.

Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist $0 = t_0 < t_1 < \cdots$ such that

$$X(t) = X(t_{k-1}) \quad \text{for } t \in [\, t_{k-1}, t_k), \; k = 1, 2, \ldots$$

for any realization (see figure next page).



Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\mathrm{I}_t(X) \equiv \sum_{k=0}^{n-1} X(t_k)[\,W(t_{k+1}) - W(t_k)\,], \tag{47}$$

  where $t_n = t$.
  - The integrand X is evaluated at t_k, not t_{k+1}.
- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (continued)

- Let $X = \{\, X(t), t \geq 0 \,\}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that X_n converges in probability to X.
- If X is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as $\delta_n \equiv \max_{1 \leq k \leq n}(t_k - t_{k-1})$ goes to zero.

Ito Integral (concluded)

- It is a fundamental fact that $\int X\,dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.
- A corollary is the mean value formula

$$E\left[\int_a^b X\,dW\right] = 0.$$

Theorem 15 *The Ito integral* $\int X\,dW$ *is a martingale.*

Discrete Approximation

- Recall Eq. (47) on p. 438.
- The following simple stochastic process $\{\,\widehat{X}(t)\,\}$ can be used in place of X to approximate the stochastic integral $\int_0^t X\,dW$,

$$\widehat{X}(s) \equiv X(t_{k-1}) \quad \text{for } s \in [\,t_{k-1}, t_k), \ k = 1, 2, \ldots, n.$$

- Note the nonanticipating feature of $\widehat{X}$.
  - The information up to time s,

$$\{\,\widehat{X}(t), W(t), 0 \leq t \leq s\,\},$$

    cannot determine the future evolution of X or W.

Discrete Approximation (concluded)

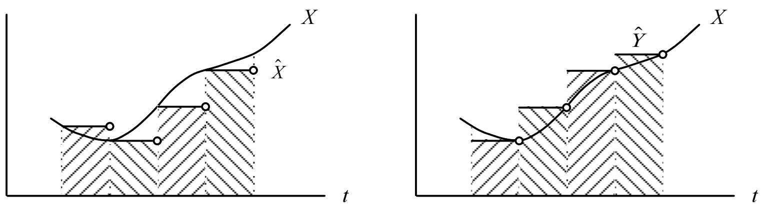
- Suppose we defined the stochastic integral as

$$\sum_{k=0}^{n-1} X(t_{k+1})[\,W(t_{k+1}) - W(t_k)\,].$$

- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \equiv X(t_k) \;\text{ for }\; s \in [\,t_{k-1}, t_k),\, k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of X.



(a) (b)

Ito Process

- The stochastic process $X = \{\, X_t, t \geq 0 \,\}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

is called an Ito process.
  - X_0 is a scalar starting point.
  - $\{\, a(X_t, t) : t \geq 0 \,\}$ and $\{\, b(X_t, t) : t \geq 0 \,\}$ are stochastic processes satisfying certain regularity conditions.
- The terms $a(X_t, t)$ and $b(X_t, t)$ are the drift and the diffusion, respectively.

Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation for the Ito differential dX_t,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{48}$$

  - Or simply $dX_t = a_t\, dt + b_t\, dW_t$.
- This is Brownian motion with an instantaneous drift a_t and an instantaneous variance b_t^2.
- X is a martingale if the drift a_t is zero by Theorem 15 (p. 440).

[a] Paul Langevin (1904).

Ito Process (concluded)

- dW is normally distributed with mean zero and variance dt.
- An equivalent form to Eq. (48) is
$$dX_t = a_t\, dt + b_t \sqrt{dt}\, \xi, \tag{49}$$
where $\xi \sim N(0,1)$.
- This formulation makes it easy to derive Monte Carlo simulation algorithms.

Euler Approximation

- The following approximation follows from Eq. (49),
$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\,\Delta W(t_n),\end{aligned} \tag{50}$$
where $t_n \equiv n\Delta t$.
- It is called the Euler or Euler-Maruyama method.
- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.
- Recall that $\Delta W(t_n)$ should be interpreted as $W(t_{n+1}) - W(t_n)$ instead of $W(t_n) - W(t_{n-1})$.

More Discrete Approximations

- Under fairly loose regularity conditions, approximation (50) on p. 447 can be replaced by
$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,Y(t_n).\end{aligned}$$
  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- A simpler discrete approximation scheme:
$$\begin{aligned}&\widehat{X}(t_{n+1})\\ =&\widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,\xi.\end{aligned}$$
  - $\text{Prob}[\,\xi = 1\,] = \text{Prob}[\,\xi = -1\,] = 1/2$.
  - Note that $E[\,\xi\,] = 0$ and $\text{Var}[\,\xi\,] = 1$.
- This clearly defines a binomial model.
- As Δt goes to zero, $\widehat{X}$ converges to X.

Trading and the Ito Integral

- Consider an Ito process $dS_t = \mu_t \, dt + \sigma_t \, dW_t$.
  - S_t is the vector of security prices at time t.
- Let ϕ_t be a trading strategy denoting the quantity of each type of security held at time t.
- Hence the stochastic process $\phi_t S_t$ is the value of the portfolio ϕ_t at time t.
- $\phi_t \, dS_t \equiv \phi_t(\mu_t \, dt + \sigma_t \, dW_t)$ represents the change in the value from security price changes occurring at time t.

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\phi) \equiv \int_0^T \phi_t \, dS_t = \int_0^T \phi_t \mu_t \, dt + \int_0^T \phi_t \sigma_t \, dW_t,$$

measures the gains realized by the trading strategy over the period $[0, T]$.

Ito's Lemma

A smooth function of an Ito process is itself an Ito process.

Theorem 16 *Suppose $f : R \to R$ is twice continuously differentiable and $dX = a_t \, dt + b_t \, dW$. Then $f(X)$ is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \; & f(X_0) + \int_0^t f'(X_s) \, a_s \, ds + \int_0^t f'(X_s) \, b_s \, dW \\ & + \frac{1}{2} \int_0^t f''(X_s) \, b_s^2 \, ds \end{aligned}$$

for $t \geq 0$.

Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$df(X) = f'(X) \, a \, dt + f'(X) \, b \, dW + \frac{1}{2} f''(X) \, b^2 \, dt. \tag{51}$$

- Compared with calculus, the interesting part is the third term on the right-hand side.
- A convenient formulation of Ito's lemma is

$$df(X) = f'(X) \, dX + \frac{1}{2} f''(X)(dX)^2.$$

Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to

$\times$	dW	dt
dW	dt	0
dt	0	0

  - The $(dW)^2 = dt$ entry is justified by a known result.
- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (continued)

Theorem 17 (Higher-Dimensional Ito's Lemma) *Let $W_1, W_2, \ldots, W_n$ be independent Wiener processes and $X \equiv (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : R^m \to R$ is twice continuously differentiable and X_i is an Ito process with $dX_i = a_i\,dt + \sum_{j=1}^n b_{ij}\,dW_j$. Then $df(X)$ is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^m f_i(X)\,dX_i + \frac{1}{2}\sum_{i=1}^m \sum_{k=1}^m f_{ik}(X)\,dX_i\,dX_k,$$

where $f_i \equiv \partial f/\partial x_i$ and $f_{ik} \equiv \partial^2 f/\partial x_i \partial x_k$.

Ito's Lemma (continued)

- The multiplication table for Theorem 17 is

$\times$	dW_i	dt
dW_k	$\delta_{ik}\,dt$	0
dt	0	0

  in which

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Ito's Lemma (continued)

Theorem 18 (Alternative Ito's Lemma) *Let $W_1, W_2, \ldots, W_m$ be Wiener processes and $X \equiv (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : R^m \to R$ is twice continuously differentiable and X_i is an Ito process with $dX_i = a_i\,dt + b_i\,dW_i$. Then $df(X)$ is the following Ito process,*

$$df(X) = \sum_{i=1}^m f_i(X)\,dX_i + \frac{1}{2}\sum_{i=1}^m \sum_{k=1}^m f_{ik}(X)\,dX_i\,dX_k.$$

Ito's Lemma (concluded)

- The multiplication table for Theorem 18 is

$\times$	dW_i	dt
dW_k	$\rho_{ik}\,dt$	0
dt	0	0

- Here, ρ_{ik} denotes the correlation between dW_i and dW_k.

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