Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
— Blaise Pascal (1623–1662)

Sensitivity Measures ("The Greeks")

- Understanding how the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- Let $x \equiv \frac{\ln(S/X)+(r+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$ (recall p. 239).
- Note that

$$N'(y) = (1/\sqrt{2\pi}\,)\,e^{-y^2/2} > 0,$$

the density function of standard normal distribution.

Delta

- Defined as $\Delta \equiv \partial f / \partial S$.
  - f is the price of the derivative.
  - S is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.
- The delta used in the BOPM is the discrete analog.

Delta (concluded)

- The delta of a European call on a non-dividend-paying stock equals
$$\frac{\partial C}{\partial S} = N(x) > 0.$$
- The delta of a European put equals
$$\frac{\partial P}{\partial S} = N(x) - 1 < 0.$$
- The delta of a long stock is 1.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
- A delta-neutral portfolio is immune to small price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  - Short Δ shares of stock to hedge a long call.
- In general, hedge a position in a security with a delta of Δ_1 by shorting Δ_1/Δ_2 units of a security with a delta of Δ_2.

Theta (Time Decay)

- Defined as the rate of change of a security's value with respect to time, or $\Theta \equiv -\partial f/\partial \tau$.
- For a European call on a non-dividend-paying stock,
$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) < 0.$$
  - The call loses value with the passage of time.
- For a European put,
$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}).$$
  - Can be negative or positive.

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \equiv \partial^2\Pi/\partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- A portfolio with a high gamma needs in practice be rebalanced more often to maintain delta neutrality.
- Delta $\sim$ duration; gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is
$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$

Vega[a] (Lambda, Kappa, Sigma)

- Defined as the rate of change of its value with respect to the volatility of the underlying asset $\Lambda \equiv \partial \Pi / \partial \sigma$.
- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\, N'(x) > 0$.
  - Higher volatility increases option value.

[a]Vega is not Greek.

Rho

- Defined as the rate of change in its value with respect to interest rates $\rho \equiv \partial \Pi / \partial r$.
- The rho of a European call on a non-dividend-paying stock is
$$X\tau e^{-r\tau} N(x - \sigma\sqrt{\tau}) > 0.$$
- The rho of a European put on a non-dividend-paying stock is
$$-X\tau e^{-r\tau} N(-x + \sigma\sqrt{\tau}) < 0.$$

Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,
$$\frac{f(S + \Delta S) - f(S - \Delta S)}{2\Delta S}.$$
- The computation time roughly doubles that for evaluating the derivative security itself.

An Alternative Numerical Delta[a]

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, f_u and f_d are computed.
- These values correspond to derivative values at stock prices Su and Sd, respectively.
- Delta is approximated by
$$\frac{f_u - f_d}{Su - Sd}.$$
- Almost zero extra computational effort.

[a]Pelsser and Vorst (1994).

Numerical Gamma

- At the stock price $(Suu + Sud)/2$, delta is approximately $(f_{uu} - f_{ud})/(Suu - Sud)$.
- At the stock price $(Sud + Sdd)/2$, delta is approximately $(f_{ud} - f_{dd})/(Sud - Sdd)$.
- Gamma is the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,

$$\frac{\frac{f_{uu}-f_{ud}}{Suu-Sud} - \frac{f_{ud}-f_{dd}}{Sud-Sdd}}{(Suu - Sdd)/2}.$$

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

$$\frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}.$$

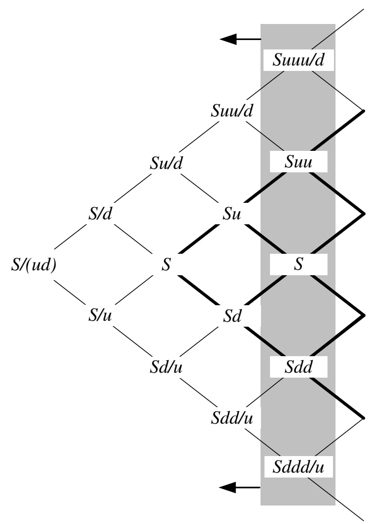
- It does not work (see text).
- Why did the binomial tree version work?

Other Numerical Greeks

- The theta can be computed as

$$\frac{f_{ud} - f}{2(\tau/n)}.$$

  - In fact, the theta of a European option will be shown to be computable from delta and gamma (see p. 502).
- For vega and rho, there is no alternative but to run the binomial tree algorithm twice.



Extensions of Options Theory

As I never learnt mathematics,
so I have had to think.
— Joan Robinson (1903–1983)

Pricing Corporate Securities[a]

- Interpret the underlying asset interpretated as the total value of the firm.
- The option pricing methodology can be applied to pricing corporate securities.
- Assume:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

[a]Black and Scholes (1973).

Risky Zero-Coupon Bonds and Stock

- Consider XYZ.com.
- Capital structure:
  - n shares of its own common stock, S.
  - Zero-coupon bonds with an aggregate par value of X.
- What is the value of the bonds, B?
- What is the value of the XYZ.com stock?

Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds' maturity date, suppose the total value of the firm V^* is less than the bondholders' claim X.
- Then the firm declares bankruptcy and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain X and the stockholders $V^* - X$.

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

Risky Zero-Coupon Bonds and Stock (continued)

- The stock is a call on the total value of the firm with a strike price of X and an expiration date equal to the bonds'.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call on the total value of the firm.
- Let V stand for the total value of the firm.
- Let C stand for the call.

Risky Zero-Coupon Bonds and Stock (continued)

- Thus $nS = C$ and $B = V - C$.
- Knowing C amounts to knowing how the value of the firm is divided between the stockholders and the bondholders.
- Whatever the value of C, the total value of the stock and bonds at maturity remains V^*.
- The relative size of debt and equity is irrelevant to the firm's current value V.

Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 8 (p. 239) and the put-call parity,

$$\begin{aligned} nS &= VN(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \\ B &= VN(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}). \end{aligned}$$

  - where
$$x \equiv \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

- The continuously compounded yield to maturity of the firm's bond is
$$\frac{\ln(X/B)}{\tau}.$$

Risky Zero-Coupon Bonds and Stock (concluded)

- Define default premium as the yield difference between risky and riskless bonds,

$$\begin{aligned} & (1/\tau)\ln(X/B) - r \\ = \; & -\frac{1}{\tau}\ln\left(N(-z) + \frac{1}{\omega}N(z - \sigma\sqrt{\tau})\right). \end{aligned}$$

  - $\omega \equiv Xe^{-r\tau}/V$.
  - $z \equiv (\ln\omega)/(\sigma\sqrt{\tau}) + (1/2)\,\sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}$.
  - Note that ω is the debt-to-total-value ratio.

A Numerical Example

- XYZ.com's assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck's market value per share is \$44.5.
- XYZ.com's securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay \$1,000 at maturity.
- $n = 1000$, $V = 44.5 \times n = 44500$, and $X = 30 \times n = 30000$.

			—Call—		—Put—	
Option	Strike	Exp.	Vol.	Last	Vol.	Last
Merck	30	Jul	328	15 1/4	...	...
44 1/2	35	Jul	150	9 1/2	10	1/16
44 1/2	40	Apr	887	4 3/4	136	1/16
44 1/2	40	Jul	220	5 1/2	297	1/4
44 1/2	40	Oct	58	6	10	1/2
44 1/2	45	Apr	3050	7/8	100	1 1/8
44 1/2	45	May	462	1 3/8	50	1 3/8
44 1/2	45	Jul	883	1 15/16	147	1 3/4
44 1/2	45	Oct	367	2 3/4	188	2 1/16

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for \$15.25.
- So XYZ.com's stock is worth $15.25 \times n = 15250$ dollars.
- The entire bond issue is worth $B = 44500 - 15250 = 29250$ dollars.
  - Or \$975 per bond.

A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $\$X$ par value plus n written European puts on Merck at a strike price of \$30.
  - By the put-call parity.
- The difference between B and the price of the default-free bond is the value of these puts.
- The table next page shows the total market values of the XYZ.com stock and bonds under various debt amounts X.

Promised payment to bondholders X	Current market value of bonds B	Current market value of stock nS	Current total value of firm V
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

A Numerical Example (continued)

- Suppose the promised payment to bondholders is \$45,000.
- Then the relevant option is the July call with a strike price of $45000/n = 45$ dollars.
- Since that option is selling for \$$1\frac{15}{16}$, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option's terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
- So parameters such volatility, dividend, and strike price are under partial control of the stockholders.

A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now $X = 45{,}000$ dollars.
- The table on p. 286 says the total market value of the bonds should be \$42,562.5.
- The *new* bondholders pay $42562.5 \times (15/45) = 14187.5$ dollars.
- The remaining stock is worth \$1,937.5.

A Numerical Example (continued)

- The stockholders therefore gain

$$14187.5 + 1937.5 - 15250 = 875$$

dollars.

- The *original* bondholders lose an equal amount,

$$29250 - \frac{30}{45} \times 42562.5 = 875. \tag{28}$$

A Numerical Example (continued)

- Suppose the stockholders distribute \$14,833.3 cash dividends by selling $(1/3) \times n$ Merck shares.
- They now have \$14,833.3 in cash plus a call on $(2/3) \times n$ Merck shares.
- The strike price remains $X = 30000$.
- This is equivalent to owning two-thirds of a call on n Merck shares with a total strike price of \$45,000.
- The n such calls are worth \$1,937.5 (p. 283).
- So the total market value of the XYZ.com stock is $(2/3) \times 1937.5 = 1291.67$ dollars.

A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence $(2/3) \times n \times 44.5 - 1291.67 = 28375$ dollars.
- Hence the stockholders gain

$$14833.3 + 1291.67 - 15250 \approx 875$$

dollars.

- The bondholders watch their value drop from \$29,250 to \$28,375, a loss of \$875.

Other Examples

- Subordinated debts as bull call spreads.
- Warrants as calls.
- Callable bonds as American calls with 2 strike prices.
- Convertible bonds.

Barrier Options[a]

- Their payoff depends on whether the underlying asset's price reaches a certain price level H.
- A knock-out option is an ordinary European option which ceases to exist if the barrier H is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if $H < S$.
- A put knock-out option is sometimes called an up-and-out option when $H > S$.

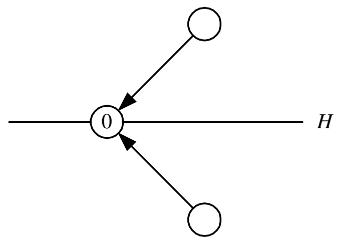
[a]A former student told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank.

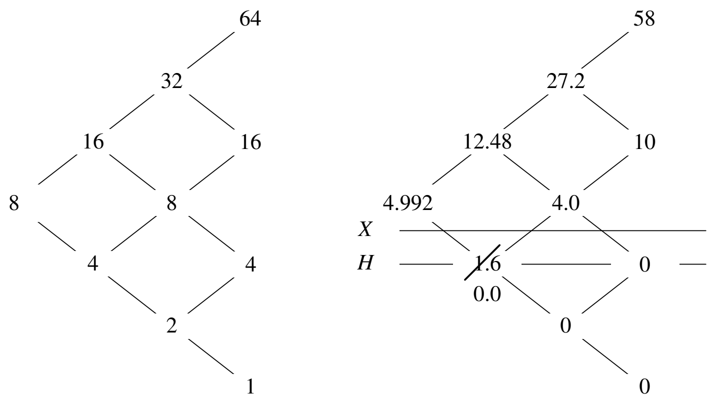
Barrier Options (concluded)

- A knock-in option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.
- Formulas exist for all kinds of barrier options.

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.

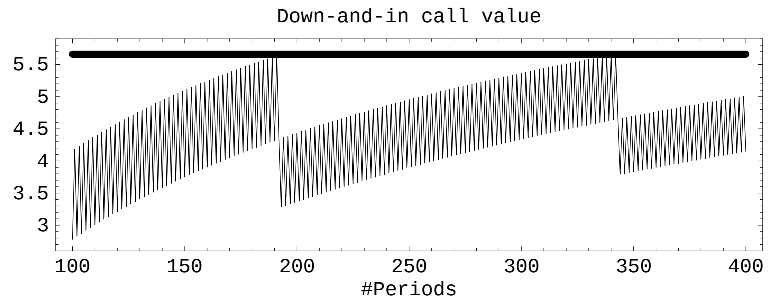




$S = 8$, $X = 6$, $H = 4$, $R = 1.25$, $u = 2$, and $d = 0.5$.
Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25$.

Binomial Tree Algorithms (concluded)

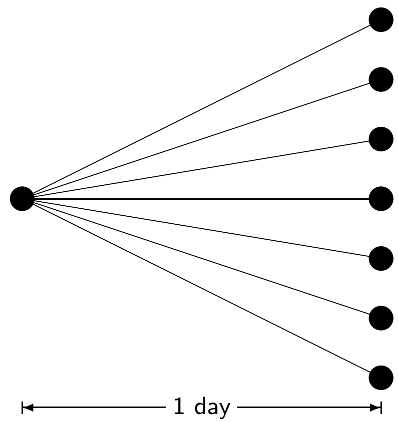
- But convergence is erratic because H is not at a price level on the tree (see plot on next page).
  - Typically, the barrier has to be adjusted to be at a price level.
- Solutions will be presented later.



Daily Monitoring

- Almost all barrier options monitor the barrier only for the daily closing prices.
- In that case, only nodes at the end of a day need to check for the barrier condition.
- We can even remove intraday nodes to create a multinomial tree.
  - A node is then followed by $d + 1$ nodes if each day is partitioned into d periods.
- This saves time and space.

A Heptanomial Tree (6 Periods Per Day)



Foreign Currencies

- S denotes the spot exchange rate in domestic/foreign terms.
- σ denotes the volatility of the exchange rate.
- r denotes the domestic interest rate.
- $\hat{r}$ denotes the foreign interest rate.
- A foreign currency is analogous to a stock paying a known dividend yield.
  - Foreign currencies pay a "continuous dividend yield" equal to $\hat{r}$ in the foreign currency.

Foreign Exchange Options

- Foreign exchange options are settled via delivery of the underlying currency.
- A primary use of foreign exchange (or forex) options is to hedge currency risk.
- Consider a U.S. company expecting to receive 100 million Japanese yen in March 2000.
- Those 100 million Japanese yen will be exchanged for U.S. dollars.

Foreign Exchange Options (continued)

- The contract size for the Japanese yen option is JPY6,250,000.
- The company purchases $100{,}000{,}000/6{,}250{,}000 = 16$ puts on the Japanese yen with a strike price of \$.0088 and an exercise month in March 2000.
- This gives the company the right to sell 100,000,000 Japanese yen for $100{,}000{,}000 \times .0088 = 880{,}000$ U.S. dollars.

Foreign Exchange Options (concluded)

- The formulas derived for stock index options in Eqs. (26) on p. 255 apply with the dividend yield equal to $\hat{r}$:

$$C = Se^{-\hat{r}\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{29}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-\hat{r}\tau} N(-x), \tag{29'}$$

  – where

$$x \equiv \frac{\ln(S/X) + (r - \hat{r} + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}.$$

Bar the roads!
Bar the paths!
Wert thou to flee from here, wert thou
to find all the roads of the world,
the way thou seekst
the path to that thou'dst find not[.]
— Richard Wagner (1813–1883), *Parsifal*

Path-Dependent Derivatives

- Let $S_0, S_1, \ldots, S_n$ denote the prices of the underlying asset over the life of the option.
- S_0 is the known price at time zero.
- S_n is the price at expiration.
- The standard European call has a terminal value depending only on the last price, $\max(S_n - X, 0)$.
- Its value thus depends only on the underlying asset's terminal price regardless of how it gets there.

Path-Dependent Derivatives (continued)

- In contrast, some derivatives are path-dependent in that their terminal payoff depends "critically" on the path.
- The (arithmetic) average-rate call has a terminal value given by

$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_i - X, 0\right).$$

- The average-rate put's terminal value is given by

$$\max\left(X - \frac{1}{n+1}\sum_{i=0}^{n} S_i, 0\right).$$

Path-Dependent Derivatives (continued)

- Average-rate options are also called Asian options.
- They are very popular.[a]
- They are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price.
- They are mostly European.
- Like painting on a canvas, each brush stroke leaves less room for the future composition.

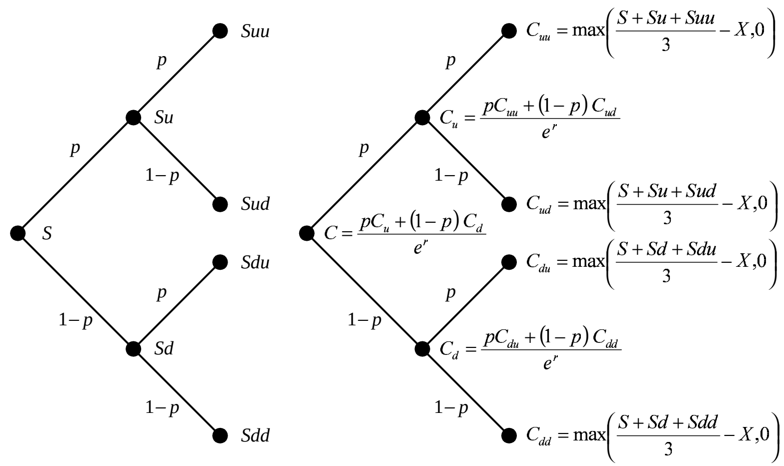
[a] As of the late 1990s, the outstanding volume was in the range of 5–10 billion U.S. dollars; see Nielsen and Sandmann (2003).

Path-Dependent Derivatives (concluded)

- A lookback call option on the minimum has a terminal payoff of $S_n - \min_{0 \leq i \leq n} S_i$.
- A lookback put option on the maximum has a terminal payoff of $\max_{0 \leq i \leq n} S_i - S_n$.
- The fixed-strike lookback option provides a payoff of $\max(\max_{0 \leq i \leq n} S_i - X, 0)$ for the call and $\max(X - \min_{0 \leq i \leq n} S_i, 0)$ for the put.
- Lookback call and put options on the average are called average-strike options.

Average-Rate Options

- Average-rate options are notoriously hard to price.
- The binomial tree for the averages does not combine.
- A straightforward algorithm is to enumerate the 2^n price paths for an n-period binomial tree and then average the payoffs.
- But the exponential complexity makes this naive algorithm impractical.
- As a result, the Monte Carlo method and approximation algorithms are some of the alternatives left.



Pricing Some Path-Dependent Options

- Not all path-dependent derivatives are hard to price.
- Barrier options are easy to price.
- When averaging is done *geometrically*, the option payoffs are

$$\max\left((S_0 S_1 \cdots S_n)^{1/(n+1)} - X, 0\right),$$
$$\max\left(X - (S_0 S_1 \cdots S_n)^{1/(n+1)}, 0\right).$$

Pricing Some Path-Dependent Options (concluded)

- The limiting analytical solutions are the Black-Scholes formulas.
  - With the volatility set to $\sigma_{\mathrm{a}} \equiv \sigma/\sqrt{3}$.
  - With the dividend yield set to $q_{\mathrm{a}} \equiv (r + q + \sigma^2/6)/2$.
- The formula is therefore

$$C = Se^{-q_{\mathrm{a}}\tau} N(x) - Xe^{-r\tau} N(x - \sigma_{\mathrm{a}}\sqrt{\tau}), \tag{30}$$

(31)

$$P = Xe^{-r\tau} N(-x + \sigma_{\mathrm{a}}\sqrt{\tau}) - Se^{-q_{\mathrm{a}}\tau} N(-x), \tag{31'}$$

  - where $x \equiv \frac{\ln(S/X) + \left(r - q_{\mathrm{a}} + \sigma_{\mathrm{a}}^2/2\right)\tau}{\sigma_{\mathrm{a}}\sqrt{\tau}}$.