Principles of Financial Computing

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References

- Yuh-Dauh Lyuu. *Financial Engineering & Computation: Principles, Mathematics, Algorithms*. Cambridge University Press. 2002.
- Official Web page is

  `www.csie.ntu.edu.tw/~lyuu/finance1.html`
- Check

  `www.csie.ntu.edu.tw/~lyuu/capitals.html`

  for some of the software.

Useful Journals

- *Financial Analysts Journal.*
- *Journal of Computational Finance.*
- *Journal of Derivatives.*
- *Journal of Economic Dynamics & Control.*
- *Journal of Finance.*
- *Journal of Financial Economics.*
- *Journal of Fixed Income.*
- *Journal of Futures Markets.*
- *Journal of Financial and Quantitative Analysis.*
- *Journal of Portfolio Management.*
- *Journal of Real Estate Finance and Economics.*
- *Management Science.*
- *Mathematical Finance.*
- *Quantitative Finance.*
- *Review of Financial Studies.*
- *Review of Derivatives Research.*
- *Risk Magazine.*

Introduction

A Very Brief History of Modern Finance

- 1900: Ph.D. thesis *Mathematical Theory of Speculation* of Bachelier (1870–1946).
- 1950s: modern portfolio theory (MPT) of Markowitz.
- 1960s: the Capital Asset Pricing Model (CAPM) of Treynor, Sharpe, Lintner (1916–1984), and Mossin.
- 1960s: the efficient markets hypothesis of Samuelson and Fama.
- 1970s: theory of option pricing of Black (1938–1995) and Scholes.
- 1970s–present: new instruments and pricing methods.

A Very Brief and Biased History of Modern Computers

- 1930s: theory of Gödel (1906–1978), Turing (1912–1954), and Church (1903–1995).
- 1940s: first computers (Z3, ENIAC, etc.) and birth of solid-state transistor (Bell Labs).
- 1950s: Texas Instruments patented integrated circuits; Backus (IBM) invented FORTRAN.
- 1960s: Internet (ARPA) and mainframes (IBM).
- 1970s: relational database (Codd) and PCs (Apple).
- 1980s: IBM PC and Lotus 1-2-3.
- 1990s: Windows 3.1 (Microsoft) and World Wide Web (Berners-Lee).

What This Course Is About

- Financial theories in pricing.
- Mathematical backgrounds.
- Derivative securities.
- Pricing models.
- Efficient algorithms in pricing financial instruments.
- Research problems.
- Finding your thesis directions.

What This Course Is *Not* About

- How to program.
- Basic mathematics in calculus, probability, and algebra.
- Details of the financial markets.
- How to be rich.
- How the markets will perform tomorrow.

Outstanding U.S. Debts (bln)

Year	Municipal	Treasury	Mortgage—related	U.S. corporate	Fed agencies	Money market	Asset—backed	Total
85	859.5	1,437.7	372.1	776.5	293.9	847.0	0.9	4,587.6
86	920.4	1,619.0	534.4	959.6	307.4	877.0	7.2	5,225.0
87	1,010.4	1,724.7	672.1	1,074.9	341.4	979.8	12.9	5,816.2
88	1,082.3	1,821.3	772.4	1,195.7	381.5	1,108.5	29.3	6,391.0
89	1,135.2	1,945.4	971.5	1,292.5	411.8	1,192.3	51.3	7,000.0
90	1,184.4	2,195.8	1,333.4	1,350.4	434.7	1,156.8	89.9	7,745.4
91	1,272.2	2,471.6	1,636.9	1,454.7	442.8	1,054.3	129.9	8,462.4
92	1,302.8	2,754.1	1,937.0	1,557.0	484.0	994.2	163.7	9,192.8
93	1,377.5	2,989.5	2,144.7	1,674.7	570.7	971.8	199.9	9,928.8
94	1,341.7	3,126.0	2,251.6	1,755.6	738.9	1,034.7	257.3	10,505.8
95	1,293.5	3,307.2	2,352.1	1,937.5	844.6	1,177.3	316.3	11,228.5
96	1,296.0	3,459.7	2,486.1	2,122.2	925.8	1,393.9	404.4	12,088.1
97	1,367.5	3,456.8	2,680.2	2,346.3	1,022.6	1,692.8	535.8	13,102.0
98	1,464.3	3,355.5	2,955.2	2,666.2	1,296.5	1,978.0	731.5	14,447.2
99	1,532.5	3,281.0	3,334.2	3,022.9	1,616.5	2,338.2	900.8	16,026.4
00	1,567.8	2,966.9	3,564.7	3,372.0	1,851.9	2,661.0	1,071.8	17,056.1
01	1,688.4	2,967.5	4,125.5	3,817.4	2,143.0	2,542.4	1,281.1	18,565.3
02	1,783.8	3,204.9	4,704.9	3,997.2	2,358.5	2,577.5	1,543.3	20,170.1

Analysis of Algorithms

It is unworthy of excellent men
to lose hours like slaves
in the labor of computation.
— Gottfried Wilhelm Leibniz (1646–1716)

Computability and Algorithms

- Algorithms are precise procedures that can be turned into computer programs.
- Uncomputable problems.
  - Does this program have infinite loops?
  - Is this program bug free?
- Computable problems.
  - Intractable problems.
  - Tractable problems.

Complexity

- Start with a set of basic operations which will be assumed to take one unit of time.
- The total number of these operations is the total work done by an algorithm (its computational complexity).
- The space complexity is the amount of memory space used by an algorithm.
- Concentrate on the abstract complexity of an algorithm instead of its detailed implementation.
- Complexity is a good guide to an algorithm's *actual* running time.

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Asymptotics

- Consider the search algorithm on p. 15.
- The worst-case complexity is n comparisons (why?).
- There are operations besides comparison.
- We care only about the asymptotic growth rate not the exact number of operations.
  - So the complexity of maintaining the loop is subsumed by the complexity of the body of the loop.
- The complexity is hence $O(n)$.

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Algorithm for Searching an Element

for $k = 1, 2, 3, \ldots, n$ **do**
    if $x = A_k$ **then**
        `return` k;
    end if
end for
`return` not-found;

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Common Complexities

- Let n stand for the "size" of the problem.
  - Number of elements, number of cash flows, etc.
- Linear time if the complexity is $O(n)$.
- Quadratic time if the complexity is $O(n^2)$.
- Cubic time if the complexity is $O(n^3)$.
- Exponential time if the complexity is $2^{O(n)}$.
- Superpolynomial if the complexity is less than exponential but higher than polynomials, say $2^{O(\sqrt{n})}$.
- It is possible for an exponential-time algorithm to perform well on "typical" inputs.

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Basic Financial Mathematics

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In the fifteenth century
mathematics was mainly concerned with
questions of commercial arithmetic and
the problems of the architect.
— Joseph Alois Schumpeter (1883–1950)

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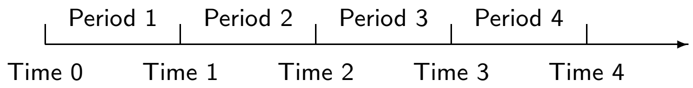
The Time Line



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Time Value of Money

$$\begin{aligned} \text{FV} &= \text{PV}(1+r)^n, \\ \text{PV} &= \text{FV} \times (1+r)^{-n}. \end{aligned}$$

- FV (future value).
- PV (present value).
- r: interest rate.

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Periodic Compounding

If interest is compounded m times per annum,

$$\text{FV} = \text{PV}\left(1 + \frac{r}{m}\right)^{nm}. \tag{1}$$

Common Compounding Methods

- Annual compounding: $m = 1$.
- Semiannual compounding: $m = 2$.
- Quarterly compounding: $m = 4$.
- Monthly compounding: $m = 12$.
- Weekly compounding: $m = 52$.
- Daily compounding: $m = 365$.

Easy Translations

- An interest rate of r compounded m times a year is "equivalent to" an interest rate of r/m per $1/m$ year.
- If a loan asks for a return of 1% per month, the annual interest rate will be 12% *with monthly compounding*.

Example

- Annual interest rate is 10% compounded twice per annum.
- Each dollar will grow to be

  $$[\,1 + (0.1/2)\,]^2 = 1.1025$$

  one year from now.
- The rate is equivalent to an interest rate of 10.25% compounded once *per annum*.

Continuous Compounding

- Let $m \to \infty$ so that
$$\left(1 + \frac{r}{m}\right)^m \to e^r$$
in Eq. (1) on p. 21.
- Then
$$\text{FV} = \text{PV} \times e^{rn},$$
where $e = 2.71828\ldots$.

Continuous Compounding (concluded)

- Continuous compounding is easier to work with.
- Suppose the annual interest rate is r_1 for n_1 years and r_2 for the following n_2 years.
- Then the FV of one dollar will be
$$e^{r_1 n_1 + r_2 n_2}.$$

Efficient Algorithms for PV and FV

- The PV of the cash flow $C_1, C_2, \ldots, C_n$ at times $1, 2, \ldots, n$ is
$$\frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_n}{(1+y)^n}.$$
- This formula and its variations are the engine behind most of financial calculations.
  - What is y?
  - What are C_i?
  - What is n?
- It can be computed by the algorithm on p. 28.

Algorithm for Evaluating PV

```
x := 0;
d := 1 + y;
for i = n, n − 1, ..., 1 do
    x := (x + C_i)/d;
end for
return x;
```

Horner's Rule: The Idea Behind p. 28

- This idea is

$$\left(\cdots \left(\left(\frac{C_n}{1+y} + C_{n-1} \right) \frac{1}{1+y} + C_{n-2} \right) \frac{1}{1+y} + \cdots \right) \frac{1}{1+y}.$$

  - Due to Horner (1786–1837) in 1819.
- The algorithm takes $O(n)$ time.
- It is the most efficient possible in terms of the absolute number of arithmetic operations.[a]

[a] Borodin and Munro (1975).

Conversion between Compounding Methods

- Suppose r_1 is the annual rate with continuous compounding.
- Suppose r_2 is the equivalent rate compounded m times per annum.
- How are they related?

Conversion between Compounding Methods (concluded)

- Both interest rates must produce the same amount of money after one year.
- That is,

$$\left(1 + \frac{r_2}{m} \right)^m = e^{r_1}.$$

- Therefore,

$$\begin{aligned} r_1 &= m \ln \left(1 + \frac{r_2}{m} \right), \\ r_2 &= m \left(e^{r_1/m} - 1 \right). \end{aligned}$$

Annuities

- An annuity pays out the same C dollars at the end of each year for n years.
- With a rate of r, the FV at the end of the nth year is

$$\sum_{i=0}^{n-1} C(1+r)^i = C\, \frac{(1+r)^n - 1}{r}. \tag{2}$$

General Annuities

- If m payments of C dollars each are received per year (the general annuity), then Eq. (2) becomes

$$C\,\frac{\left(1+\frac{r}{m}\right)^{nm}-1}{\frac{r}{m}}.$$

- The PV of a general annuity is

$$\sum_{i=1}^{nm} C\left(1+\frac{r}{m}\right)^{-i} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-nm}}{\frac{r}{m}}. \qquad (3)$$

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Amortization

- It is a method of repaying a loan through regular payments of interest *and* principal.
- The size of the loan (the original balance) is reduced by the principal part of each payment.
- The interest part of each payment pays the interest incurred on the remaining principal balance.
- As the principal gets paid down over the term of the loan, the interest part of the payment diminishes.

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Example: Home Mortgage

- By paying down the principal consistently, the risk to the lender is lowered.
- When the borrower sells the house, the remaining principal is due the lender.
- Consider the equal-payment case, i.e., fixed-rate, level-payment, fully amortized mortgages.
  - They are called traditional mortgages in the U.S.

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A Numerical Example

- Consider a 15-year, \$250,000 loan at 8.0% interest rate.
- Solving Eq. (3) on p. 33 with $\mathrm{PV} = 250000$, $n = 15$, $m = 12$, and $r = 0.08$ gives a monthly payment of $C = 2389.13$.
- The amortization schedule is shown on p. 37.
- In every month (1) the principal and interest parts add up to \$2,389.13, (2) the remaining principal is reduced by the amount indicated under the Principal heading, and (3) the interest is computed by multiplying the remaining balance of the previous month by 0.08/12.

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Month	Payment	Interest	Principal	Remaining principal
				250,000.000
1	2,389.13	1,666.667	722.464	249,277.536
2	2,389.13	1,661.850	727.280	248,550.256
3	2,389.13	1,657.002	732.129	247,818.128
		...		
178	2,389.13	47.153	2,341.980	4,730.899
179	2,389.13	31.539	2,357.591	2,373.308
180	2,389.13	15.822	2,373.308	0.000
Total	430,043.438	180,043.438	250,000.000	

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Method 1 of Calculating the Remaining Principal

- Go down the amortization schedule until you reach the particular month you are interested in.
  - A month's remaining principal equals the previous month's remaining principal times the monthly interest rate and subtracted by the monthly payment.
- This method is relatively slow but is universal in its applicability.
  - It can, for example, accommodate prepayment and variable interest rates.

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Method 2 of Calculating the Remaining Principal

- Right after the kth payment, the remaining principal is the PV of the future $nm - k$ cash flows,

$$\sum_{i=1}^{nm-k} C\left(1+\frac{r}{m}\right)^{-i} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-nm+k}}{\frac{r}{m}}.$$

- This method is faster but more limited in applications.

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