Term Structure Dynamics

- An n-period zero-coupon bond's price can be computed by assigning \$1 to every node at period n and then applying backward induction.
- Repeating this step for $n = 1, 2, \ldots$, one obtains the market discount function implied by the tree.
- The tree therefore determines a term structure.
- It also contains a term structure dynamics as taking any node in the tree as the current state induces a binomial interest rate tree and, again, a term structure.
- It defines how the term structure evolves over time.

Sample Term Structure

- We shall construct interest rate trees consistent with the sample term structure in the following table.
- Assume the short rate volatility is such that $v \equiv r_{\mathrm{h}}/r_{\ell} = 1.5$, independent of time.

Period	1	2	3
Spot rate (%)	4	4.2	4.3
One-period forward rate (%)	4	4.4	4.5
Discount factor	0.96154	0.92101	0.88135

An Approximate Calibration Scheme

- Start with the implied one-period forward rates and then equate the expected short rate with the forward rate (see Exercise 5.6.6 in text).
- For the first period, the forward rate is today's one-period spot rate.
- In general, let f_j denote the forward rate in period j.
- This forward rate can be derived from the market discount function via $f_j = (d(j)/d(j+1)) - 1$ (see Exercise 5.6.3 in text).

An Approximate Calibration Scheme (continued)

- Since the ith short rate, $1 \leq i \leq j$, occurs with probability $2^{-(j-1)} \binom{j-1}{i-1}$, this means

$$\sum_{i=1}^{j} 2^{-(j-1)} \binom{j-1}{i-1} r_j v_j^{i-1} = f_j.$$

- Thus

$$r_j = \left(\frac{2}{1+v_j} \right)^{j-1} f_j. \tag{84}$$

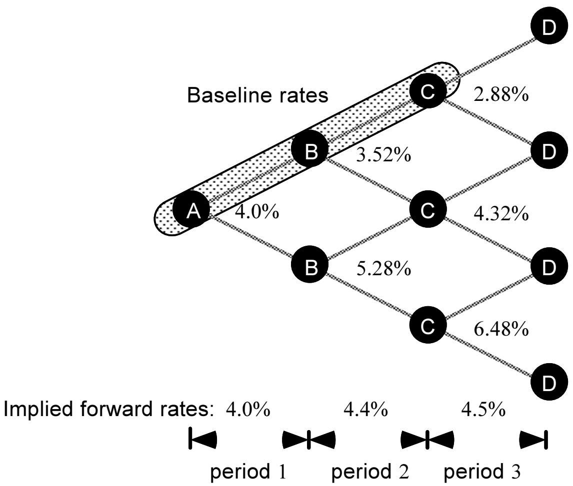
- The binomial interest rate tree is trivial to set up.

An Approximate Calibration Scheme (concluded)

- The ensuing tree for the sample term structure appears in figure next page.
- For example, the price of the zero-coupon bond paying \$1 at the end of the third period is

  $$\frac{1}{4} \times \frac{1}{1.04} \times \left(\frac{1}{1.0352} \times \left(\frac{1}{1.0288} + \frac{1}{1.0432} \right) + \frac{1}{1.0528} \times \left(\frac{1}{1.0432} + \frac{1}{1.0648} \right) \right)$$

  or 0.88155, which exceeds discount factor 0.88135.
- The tree is thus not calibrated.
- Indeed, this bias is inherent (see text).



Issues in Calibration

- The model prices generated by the binomial interest rate tree should match the observed market prices.
- Perhaps the most crucial aspect of model building.
- Treat the backward induction for the model price of the m-period zero-coupon bond as computing some function of the unknown baseline rate r_m called $f(r_m)$.
- A root-finding method is applied to solve $f(r_m) = P$ for r_m given the zero's price P and $r_1, r_2, \ldots, r_{m-1}$.
- This procedure is carried out for $m = 1, 2, \ldots, n$.
- Runs in cubic time, hopelessly slow.

Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in quadratic time by the use of forward induction (Jamshidian, 1991).
- The scheme records how much \$1 at a node contributes to the model price.
- This number is called the state price.
  - It stands for the price of a state contingent claim that pays \$1 at that particular node (state) and 0 elsewhere.
- The column of state prices will be established by moving *forward* from time 1 to time n.

Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at time j and there are $j+1$ nodes.
  - The baseline rate for period j is $r \equiv r_j$.
  - The multiplicative ratio be $v \equiv v_j$.
  - $P_1, P_2, \ldots, P_j$ are the state prices a period prior, corresponding to rates $r, rv, \ldots, rv^{j-1}$.
- By definition, $\sum_{i=1}^{j} P_i$ is the price of the $(j-1)$-period zero-coupon bond.

Binomial Interest Rate Tree Calibration (continued)

- One dollar at time j has a known market value of $1/[\,1+S(j)\,]^j$, where $S(j)$ is the j-period spot rate.
- Alternatively, this dollar has a present value of

$$g(r) \equiv \frac{P_1}{(1+r)} + \frac{P_2}{(1+rv)} + \frac{P_3}{(1+rv^2)} + \cdots + \frac{P_j}{(1+rv^{j-1})}.$$

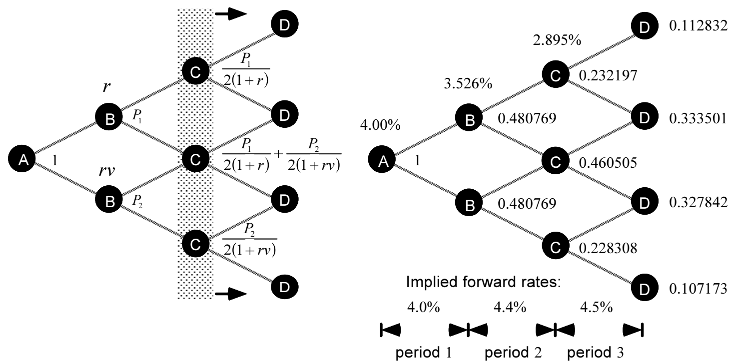
- So we solve

$$g(r) = \frac{1}{[\,1+S(j)\,]^j} \tag{85}$$

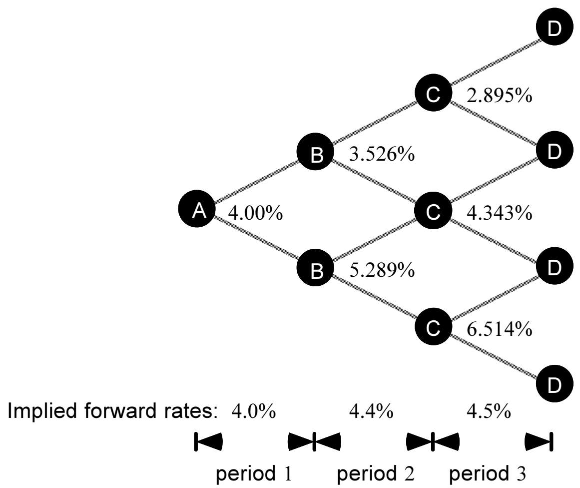
for r.

Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive solution for r is guaranteed.
- The state prices at time j can now be calculated (see figure (a) next page).
- We call a tree with these state prices a binomial state price tree (see figure (b) next page).
- The calibrated tree is depicted in on p. 734.



(a) (b)



Binomial Interest Rate Tree Calibration (concluded)

- The Newton-Raphson method can be used to solve for the r in Eq. (85) on p. 731 as $g'(r)$ is easy to evaluate.
- The monotonicity and the convexity of $g(r)$ also facilitate root finding.
- The above idea is straightforward to implement.
- The total running time is $O(\mathcal{C}n^2)$, where $\mathcal{C}$ is the maximum number of times the root-finding routine iterates, each consuming $O(n)$ work.
- With a good initial guess, the Newton-Raphson method converges in only a few steps (Lyuu, 1999).

A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.
- The baseline rate for the second period, r_2, satisfies

$$\frac{0.480769}{1+r_2} + \frac{0.480769}{1+1.5\times r_2} = 0.92101.$$

- The result is $r_2 = 3.526\%$.
- This is used to derive the next column of state prices shown in figure (b) on p. 733 as 0.232197, 0.460505, and 0.228308.
- Their sum gives the correct market discount factor 0.92101.

A Numerical Example (concluded)

- The baseline rate for the third period, r_3, satisfies

$$\frac{0.232197}{1+r_3} + \frac{0.460505}{1+1.5\times r_3} + \frac{0.228308}{1+(1.5)^2\times r_3} = 0.88135.$$

- The result is $r_3 = 2.895\%$.
- Now, redo the calculation on p. 726 using the new rates:

$$\frac{1}{4}\times\frac{1}{1.04}\times\left[\frac{1}{1.03526}\times\left(\frac{1}{1.02895}+\frac{1}{1.04343}\right)+\frac{1}{1.05289}\times\left(\frac{1}{1.04343}+\frac{1}{1.06514}\right)\right],$$

which equals 0.88135, an exact match.

- The tree on p. 734 prices without bias the benchmark securities.
- The term structure dynamics is shown on p. 738.

$$
\begin{bmatrix} 0.96154 \\ 0.92101 \\ 0.88135 \end{bmatrix}
\begin{matrix} \nearrow \\ \\ \searrow \end{matrix}
\begin{matrix} \begin{bmatrix} 0.965941 \\ 0.932250 \end{bmatrix} \\ \\ \begin{bmatrix} 0.949767 \\ 0.900959 \end{bmatrix} \end{matrix}
\begin{matrix} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{matrix}
\begin{matrix} [\,0.971865\,] \\ \\ [\,0.958378\,] \\ \\ [\,0.938844\,] \end{matrix}
$$

Spread of Nonbenchmark Bonds

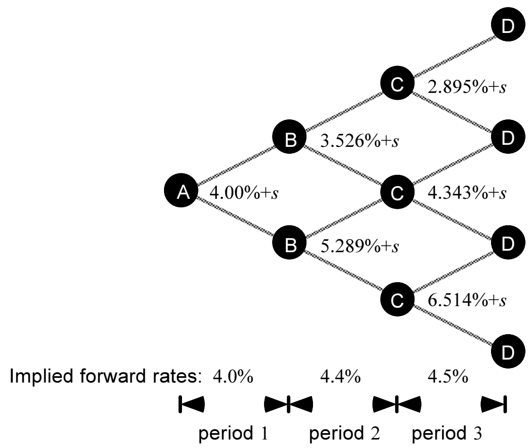
- Model prices calculated by the calibrated tree as a rule do not match market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- We look for the spread that, when added uniformly over the short rates in the tree, makes the model price equal the market price.
- We will apply the spread concept to option-free bonds here.

Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 741.
- Consider a security with cash flow C_i at time i for $i = 1, 2, 3$.
- Its model price is $p(s)$, which is equal to

$$
\frac{1}{1.04+s} \times \left[C_1 + \frac{1}{2} \times \frac{1}{1.03526+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.02895+s} + \frac{C_3}{1.04343+s} \right) \right) + \frac{1}{2} \times \frac{1}{1.05289+s} \times \left(C_2 + \frac{1}{2} \left(\frac{C_3}{1.04343+s} + \frac{C_3}{1.06514+s} \right) \right) \right].
$$

- Given a market price of P, the spread is the s that solves $P = p(s)$.



Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of s.
- We will employ the Newton-Raphson root-finding method to solve $p(s) - P = 0$ for s.
- But a quick look at the equation above reveals that evaluating $p'(s)$ directly is infeasible.
- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.

Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate r.
- In the process of computing the model price $p(s)$, a price $p_{\mathrm{A}}(s)$ is computed at A.
- Prices computed at A's two successor nodes B and C are discounted by $r + s$ to obtain $p_{\mathrm{A}}(s)$ as follows,

$$p_{\mathrm{A}}(s) = c + \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1 + r + s)},$$

where c denotes the cash flow at A.

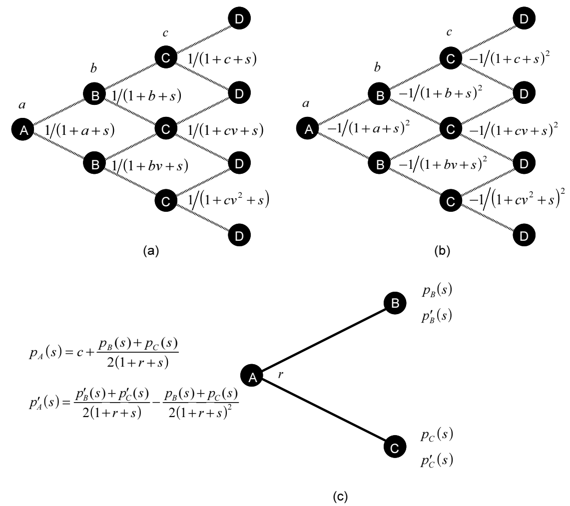
Spread of Nonbenchmark Bonds (continued)

- To compute $p'_{\mathrm{A}}(s)$ as well, node A calculates

$$p'_{\mathrm{A}}(s) = \frac{p'_{\mathrm{B}}(s) + p'_{\mathrm{C}}(s)}{2(1 + r + s)} - \frac{p_{\mathrm{B}}(s) + p_{\mathrm{C}}(s)}{2(1 + r + s)^2}. \qquad (86)$$

- This is easy if $p'_{\mathrm{B}}(s)$ and $p'_{\mathrm{C}}(s)$ are also computed at nodes B and C.
- Apply the above procedure inductively to yield $p(s)$ and $p'(s)$ at the root (see p. 745).
- This is called the differential tree method.[a]

[a]Lyuu (1999).



Spread of Nonbenchmark Bonds (continued)

- Let $\mathcal{C}$ represent the number of times the tree is traversed, which takes $O(n^2)$ time.
- The total running time is $O(\mathcal{C}n^2)$.
- In practice $\mathcal{C}$ is a small constant.
- The memory requirement is $O(n)$.

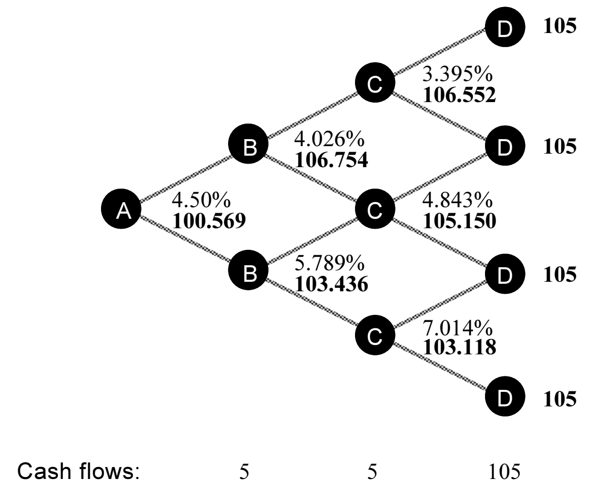
Spread of Nonbenchmark Bonds (continued)

Number of partitions n	Running time (s)	Number of iterations	Number of partitions	Running time (s)	Number of iterations
500	7.850	5	10500	3503.410	5
1500	71.650	5	11500	4169.570	5
2500	198.770	5	12500	4912.680	5
3500	387.460	5	13500	5714.440	5
4500	641.400	5	14500	6589.360	5
5500	951.800	5	15500	7548.760	5
6500	1327.900	5	16500	8502.950	5
7500	1761.110	5	17500	9523.900	5
8500	2269.750	5	18500	10617.370	5
9500	2834.170	5	....	....	....

75MHz Sun SPARCstation 20.

Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread can be shown to be 50 basis points over the tree (see p. 749).
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread and static spread of the nonbenchmark bond over an otherwise identical benchmark bond.



More Applications of the Differential Tree: Calibrating Black-Derman-Toy (in seconds)

Number of years	Running time	Number of years	Running time	Number of years	Running time
3000	398.880	39000	8562.640	75000	26182.080
6000	1697.680	42000	9579.780	78000	28138.140
9000	2539.040	45000	10785.850	81000	30230.260
12000	2803.890	48000	11905.290	84000	32317.050
15000	3149.330	51000	13199.470	87000	34487.320
18000	3549.100	54000	14411.790	90000	36795.430
21000	3990.050	57000	15932.370	120000	63767.690
24000	4470.320	60000	17360.670	150000	98339.710
27000	5211.830	63000	19037.910	180000	140484.180
30000	5944.330	66000	20751.100	210000	190557.420
33000	6639.480	69000	22435.050	240000	249138.210
36000	7611.630	72000	24292.740	270000	313480.390

75MHz Sun SPARCstation 20, one period per year.

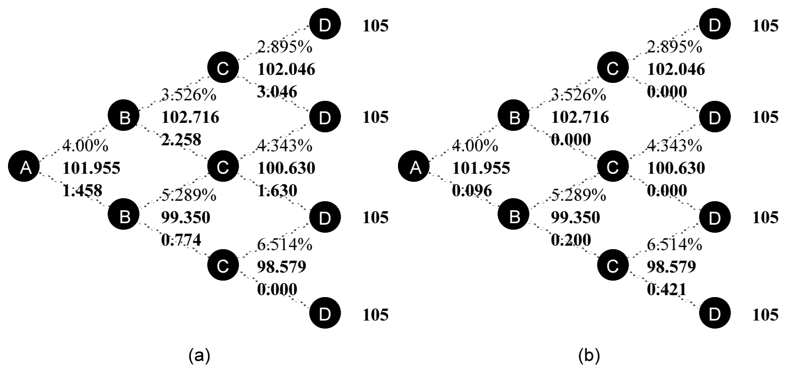
More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)

American call			American put		
Number of partitions	Running time	Number of iterations	Number of partitions	Running time	Number of iterations
100	0.008210	2	100	0.013845	3
200	0.033310	2	200	0.036335	3
300	0.072940	2	300	0.120455	3
400	0.129180	2	400	0.214100	3
500	0.201850	2	500	0.333950	3
600	0.290480	2	600	0.323260	2
700	0.394090	2	700	0.435720	2
800	0.522040	2	800	0.569605	2

Intel 166MHz Pentium, running on Microsoft Windows 95.

Fixed-Income Options

- Consider a two-year 99 European call on the three-year, 5% Treasury.
- Assume the Treasury pays annual interest.
- From p. 753 the three-year Treasury's price minus the $5 interest could be $102.046, $100.630, or $98.579 two years from now.
- Since these prices do not include the accrued interest, we should compare the strike price against them.
- The call is therefore in the money in the first two scenarios, with values of $3.046 and $1.630, and out of the money in the third scenario.



Fixed-Income Options (continued)

- The option value is calculated to be \$1.458 on p. 753(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only if the Treasury is worth \$98.579 without the accrued interest.
- The option value is computed to be \$0.096 on p. 753(b).

Fixed-Income Options (concluded)

- The present value of the strike price is $\mathrm{PV}(X) = 99 \times 0.92101 = 91.18$.
- The Treasury is worth $B = 101.955$.
- The present value of the interest payments during the life of the options is

$$\mathrm{PV}(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275.$$

- The call and the put are worth $C = 1.458$ and $P = 0.096$, respectively.
- Hence the put-call parity is preserved:

$$C = P + B - \mathrm{PV}(I) - \mathrm{PV}(X).$$

Delta or Hedge Ratio

- How much does the option price change in response to changes in the price of the underlying bond?
- This relation is called delta (or hedge ratio) defined as

$$\frac{O_{\mathrm{h}} - O_\ell}{P_{\mathrm{h}} - P_\ell}.$$

- In the above P_{h} and P_ℓ denote the bond prices if the short rate moves up and down, respectively.
- Similarly, O_{h} and O_ℓ denote the option values if the short rate moves up and down, respectively.

Delta or Hedge Ratio (concluded)

- Since delta measures the sensitivity of the option value to changes in the underlying bond price, it shows how to hedge one with the other.
- Take the call and put on p. 753 as examples.
- Their deltas are

$$\begin{aligned} \frac{0.774 - 2.258}{99.350 - 102.716} &= 0.441, \\ \frac{0.200 - 0.000}{99.350 - 102.716} &= -0.059, \end{aligned}$$

respectively.

Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an n-period zero-coupon bond.
- First find its yield to maturity y_{h} (y_ℓ, respectively) at the end of the initial period if the rate rises (declines, respectively).
- The yield volatility for our model is defined as $(1/2) \ln(y_{\mathrm{h}}/y_\ell)$.

Volatility Term Structures (continued)

- For example, based on the tree on p. 734, the two-year zero's yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.
- Its yield volatility is therefore

$$\frac{1}{2} \ln \left(\frac{0.05289}{0.03526} \right) = 20.273\%.$$

Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the rate rises, the price of the zero one year from now will be

$$\frac{1}{2} \times \frac{1}{1.05289} \times \left(\frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.$$

- Thus its yield is $\sqrt{\frac{1}{0.90096}} - 1 = 0.053531$.
- If the rate declines, the price of the zero one year from now will be

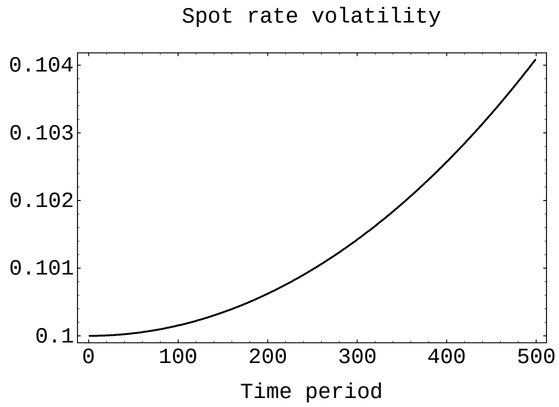
$$\frac{1}{2} \times \frac{1}{1.03526} \times \left(\frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.$$

Volatility Term Structures (continued)

- Thus its yield is $\sqrt{\frac{1}{0.93225}} - 1 = 0.0357$.
- The yield volatility is hence

$$\frac{1}{2} \ln \left(\frac{0.053531}{0.0357} \right) = 20.256\%,$$

  slightly less than the one-year yield volatility.
- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.



Short rate volatility given flat %10 volatility term structure.

Volatility Term Structures (continued)

- We started with v_i and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The v_i—hence the short rate volatilities via Eq. (82) on p. 714—and the r_i are then simultaneously determined.
- The result is the Black-Derman-Toy model.

Volatility Term Structures (concluded)

- Suppose the user supplies the volatility term structure which results in $(v_1, v_2, v_3, \ldots)$ for the tree.
- The volatility term structure one period from now will be determined by $(v_2, v_3, v_4, \ldots)$ not $(v_1, v_2, v_3, \ldots)$.
- The volatility term structure supplied by the user is hence not maintained through time.
- This issue will be addressed by other types of (complex) models.

Foundations of Term Structure Modeling

[Meriwether] scoring especially high marks
in mathematics — an indispensable subject
for a bond trader.
— Roger Lowenstein,
When Genius Failed (2000)

Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time t, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

t: a point in time.

$r(t)$: the one-period riskless rate prevailing at time t for repayment one period later (the instantaneous spot rate, or short rate, at time t).

$P(t, T)$: the present value at time t of one dollar at time T.

Standard Notations (continued)

$r(t, T)$: the $(T - t)$-period interest rate prevailing at time t stated on a per-period basis and compounded once per period—in other words, the $(T - t)$-period spot rate at time t.

- The long rate is defined as $r(t, \infty)$.

$F(t, T, M)$: the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \geq T$.

Standard Notations (concluded)

$f(t, T, L)$**:** the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.

$f(t, T)$**:** the one-period or instantaneous forward rate at time T as seen at time t stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$.

Fundamental Relations

- The price of a zero-coupon bond equals

$$P(t, T) = \begin{cases} (1 + r(t, T))^{-(T-t)} & \text{in discrete time,} \\ e^{-r(t,T)(T-t)} & \text{in continuous time.} \end{cases}$$

- $r(t, T)$ as a function of T defines the spot rate curve at time t.
- By definition,

$$f(t, t) = \begin{cases} r(t, t+1) & \text{in discrete time,} \\ r(t, t) & \text{in continuous time.} \end{cases}$$

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

$$F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \tag{87}$$

  – The forward price equals the future value at time T of the underlying asset (see text for proof).

- Equation (87) holds whether the model is discrete-time or continuous-time, and it implies

$$F(t, T, M) = F(t, T, S)\, F(t, S, M), \quad T \leq S \leq M.$$

Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

$$f(t, T, L) = \left(\frac{1}{F(t, T, T+L)} \right)^{1/L} - 1 = \left(\frac{P(t, T)}{P(t, T+L)} \right)^{1/L} - 1 \tag{88}$$

  in discrete time.

  – $f(t, T, L) = \frac{1}{L} \left(\frac{P(t,T)}{P(t,T+L)} - 1 \right)$ is the analog to Eq. (88) under simple compounding.

Fundamental Relations (continued)

- In continuous time,

$$f(t,T,L) = -\frac{\ln F(t,T,T+L)}{L} = \frac{\ln(P(t,T)/P(t,T+L))}{L} \tag{89}$$

by Eq. (87) on p. 772.

- Furthermore,

$$\begin{aligned} f(t,T,\Delta t) &= \frac{\ln(P(t,T)/P(t,T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t,T)}{\partial T} \\ &= -\frac{\partial P(t,T)/\partial T}{P(t,T)}. \end{aligned}$$

Fundamental Relations (continued)

- So

$$f(t,T) \equiv \lim_{\Delta t \to 0} f(t,T,\Delta t) = -\frac{\partial P(t,T)/\partial T}{P(t,T)}, \quad t \leq T. \tag{90}$$

- Because Eq. (90) is equivalent to

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds}, \tag{91}$$

the spot rate curve is

$$r(t,T) = \frac{1}{T-t} \int_t^T f(t,s)\,ds.$$

Fundamental Relations (concluded)

- The discrete analog to Eq. (91) is

$$P(t,T) = \frac{1}{(1+r(t))(1+f(t,t+1))\cdots(1+f(t,T-1))}. \tag{92}$$

- The short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t,T)}{\partial T}\right|_{T=t}.$$

  - This can be verified with Eq. (90) on p. 775 and the observation that $P(t,t) = 1$ and $r(t) = f(t,t)$.

Risk-Neutral Pricing

- Under the local expectations theory, the expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t+1 < T$,

$$\frac{E_t[\,P(t+1,T)\,]}{P(t,T)} = 1 + r(t). \tag{93}$$

  - Relation (93) in fact follows from the risk-neutral valuation principle, Theorem 18 (p. 428).

Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability π.
- Rewrite Eq. (93) as

$$\frac{E_t^{\pi}[\,P(t+1,T)\,]}{1+r(t)} = P(t,T).$$

  - It says the current spot rate curve equals the expected spot rate curve one period from now discounted by the short rate.

Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

$$\begin{aligned}
& P(t,T) \\
= \;& E_t^{\pi}\left[\frac{P(t+1,T)}{1+r(t)}\right] \\
= \;& E_t^{\pi}\left[\frac{E_{t+1}^{\pi}[\,P(t+2,T)\,]}{(1+r(t))(1+r(t+1))}\right] = \cdots \\
= \;& E_t^{\pi}\left[\frac{1}{(1+r(t))(1+r(t+1))\cdots(1+r(T-1))}\right]. \qquad (94)
\end{aligned}$$

Risk-Neutral Pricing (concluded)

- Equation (93) on p. 777 can also be expressed as

$$E_t[\,P(t+1,T)\,] = F(t,t+1,T).$$

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.

Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies

$$P(t,T) = E_t\left[e^{-\int_t^T r(s)\,ds}\right], \quad t < T. \qquad (95)$$

- Note that $e^{\int_t^T r(s)\,ds}$ is the bank account process, which denotes the rolled-over money market account.
- When the local expectations theory holds, riskless arbitrage opportunities are impossible.