Coupon Effect on the Yield to Maturity

- Under a normal spot rate curve, a coupon bond has a lower yield than a zero-coupon bond of equal maturity.
- Picking a zero-coupon bond over a coupon bond based purely on the zero's higher yield to maturity is flawed.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But this is only a trend not a mathematical truth.
- See a counterexample in the textbook.

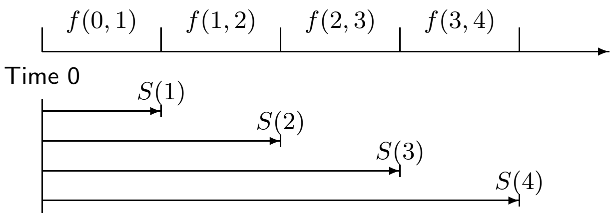
Forward Rates

- The yield curve contains information regarding future interest rates currently "expected" by the market.
- Invest \$1 for j periods to end up with $[\,1+S(j)\,]^j$ dollars at time j.
  - The maturity strategy.
- Invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j-i$ periods where $j>i$.
- Will have $[\,1+S(i)\,]^i[\,1+S(i,j)\,]^{j-i}$ dollars at time j.
  - $S(i,j)$: $(j-i)$-period spot rate i periods from now.
  - The rollover strategy.

Forward Rates (concluded)

- When $S(i,j)$ equals
$$f(i,j) \equiv \left[\frac{(1+S(j))^j}{(1+S(i))^i}\right]^{1/(j-i)} - 1, \qquad (11)$$
we will end up with $[\,1+S(j)\,]^j$ dollars again.
- By definition, $f(0,j)=S(j)$.
- $f(i,j)$ is called the (implied) forward rates.
  - More precisely, the $(j-i)$-period forward rate i periods from now.

Time Line



Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i,j)$ and future spot rate $S(i,j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate two investment strategies.
- $f(i,i+1)$ are instantaneous forward rates or one-period forward rates.

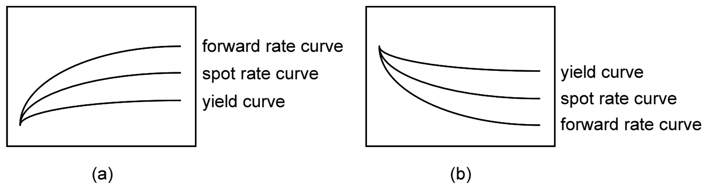
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i,j) > S(j) > \cdots > S(i).$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i,j) < S(j) < \cdots < S(i).$$



Forward Rates=Spot Rates=Yield Curve

- The FV of \$1 at time n can be derived in two ways.
- Buy n-period zero-coupon bonds and receive $[\,1 + S(n)\,]^n$.
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is $[\,1 + S(1)\,][\,1 + f(1,2)\,]\cdots[\,1 + f(n-1,n)\,]$.

Forward Rates=Spot Rates=Yield Curve (concluded)

- Since they are identical,

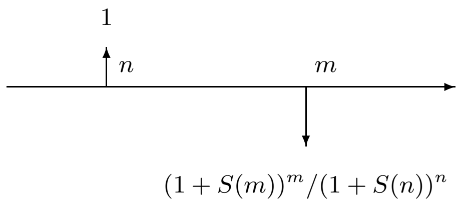
$$S(n) = ((1 + S(1))(1 + f(1,2)) \cdots (1 + f(n-1,n)))^{1/n} - 1. \qquad (12)$$

- Hence, the forward rates, specifically the one-period forward rates, determine the spot rate curve.
- Other equivalency can be derived similarly, such as

$$f(T, T+1) = d(T)/d(T+1) - 1.$$

Locking in the Forward Rate $f(n,m)$

- Buy one n-period zero-coupon bond for $1/(1 + S(n))^n$.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ m-year zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow $1/(1 + S(n))^n$.
- At time n there will be a cash inflow of \$1.
- At time m there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.
- This implies the rate $f(n,m)$ between times n and m.



Forward Contracts

- We generated the cash flow of a financial instrument called forward contract.
- Agreed upon today, it enables one to borrow money at time n in the future and repay the loan at time $m > n$ with an interest rate equal to the forward rate $f(n, m)$.
- Can the spot rate curve be an arbitrary curve?[a]

[a] Contributed by Mr. Dai, Tian-Shyr (R86526008, D8852600) in 1998.

Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}.$$

- The spot rate is an arithmetic average of forward rates,

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}.$$

Spot and Forward Rates under Continuous Compounding (concluded)

- The formula for the forward rate:

$$f(i, j) = \frac{jS(j) - iS(i)}{j - i}.$$

- The one-period forward rate:

$$f(j, j+1) = -\ln \frac{d(j+1)}{d(j)}.$$

- $$f(T) \equiv \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T\,\frac{\partial S}{\partial T}.$$
- $f(T) > S(T)$ if and only if $\partial S / \partial T > 0$.

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a, b) = E[\, S(a, b) \,]. \tag{13}$$

- Does not imply that the forward rate is an accurate predictor for the future spot rate.
- Implies that the maturity strategy and the rollover strategy produce the same result at the horizon on the average.

Unbiased Expectations Theory and Spot Rate Curve

- Implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from Eq. (11) on p. 109.
  - So $E[\,S(j, j+1)\,] > S(j+1) > \cdots > S(1)$ if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

More Implications

- The theory has been rejected by most empirical studies with the possible exception of the period prior to 1915.
- Since the term structure has been upward sloping about 80% of the time, the theory would imply that investors have expected interest rates to rise 80% of the time.
- Riskless bonds, regardless of their different maturities, are expected to earn the same return on the average.
- That would mean investors are indifferent to risk.

A "Bad" Expectations Theory

- The expected returns on all possible riskless bond strategies are equal for *all* holding periods.
- So

$$(1 + S(2))^2 = (1 + S(1))\, E[\,1 + S(1, 2)\,] \tag{14}$$

  because of the equivalency between buying a two-period bond and rolling over one-period bonds.
- After rearrangement, $E[\,1 + S(1, 2)\,] = (1 + S(2))^2/(1 + S(1))$.

A "Bad" Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond and sells it after one period.
  - The expected return is $E[\,(1 + S(1, 2))^{-1}\,]\,(1 + S(2))^2$.
  - Strategy two buys a one-period bond with a return of $1 + S(1)$.
- The theory says the returns are equal:

$$\frac{(1 + S(2))^2}{1 + S(1)} = \frac{1}{E\,[\,(1 + S(1, 2))^{-1}\,]}.$$

A "Bad" Expectations Theory (concluded)

- Combine this with Eq. (14) to obtain

$$E\left[\frac{1}{1+S(1,2)}\right]=\frac{1}{E[\,1+S(1,2)\,]}.$$

- But this is impossible save for a certain economy.
  - Jensen's inequality states that $E[\,g(X)\,]>g(E[\,X\,])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x)>0$).
  - Use $g(x)\equiv(1+x)^{-1}$ to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}}=1+S(1)\quad\text{for } \textit{all } n>1.$$

- This theory is the basis of many interest rate models.

Duration Revisited

- Let $P(y)\equiv\sum_i C_i/(1+S(i)+y)^i$ be the price associated with the cash flow $C_1, C_2, \ldots$.
- Define duration as

$$-\left.\frac{\partial P(y)/P(0)}{\partial y}\right|_{y=0}=\frac{\sum_i \frac{iC_i}{(1+S(i))^{i+1}}}{\sum_i \frac{C_i}{(1+S(i))^i}}.$$

  - The curve is shifted in parallel to $S(1)+\Delta y, S(2)+\Delta y, \ldots$ before letting Δy go to zero.
- The percentage price change roughly equals duration times the size of the parallel shift in the spot rate curve.

Duration Revisited (continued)

- The simple linear relation between duration and MD in Eq. (9) on p. 79 breaks down.
- One way to regain it is to resort to a different kind of shift, the proportional shift:

$$\frac{\Delta(1+S(i))}{1+S(i)}=\frac{\Delta(1+S(1))}{1+S(1)}$$

for all i.
  - $\Delta(x)$ denotes the change in x when the short-term rate is shifted by Δy.

Duration Revisited (continued)

- Duration now becomes

$$\frac{1}{1+S(1)}\left[\frac{\sum_i \frac{iC_i}{(1+S(i))^i}}{\sum_i \frac{C_i}{(1+S(i))^i}}\right]. \tag{15}$$

- Define Macaulay's second duration to be the number within the brackets in Eq. (15).
- Then

$$\text{duration} = \frac{\text{Macaulay's second duration}}{(1+S(1))}.$$

Duration Revisited (concluded)

- To handle more general types of spot rate curve changes, define a vector $[\,c_1, c_2, \ldots, c_n\,]$ that characterizes the perceived type of change.
  - Parallel shift: $[\,1, 1, \ldots, 1\,]$.
  - Twist: $[\,1, 1, \ldots, 1, -1, \ldots, -1\,]$.
- Let $P(y) \equiv \sum_i C_i/(1 + S(i) + yc_i)^i$ be the price associated with the cash flow $C_1, C_2, \ldots$.
- Define duration as

$$-\frac{\partial P(y)/P(0)}{\partial y}\bigg|_{y=0}.$$

Fundamental Statistical Concepts

There are three kinds of lies:
lies, damn lies, and statistics.
— Benjamin Disraeli (1804–1881)

One death is a tragedy,
but a million deaths are a statistic.
— Josef Stalin (1879–1953)

Moments

- The variance of a random variable X is defined as
$$\mathrm{Var}[\,X\,] \equiv E\left[\,(X - E[\,X\,])^2\,\right].$$
- The covariance between random variables X and Y is
$$\mathrm{Cov}[\,X, Y\,] \equiv E\,[\,(X - \mu_X)(Y - \mu_Y)\,]\,,$$
where μ_X and μ_Y are the means of X and Y, respectively.
- Random variables X and Y are uncorrelated if $\mathrm{Cov}[\,X, Y\,] = 0$.

Variance of Sum

- Variance of a weighted sum of random variables equals
$$\mathrm{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \,\mathrm{Cov}[\,X_i, X_j\,].$$
- It becomes
$$\sum_{i=1}^{n} a_i^2\,\mathrm{Var}[\,X_i\,]$$
when X_i are uncorrelated.

Conditional Expectation

- "$X \mid I$" denotes X conditional on the information set I.
- The information set can be another random variable's value or the past values of X, say.
- The conditional expectation $E[\,X \mid I\,]$ is the expected value of X conditional on I; it is a random variable.
- The law of iterated conditional expectations:
$$E[\,X\,] = E[\,E[\,X \mid I\,]\,].$$
- If I_2 contains at least as much information as I_1, then
$$E[\,X \mid I_1\,] = E[\,E[\,X \mid I_2\,] \mid I_1\,]. \tag{16}$$

The Normal Distribution

- A random variable X has the normal distribution with mean μ and variance σ^2 if its probability density function is
$$e^{-(x-\mu)^2/(2\sigma^2)}/(\sigma\sqrt{2\pi})\,.$$
- This is expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the distribution function
$$\mathrm{Prob}[\,X \le z\,] = N(z) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-x^2/2}\,dx.$$

Moment Generating Function

- The moment generating function of random variable X is
$$\theta_X(t) \equiv E[\, e^{tX}\,].$$
- The moment generating function of $X \sim N(\mu, \sigma^2)$ is
$$\theta_X(t) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]. \tag{17}$$

Distribution of Sum

- If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then $\sum_i X_i \sim N(\sum_i \mu_i, \sum_i \sigma_i^2)$.
- Let $X_i \sim N(\mu_i, \sigma_i^2)$, which may not be independent.
- Then $\sum_{i=1}^n t_i X_i \sim N(\sum_{i=1}^n t_i\, \mu_i, \sum_{i=1}^n \sum_{j=1}^n t_i t_j \operatorname{Cov}[\, X_i, X_j\,])$.
- These X_i are said to have a multivariate normal distribution.

Generation of Univariate Normal Distributions

- Let X be uniformly distributed over $(0, 1\,]$ so that $\text{Prob}[\, X \leq x\,] = x$ for $0 < x \leq 1$.
- Repeatedly draw two samples x_1 and x_2 from X until
$$\omega \equiv (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1.$$
- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where
$$c \equiv \sqrt{-2(\ln \omega)/\omega}\,.$$

A Dirty Trick and a Right Attitude

- Let ξ_i are independent and uniformly distributed over $(0, 1)$.
- A simple method to generate the standard normal variable is to calculate
$$\sum_{i=1}^{12} \xi_i - 6.$$
- But "this is not a highly accurate approximation and should only be used to establish ballpark estimates."[a]

[a] Jäckel, *Monte Carlo Methods in Finance* (2002).

A Dirty Trick and a Right Attitude (concluded)

- Always blame your random number generator last.[a]
- Instead, check your programs first.

[a] "The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings." William Shakespeare (1564–1616), *Julius Caesar*.

Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation ρ can be generated.
- Let X_1 and X_2 be independent standard normal variables.
- Then

$$\begin{aligned} U &\equiv aX_1, \\ V &\equiv \rho U + \sqrt{1-\rho^2}\, aX_2, \end{aligned}$$

are the desired random variables with $\text{Var}[U] = \text{Var}[V] = a^2$ and $\text{Cov}[U, V] = \rho a^2$.

The Lognormal Distribution

- A random variable Y is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \equiv e^X$.
- The mean and variance of Y are

$$\mu_Y = e^{\mu+\sigma^2/2} \text{ and } \sigma_Y^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right), \tag{18}$$

respectively.
  - They follow from $E[Y^n] = e^{n\mu+n^2\sigma^2/2}$.

Option Basics

The shift toward options as
the center of gravity of finance [. . .]
— Merton H. Miller (1923–2000)

Calls and Puts

- A call gives its holder the right to buy a number of the underlying asset by paying a strike price.
- A put gives its holder the right to sell a number of the underlying asset for the strike price.
- An embedded option has to be traded along with the underlying asset.
- How to price options?

Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- An option can be exercised prior to the expiration date: early exercise.

American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option because of the early exercise feature.

Convenient Conventions

- C: call value.
- P: put value.
- X: strike price.
- S: stock price.
- D: dividend.

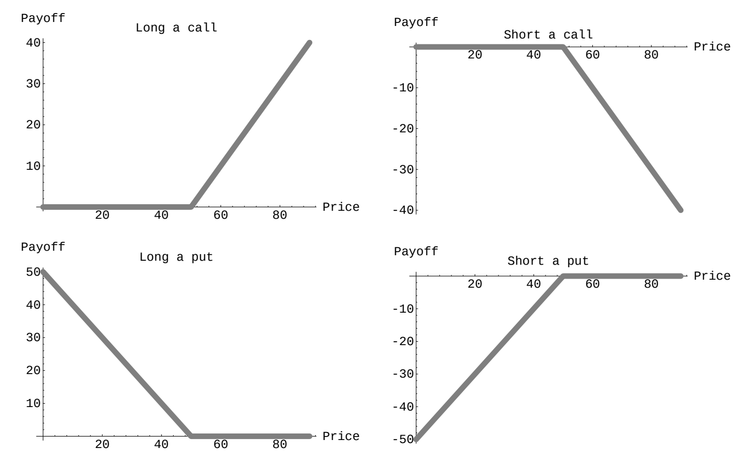
Payoff

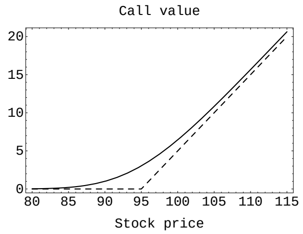
- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.
- The payoff of a call at expiration is $C = \max(0, S - X)$.
- The payoff of a put at expiration is $P = \max(0, X - S)$.
- At any time t before the expiration date, we call $\max(0, S_t - X)$ the intrinsic value of a call.
- At any time t before the expiration date, we call $\max(0, X - S_t)$ the intrinsic value of a put.

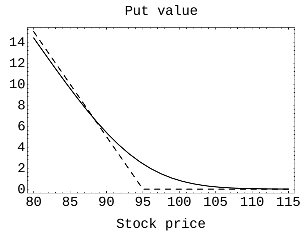
Payoff (concluded)

- A call is in the money if $S > X$, at the money if $S = X$, and out of the money if $S < X$.
- A put is in the money if $S < X$, at the money if $S = X$, and out of the money if $S > X$.
- Options that are in the money at expiration should be exercised.[a]
- Finding an option's value at any time before expiration is a major intellectual breakthrough.

[a] 11% of option holders let in-the-money options expire worthless.







Cash Dividends

- Exchange-traded stock options are not cash dividend-protected (or simply protected).
  - The option contract is not adjusted for cash dividends.
- The stock price falls by an amount roughly equal to the amount of the cash dividend as it goes ex-dividend.
- Cash dividends are detrimental for calls.
- The opposite is true for puts.

Stock Splits and Stock Dividends

- Options are adjusted for stock splits.
- After an n-for-m stock split, the strike price is only m/n times its previous value, and the number of shares covered by one contract becomes n/m times its previous value.
- Exchange-traded stock options are adjusted for stock dividends.
- Options are assumed to be unprotected.

Example

- Consider an option to buy 100 shares of a company for \$50 per share.
- A 2-for-1 split changes the term to a strike price of \$25 per share for 200 shares.

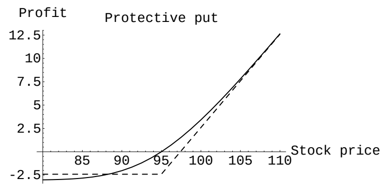
Short Selling

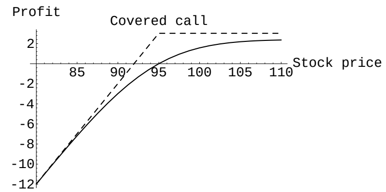
- Short selling (or simply shorting) involves selling an asset that is *not* owned with the intention of buying it back later.
  - If you short 1,000 XYZ shares, the broker borrows them from another client to sell them in the market.
  - This action generates proceeds for the investor.
  - The investor can close out the short position by buying 1,000 XYZ shares.
  - Clearly, the investor profits if the stock price falls.
- Not all assets can be shorted.

Covered Position: Hedge

- A hedge combines an option with its underlying stock in such a way that one protects the other against loss.
- Protective put: A long position in stock with a long put.
- Covered call: A long position in stock with a short call.[a]
- Both strategies break even only if the stock price rises, so they are bullish.
- Writing a cash-secured put means writing a put while putting in enough money to cover the strike price if the put is exercised.

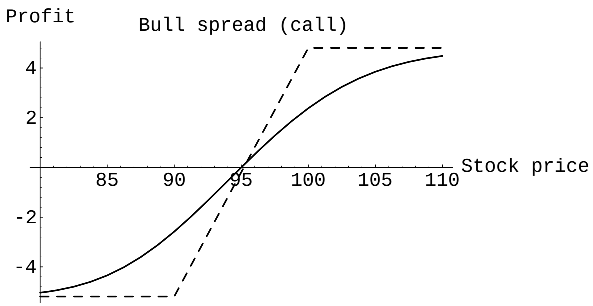
[a]A short position has a payoff opposite in sign to that of a long position.





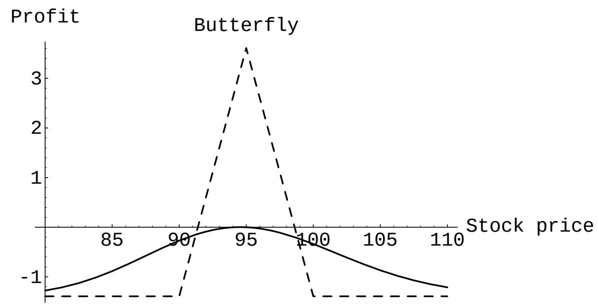
Covered Position: Spread

- A spread consists of options of the same type and on the same underlying asset but with different strike prices or expiration dates.
- We use X_L, X_M, and X_H to denote the strike prices with $X_L < X_M < X_H$.
- A bull call spread consists of a long X_L call and a short X_H call with the same expiration date.
  - The initial investment is $C_L - C_H$.
  - The maximum profit is $(X_H - X_L) - (C_L - C_H)$.
  - The maximum loss is $C_L - C_H$.



Covered Position: Spread (continued)

- Writing an X_H put and buying an X_L put with identical expiration date creates the bull put spread.
- A bear spread amounts to selling a bull spread.
- It profits from declining stock prices.
- Three calls or three puts with different strike prices and the same expiration date create a butterfly spread.
  - The spread is long one X_L call, long one X_H call, and short two X_M calls.

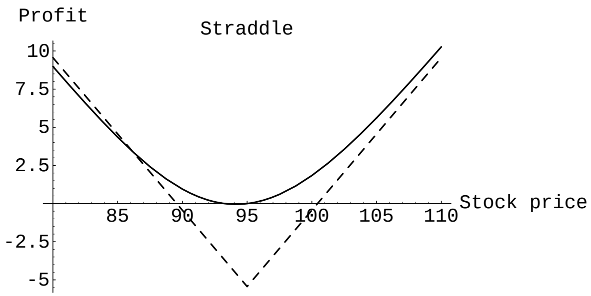


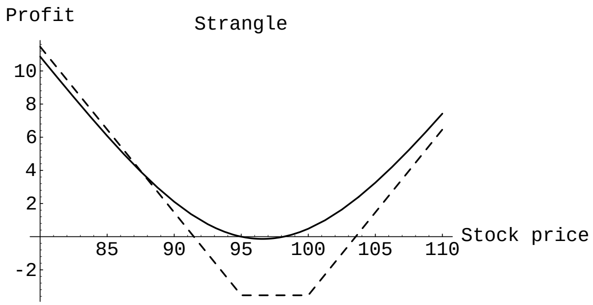
Covered Position: Spread (concluded)

- A butterfly spread pays off a positive amount at expiration only if the asset price falls between X_L and X_H.
- A butterfly spread with a small $X_H - X_L$ approximates a state contingent claim, which pays $1 only when a particular price results.
- The price of a state contingent claim is called a state price.

Covered Position: Combination

- A combination consists of options of different types on the same underlying asset, and they are either all bought or all written.
- Straddle: A long call and a long put with the same strike price and expiration date.
- Since it profits from high volatility, a person who buys a straddle is said to be long volatility.
- Selling a straddle benefits from low volatility.
- Strangle: Identical to a straddle except that the call's strike price is higher than the put's.





Arbitrage in Option Pricing

All general laws are attended with inconveniences,
when applied to particular cases.
— David Hume (1711–1776)

Arbitrage

- The no-arbitrage principle says there should be no free lunch.
- It supplies the argument for option pricing.
- A riskless arbitrage opportunity is one that, without any initial investment, generates nonnegative returns under all circumstances and positive returns under some.
- In an efficient market, such opportunities do not exist.
- The portfolio dominance principle says portfolio A should be more valuable than B if A's payoff is at least as good under all circumstances and better under some.

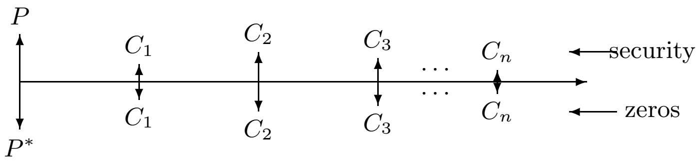
A Corollary

- A portfolio yielding a zero return in every possible scenario must have a zero PV.
  - Short the portfolio if its PV is positive.
  - Buy it if its PV is negative.
  - In both cases, a free lunch is created.

The PV Formula Justified

$P = \sum_{i=1}^{n} C_i d(i)$ for a certain cash flow $C_1, C_2, \ldots, C_n$.

- If the price $P^* < P$, short the zeros that match the security's n cash flows and use P^* of the proceeds P to buy the security.
- Since the cash inflows of the security will offset exactly the obligations of the zeros, a riskless profit of $P - P^*$ dollars has been realized now.
- If the price $P^* > P$, a riskless profit can be realized by reversing the trades.



Two More Examples

- An American option cannot be worth less than the intrinsic value.
  - Otherwise, one can buy the option, promptly exercise it and sell the stock with a profit.
- A put or a call must have a nonnegative value.
  - Otherwise, one can buy it for a positive cash flow now and end up with a nonnegative amount at expiration.

Relative Option Prices

- These relations hold regardless of the probabilistic model for stock prices.
- Assume, among other things, that there are no transactions costs or margin requirements, borrowing and lending are available at the riskless interest rate, interest rates are nonnegative, and there are no arbitrage opportunities.
- Let the current time be time zero.
- $\mathrm{PV}(x)$ stands for the PV of x dollars at expiration; hence $\mathrm{PV}(x) = xd(\tau)$ where τ is the time to expiration.

Put-Call Parity (Castelli, 1877)

$$C = P + S - \mathrm{PV}(X). \tag{19}$$

- Consider the portfolio of one short European call, one long European put, one share of stock, and a loan of $\mathrm{PV}(X)$.
- All options are assumed to carry the same strike price and time to expiration, τ.
- The initial cash flow is therefore $C - P - S + \mathrm{PV}(X)$.
- At expiration, if the stock price $S_\tau \leq X$, the put will be worth $X - S_\tau$ and the call will expire worthless.

The Proof (concluded)

- On the other hand, if $S_\tau > X$, the call will be worth $S_\tau - X$ and the put will expire worthless.
- After the loan, now X, is repaid, the net future cash flow is zero in either case.
- The no-arbitrage principle implies that the initial investment to set up the portfolio must be nil as well.

Consequences of Put-Call Parity

- There is only one kind of European option because the other can be replicated from it in combination with the underlying stock and riskless lending or borrowing.
  - Combinations such as this create synthetic securities.
- $S = C - P + \mathrm{PV}(X)$ says a stock is equivalent to a portfolio containing a long call, a short put, and lending $\mathrm{PV}(X)$.
- $C - P = S - \mathrm{PV}(X)$ implies a long call and a short put amount to a long position in stock and borrowing the PV of the strike price (buying stock on margin).

Intrinsic Value

Lemma 1 *An American call or a European call on a non-dividend-paying stock is never worth less than its intrinsic value.*

- The put-call parity implies $C = (S - X) + (X - \mathrm{PV}(X)) + P \geq S - X$.
- Since $C \geq 0$, it follows that $C \geq \max(S - X, 0)$, the intrinsic value.
- An American call also cannot be worth less than its intrinsic value.

Intrinsic Value (concluded)

A European put on a non-dividend-paying stock may be worth less than its intrinsic value, but:

Lemma 2 *For European puts,* $P \geq \max(\mathrm{PV}(X) - S, 0)$.

- Prove it with the put-call parity.
- Can explain the right figure on p. 154 why $P < X - S$ when S is small.

Early Exercise of American Calls

European calls and American calls are identical when the underlying stock pays no dividends.

Theorem 3 (Merton, 1973) *An American call on a non-dividend-paying stock should not be exercised before expiration.*

- By an exercise in text, $C \geq \max(S - \mathrm{PV}(X), 0)$.
- If the call is exercised, the value is the smaller $S - X$.

Remarks

- The above theorem does not mean American calls should be kept until maturity.
- What it does imply is that when early exercise is being considered, a *better* alternative is to sell it.
- Early exercise may become optimal for American calls on a dividend-paying stock.
  - Stock price declines as the stock goes ex-dividend.

Early Exercise of American Calls: Dividend Case

Surprisingly, an American call should be exercised only at a few dates.

Theorem 4 *An American call will only be exercised at expiration or just before an ex-dividend date.*

It might be optimal to exercise an American put even if the underlying stock does not pay dividends.

Convexity of Option Prices

Lemma 5 *For three otherwise identical calls with strike prices* $X_1 < X_2 < X_3$,

$$\begin{aligned} C_{X_2} &\leq \omega C_{X_1} + (1-\omega)\, C_{X_3} \\ P_{X_2} &\leq \omega P_{X_1} + (1-\omega)\, P_{X_3} \end{aligned}$$

Here $\omega \equiv (X_3 - X_2)/(X_3 - X_1)$. (*Equivalently,* $X_2 = \omega X_1 + (1-\omega)\, X_3$.)

Option on Portfolio vs. Portfolio of Options

An option on a portfolio of stocks is cheaper than a portfolio of options.

Theorem 6 *Consider a portfolio of non-dividend-paying assets with weights* ω_i*. Let* C_i *denote the price of a European call on asset* i *with strike price* X_i*. Then the call on the portfolio with a strike price* $X \equiv \sum_i \omega_i X_i$ *has a value at most* $\sum_i \omega_i C_i$*. All options expire on the same date.*

The same result holds for European puts.