Yields

- The term yield denotes the return of investment.
- Two widely used yields are the bond equivalent yield (BEY) and the mortgage equivalent yield (MEY).
- BEY corresponds to the $r$ in Eq. (1) on p. 21 that equates PV with FV when $m = 2$.
- MEY corresponds to the $r$ in Eq. (1) on p. 21 that equates PV with FV when $m = 12$.

Internal Rate of Return (IRR)

- It is the interest rate which equates an investment’s PV with its price $P$,
  \[ P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_n}{(1+y)^n}. \]
- The above formula is the foundation upon which pricing methodologies are built.

Numerical Methods for Yields

- Solve $f(y) = 0$ for $y \geq -1$, where
  \[ f(y) = \sum_{t=1}^{n} \frac{C_t}{(1+y)^t} - P. \]
  - $P$ is the market price.
- The function $f(y)$ is monotonic in $y$ if $C_t > 0$ for all $t$.
- A unique solution exists for a monotonic $f(y)$.

The Bisection Method

- Start with $a$ and $b$ where $a < b$ and $f(a)f(b) < 0$.
- Then $f(\xi)$ must be zero for some $\xi \in [a, b]$.
- If we evaluate $f$ at the midpoint $c \equiv (a + b)/2$, either (1) $f(c) = 0$, (2) $f(a)f(c) < 0$, or (3) $f(c)f(b) < 0$.
- In the first case we are done, in the second case we continue the process with the new bracket $[a, c]$, and in the third case we continue with $[c, b]$.
- The bracket is halved in the latter two cases.
- After $n$ steps, we will have confined $\xi$ within a bracket of length $(b - a)/2^n$. 
The Newton-Raphson Method

- Converges faster than the bisection method.
- Start with a first approximation $x_0$ to a root of $f(x) = 0$.
- Then
  \[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \]
- When computing yields,
  \[ f'(x) = -\sum_{t=1}^{n} \frac{tC_t}{(1+x)^{t+1}}. \]

The Secant Method

- A variant of the Newton-Raphson method.
- Replace differentiation with difference.
- Start with two approximations $x_0$ and $x_1$.
- Then compute the $(k+1)$st approximation with
  \[ x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}. \]
The Secant Method (concluded)

- Its convergence rate, 1.618.
- This is slightly worse than the Newton-Raphson method’s 2.
- But the secant method does not need to evaluate $f'(x_k)$ needed by the Newton-Raphson method.
- This saves about 50% in computation efforts per iteration.
- The convergence rate of the bisection method is 1.

Solving Systems of Nonlinear Equations (concluded)

- The $(k+1)$st approximation $(x_{k+1}, y_{k+1})$ satisfies the following linear equations,

$$
\begin{bmatrix}
\frac{\partial f(x_k, y_k)}{\partial x} & \frac{\partial f(x_k, y_k)}{\partial y} \\
\frac{\partial g(x_k, y_k)}{\partial x} & \frac{\partial g(x_k, y_k)}{\partial y}
\end{bmatrix}
\begin{bmatrix}
\Delta x_{k+1} \\
\Delta y_{k+1}
\end{bmatrix}
= -
\begin{bmatrix}
f(x_k, y_k) \\
g(x_k, y_k)
\end{bmatrix},
$$

where $\Delta x_{k+1} \equiv x_{k+1} - x_k$ and $\Delta y_{k+1} \equiv y_{k+1} - y_k$.
- The above has a unique solution for $(\Delta x_{k+1}, \Delta y_{k+1})$ when the $2 \times 2$ matrix is invertible.
- Set $(x_{k+1}, y_{k+1}) = (x_k + \Delta x_{k+1}, y_k + \Delta y_{k+1})$.

Zero-Coupon Bonds (Pure Discount Bonds)

- The price of a zero-coupon bond that pays $F$ dollars in $n$ periods is

$$F / (1 + r)^n,$$

where $r$ is the interest rate per period.
- Can meet future obligations without reinvestment risk.
- Coupon bonds can be thought of as a matching package of zero-coupon bonds, at least theoretically.

Solving Systems of Nonlinear Equations

- It is not easy to extend the bisection method to higher dimensions.
- But the Newton-Raphson method can be extended to higher dimensions.
- Let $(x_k, y_k)$ be the $k$th approximation to the solution of the two simultaneous equations,

$$f(x, y) = 0,$$
$$g(x, y) = 0.$$
Example
• The interest rate is 8\% compounded semiannually.
• A zero-coupon bond that pays the par value 20 years from now will be priced at \(1/(1.04)^{40}\), or 20.83\%, of its par value.
• It will be quoted as 20.83.
• If the bond matures in 10 years instead of 20, its price would be 45.64.

Level-Coupon Bonds
• Coupon rate.
• Par value, paid at maturity.
• \(F\) denotes the par value and \(C\) denotes the coupon.
• Cash flow:

\[
\begin{array}{cccccc}
C & C & C & \ldots & C + F \\
1 & 2 & 3 & \ldots & n \\
\end{array}
\]

Pricing Formula
\[
P = \sum_{i=1}^{n} \frac{C}{(1 + \frac{r}{m})^{i}} + \frac{F}{(1 + \frac{r}{m})^{n}}
\]
\[
= C \frac{1 - (1 + \frac{r}{m})^{-n}}{\frac{r}{m}} + \frac{F}{(1 + \frac{r}{m})^{n}}. \tag{4}
\]
• \(n\): number of cash flows.
• \(m\): number of payments per year.
• \(r\): annual rate compounded \(m\) times per annum.
• \(C = Fc/m\) when \(c\) is the annual coupon rate.
• Price \(P\) can be computed in \(O(1)\) time.

Yields to Maturity
• The \(r\) that satisfies Eq. (4) on p. 54 with \(P\) being the bond price.
• For a 15\% BEY, a 10-year bond with a coupon rate of 10\% paid semiannually sells for

\[
5 \times \frac{1 - [1 + (0.15/2)]^{-2 \times 10}}{0.15/2} + \frac{100}{[1 + (0.15/2)]^{2 \times 10}} = 74.5138
\]
percent of par.
Price Behavior (1)

- Bond prices fall when interest rates rise, and vice versa.
- “Only 24 percent answered the question correctly.”

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Price Behavior (2)

- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate is above the market interest rate;
  - at par (at its par value) when its coupon rate is equal to the market interest rate;
  - at a discount (below its par value) when its coupon rate is below the market interest rate.

---

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<th>Yield (%)</th>
<th>Price (% of par)</th>
</tr>
</thead>
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</tr>
<tr>
<td>8.0</td>
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</tr>
<tr>
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<td>92.31</td>
</tr>
<tr>
<td>10.5</td>
<td>88.79</td>
</tr>
</tbody>
</table>

---

Terminology

- Bonds selling at par are called par bonds.
- Bonds selling at a premium are called premium bonds.
- Bonds selling at a discount are called discount bonds.
Day Count Conventions: Actual/Actual

- The first "actual" refers to the actual number of days in a month.
- The second refers to the actual number of days in a coupon period.
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ to date $D_2 \equiv (y_2, m_2, d_2)$ is
  \[360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1)\].

Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

\[
\omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}.
\]  

(5)

- The price is now calculated by

\[
PV = \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}.
\]

(6)
Accrued Interest

- The buyer pays the quoted price plus the accrued interest

\[
C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega).
\]

- The yield to maturity is the \( r \) satisfying (6) when \( P \) is the invoice price, the sum of the quoted price and the accrued interest.

- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.

Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.

Example ("30/360") (concluded)

- The accrued interest is \((10/2) \times \frac{180 - 60}{180} = 3.3333\) per $100 of par value.
- The yield to maturity is 3%.
- This can be verified by Eq. (6) with \( \omega = 60/180 \), \( m = 2 \), \( C = 5 \), \( PV = 111.2891 + 3.3333 \), and \( r = 0.03 \).
Price Behavior (2) Revisited

• Before: A bond selling at par if the yield to maturity equals the coupon rate.
• But it assumed that the settlement date is on a coupon payment date.
• Now suppose the settlement date for a bond selling at par (i.e., the quoted price is equal to the par value) falls between two coupon payment dates.
• Then its yield to maturity is less than the coupon rate.
  – The short reason: Exponential growth is replaced by linear growth, hence “overpaying” the coupon.

“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy

Bond Price Volatility

• Volatility measures how bond prices respond to interest rate changes.
• It is key to the risk management of interest-rate-sensitive securities.
• Assume level-coupon bonds throughout.
Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?
- Define price volatility by
  \[ \frac{\partial P}{\partial y} P. \]

Price Volatility of Bonds

- The price volatility of a coupon bond is
  \[ -\frac{(C/y) n - (C/y^2) ((1 + y)^{n+1} - (1 + y)) - nF}{(C/y) ((1 + y)^{n+1} - (1 + y)) + F(1 + y)}, \]
  where \( F \) is the par value, and \( C \) is the coupon payment per period.
- For bonds without embedded options,
  \[ \frac{\partial P}{\partial y} > 0. \]

Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.
- The weights are the cash flows’ PVs divided by the asset’s price.
- Formally,
  \[ \text{MD} = \frac{1}{P} \sum_{i=1}^{n} \frac{iC_i}{(1 + y)^i}. \]
- The Macaulay duration, in periods, is equal to
  \[ \text{MD} = -(1 + y) \frac{\partial P}{\partial y} \frac{1}{P}. \]

MD of Bonds

- The MD of a coupon bond is
  \[ \text{MD} = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1 + y)^i} + \frac{n F}{(1 + y)^n} \right]. \]
- It can be simplified to
  \[ \text{MD} = \frac{c(1 + y) [(1 + y)^n - 1] + ny(y - c)}{cy [(1 + y)^n - 1] + y^2}, \]
  where \( c \) is the period coupon rate.
- The MD of a zero-coupon bond equals its term to maturity \( n \).
- The MD of a coupon bond is less than its maturity.
Finesse

• Equations (7) on p. 74 and (8) on p. 75 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
• That is, if the cash flow is independent of yields.
• To see this point, suppose the market yield declines.
• The MD will be lengthened.
• But for securities whose maturity actually decreases as a result, the MD may actually decrease.

How Not To Think about MD

• The MD has its origin in measuring the length of time a bond investment is outstanding.
• But you use it that way at your peril.
• The MD should be seen mainly as measuring price volatility.
• Many, if not most, duration-related terminology cannot be comprehended otherwise.

Conversion

• For the MD to be year-based, modify Eq. (8) on p. 75 to
\[
\frac{1}{P} \left[ \sum_{i=1}^{n} i \frac{C}{k} \left(1 + \frac{y}{k}\right)^i + \frac{n}{k} \frac{F}{(1 + \frac{y}{k})^n} \right],
\]
where $y$ is the annual yield and $k$ is the compounding frequency per annum.
• Equation (7) on p. 74 also becomes
\[
\text{MD} = - \left(1 + \frac{y}{k}\right) \frac{\partial P}{\partial y} \frac{1}{P}.
\]
• By definition, MD (in years) = $\frac{\text{MD} (in\ periods)}{k}$.

Modified Duration

• Modified duration is defined as
\[
\text{modified duration} = - \frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}.
\]
• By Taylor expansion,
\[
\text{percent price change} \approx -\text{modified duration} \times \text{yield change}.
\]
Example
- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be
  \[ -11.54 \times 0.001 = -0.01154 = -1.154\%. \]

Modified Duration of a Portfolio
- The modified duration of a portfolio equals
  \[ \sum_i \omega_i D_i, \]
  - \( D_i \) is the modified duration of the \( i \)th asset.
  - \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.

Effective Duration
- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as
  \[ \frac{P_- - P_+}{P_0(y+ - y-)}. \]
  - \( P_- \) is the price if the yield is decreased by \( \Delta y \).
  - \( P_+ \) is the price if the yield is increased by \( \Delta y \).
  - \( P_0 \) is the initial price, \( y \) is the initial yield.
  - \( \Delta y \) is small.

Effective Duration (concluded)
- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use
  \[ \frac{P_0 - P_+}{P_0 \Delta y}. \]
  - More economical but less accurate.
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as
  \[ \text{modified duration} \times \text{price (\% of par)} = -\frac{\partial P}{\partial y} \]
- The approximate dollar price change per $100 of par value is
  \[ \text{price change} \approx -\text{dollar duration} \times \text{yield change} \]

Convexity

- Convexity is defined as
  \[ \text{convexity (in periods)} = \frac{\partial^2 P}{\partial y^2} \frac{1}{P} \]
- The convexity of a coupon bond is positive (prove it!).
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude.
- Hence, between two bonds with the same duration, the one with a higher convexity is more valuable.

Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by
  \[ \text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2} \]
  when there are \( k \) periods per annum.
Use of Convexity

- The approximation $\Delta P / P \approx - \text{duration} \times \text{yield change}$ works for small yield changes.
- To improve upon it for larger yield changes, use
  \[
  \frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2
  \]
  \[= - \text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.\]
- Recall the figure on p. 86.

Effective Convexity

- The effective convexity is defined as
  \[
  \frac{P_+ - P_- - 2P_0}{P_0 (0.5 \times (y_+ - y_-))^2},
  \]
  - $P_-$ is the price if the yield is decreased by $\Delta y$.
  - $P_+$ is the price if the yield is increased by $\Delta y$.
  - $P_0$ is the initial price, $y$ is the initial yield.
  - $\Delta y$ is small.
- Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

Term Structure of Interest Rates

Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)
Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds forms the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.

Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots yields to maturity against maturity.
- A par yield curve is constructed from bonds trading near par.

Four Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.
- The PV of one dollar $i$ periods from now is $\left[1 + S(i)\right]^{-i}$.
- The one-period spot rate is called the short rate.
- A spot rate curve is a plot of spot rates against maturity.

Problems with the PV Formula

- In the bond price formula,
  \[ \sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n}, \]
  every cash flow is discounted at the same yield $y$.
- Consider two riskless bonds with different yields to maturity because of their different cash flow streams.
- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.
- But shouldn’t they be discounted at the same rate?

Discount Factors

- In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of
  \[ P = \sum_{i=1}^{n} C_i d(i). \]
  Above, $d(i) \equiv \left[1 + S(i)\right]^{-i}, i = 1, 2, \ldots, n$, are called discount factors.
- $d(i)$ is the PV of one dollar $i$ periods from now.
- The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

• Start with the short rate $S(1)$.
  — Note that short-term Treasuries are zero-coupon bonds.

• Compute $S(2)$ from the two-period coupon bond price $P$ by solving
  
  $$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.$$ 

Extracting Spot Rates from Yield Curve (concluded)

• Inductively, we are given the market price $P$ of the $n$-period coupon bond and $S(1), S(2), \ldots, S(n-1)$.

• Then $S(n)$ can be computed from Eq. (10), repeated below,
  
  $$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.$$ 

• The running time is $O(n)$.

• The procedure is called bootstrapping.

Some Problems

• Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).

• Some maturities might be missing from the data points (the incompleteness problem).

• Treasuries might not be of the same quality.

• Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.
  — Lack economic justifications.

Yield Spread

• Consider a risky bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.

• Were this bond riskless, it would fetch
  
  $$P^* = \sum_{i=1}^{n} \frac{C_i}{[1 + S(i)]^i}.$$ 

• Since riskiness must be compensated, $P < P^*$.

• Yield spread is the difference between the IRR of the risky bond and that of a riskless bond with comparable maturity.
Static Spread

- The static spread is the amount $s$ by which the spot rate curve has to shift in parallel in order to price the risky bond correctly,

$$P = \sum_{t=1}^{n} \frac{C_t}{[1 + s + S(t)]^t}.$$ 

- Unlike the yield spread, the static spread incorporates information from the term structure.

Of Spot Rate Curve and Yield Curve

- $y_k$: yield to maturity for the $k$-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.