Forwards, Futures, Futures Options, Swaps

Terms

- r will denote the riskless interest rate.
- The current time is t.
- The maturity date is T.
- The remaining time to maturity is $\tau \equiv T - t$ (all measured in years).
- The spot price S, the spot price at maturity is S_T.
- The delivery price is X.

Terms (concluded)

- The forward or futures price is F for a newly written contract.
- The value of the contract is f.
- A price with a subscript t usually refers to the price at time t.
- Continuous compounding will be assumed throughout this chapter.

Forward Contracts

- Forward contracts are for the delivery of the underlying asset for a certain delivery price on a specific time.
  - Foreign currencies, bonds, corn, etc.
- Ideal for hedging purposes.
- A farmer enters into a forward contract with a food processor to deliver 100,000 bushels of corn for $2.5 per bushel on September 27, 1995.
- The farmer is assured of a buyer at an acceptable price.
- The processor knows the cost of corn in advance.

Forward Contracts (concluded)

- A forward agreement limits both risk and rewards.
  - If the spot price of corn rises on the delivery date, the farmer will miss the opportunity of extra profits.
  - If the price declines, the processor will be paying more than it would.
- Either side has an incentive to default.
- Other problems: The food processor may go bankrupt, the farmer can go bust, the farmer might not be able to harvest 100,000 bushels of corn because of bad weather, the cost of growing corn may skyrocket, etc.

Spot and Forward Exchange Rates

- Let S denote the spot exchange rate.
- Let F denote the forward exchange rate one year from now (both in domestic/foreign terms).
- r_{f} denotes the annual interest rates of the foreign currency.
- r_{ℓ} denotes the annual interest rates of the local currency.
- Arbitrage opportunities will arise unless these four numbers satisfy an equation.

Interest Rate Parity[a]

$$\frac{F}{S} = e^{r_{\ell} - r_{\mathrm{f}}}. \tag{30}$$

- A holder of the local currency can do either of:
  - Lend the money in the domestic market to receive $e^{r_{\ell}}$ one year from now.
  - Convert local currency for foreign currency, lend for 1 year in foreign market, and convert foreign currency into local currency at the fixed forward exchange rate, F, by selling forward foreign currency now.

[a]Keynes (1923). John Maynard Keynes (1883–1946) was one of the greatest economists in history.

Interest Rate Parity (concluded)

- No money changes hand in entering into a forward contract.
- One unit of local currency will hence become $F e^{r_{\mathrm{f}}}/S$ one year from now in the 2nd case.
- If $F e^{r_{\mathrm{f}}}/S > e^{r_{\ell}}$, an arbitrage profit can result from borrowing money in the domestic market and lending it in the foreign market.
- If $F e^{r_{\mathrm{f}}}/S < e^{r_{\ell}}$, an arbitrage profit can result from borrowing money in the foreign market and lending it in the domestic market.

Forward Price

- The payoff of a forward contract at maturity is

$$S_T - X.$$

- Forward contracts do not involve any initial cash flow.
- The forward price is the delivery price which makes the forward contract zero valued.
  - That is, $f = 0$ when $F = X$.

Forward Price (concluded)

- The delivery price cannot change because it is written in the contract.
- But the forward price may change after the contract comes into existence.
  - The value of a forward contract, f, is 0 at the outset.
  - It will fluctuate with the spot price thereafter.
  - This value is enhanced when the spot price climbs and depressed when the spot price declines.
- The forward price also varies with the maturity of the contract.

Forward Price: Underlying Pays No Income

Lemma 13 *For a forward contract on an underlying asset providing no income,*

$$F = Se^{r\tau}. \tag{31}$$

- If $F > Se^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.
- At maturity, sell the asset for F and use $Se^{r\tau}$ to repay the loan, leaving an arbitrage profit of $F - Se^{r\tau} > 0$.
- If $F < Se^{r\tau}$, do the opposite.

Example

- r is the annualized 3-month riskless interest rate.
- S is the spot price of the 6-month zero-coupon bond.
- A new 3-month forward contract on a 6-month zero-coupon bond should command a delivery price of $Se^{r/4}$.
- So if $r = 6\%$ and $S = 970.87$, then the delivery price is

$$970.87 \times e^{0.06/4} = 985.54.$$

Contract Value: The Underlying Pays No Income

The value of a forward contract is

$$f = S - Xe^{-r\tau}.$$

- Consider a portfolio of one long forward contract, cash amount $Xe^{-r\tau}$, and one short position in the underlying asset.
- The cash will grow to X at maturity, which can be used to take delivery of the forward contract.
- The delivered asset will then close out the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.

Forward Price: Underlying Pays Predictable Income

Lemma 14 *For a forward contract on an underlying asset providing a predictable income with a PV of* I,

$$F = (S - I)\, e^{r\tau}. \tag{32}$$

- If $F > (S - I)\, e^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.
- At maturity, the asset is sold for F, and $(S - I)\, e^{r\tau}$ is used to repay the loan, leaving an arbitrage profit of $F - (S - I)\, e^{r\tau} > 0$.
- If $F < (S - I)\, e^{r\tau}$, reverse the above.

Example

- Consider a 10-month forward contract on a \$50 stock.
- The stock pays a dividend of \$1 every 3 months.
- The forward price is

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3 \times r_9/4} \right) e^{r_{10} \times (10/12)}.$$

  - r_i is the annualized i-month interest rate.

Underlying Pays a Continuous Dividend Yield of q

- The value of a forward contract at any time prior to T is

$$f = Se^{-q\tau} - Xe^{-r\tau}. \tag{33}$$

  - See text for proof.
- One consequence of Eq. (33) is that the forward price is

$$F = Se^{(r-q)\,\tau}. \tag{34}$$

Futures Contracts vs. Forward Contracts

- They are traded on a central exchange.
- A clearinghouse.
  - Credit risk is minimized.
- Futures contracts are standardized instruments.
- Gains and losses are marked to market daily.
  - Adjusted at the end of each trading day based on the settlement price.
  - The settlement price is some kind of average traded price immediately before the end of trading.

Size of a Futures Contract

- The amount of the underlying asset to be delivered under the contract.
  - 5,000 bushels for the corn futures on the CBT.
  - One million U.S. dollars for the Eurodollar futures on the CME.
- A position can be closed out (or offset) by entering into a reversing trade to the original one.
- Most futures contracts are closed out in this way rather than have the underlying asset delivered.
  - Forward contracts are meant for delivery.

Daily Settlements

- Price changes in the futures contract are settled daily.
- Hence the spot price rather than the initial futures price is paid on the delivery date.
- Marking to market nullifies any financial incentive for not making delivery.
  - A farmer enters into a forward contract to sell a food processor 100,000 bushels of corn at $2.00 per bushel in November.
  - Suppose the price of corn rises to $2.5 by November.

Daily Settlements (concluded)

- (continued)
  - The farmer has incentive to sell his harvest in the spot market at $2.5.
  - With marking to market, the farmer has transferred $0.5 per bushel from his futures account to that of the food processor by November.
  - When the farmer makes delivery, he is paid the spot price, $2.5 per bushel.
  - The farmer has little incentive to default.
  - The net price remains $2.00 per bushel, the original delivery price.

Delivery and Hedging

- Delivery ties the futures price to the spot price.
- On the delivery date, the settlement price of the futures contract is determined by the spot price.
- Hence, when the delivery period is reached, the futures price should be very close to the spot price.
- Changes in futures prices usually track those in spot prices.
- This makes hedging possible.
- Before the delivery date, the futures price could be above or below the spot price.

Daily Cash Flows

- Let F_i denote the futures price at the end of day i.
- The contract's cash flow on day i is $F_i - F_{i-1}$.
- The net cash flow over the life of the contract is

$$(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1}) = F_n - F_0 = S_T - F_0.$$

- A futures contract has the same accumulated payoff $S_T - F_0$ as a forward contract.
- The actual payoff may differ because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed.

Forward and Futures Prices[a]

Futures price equals forward price if interest rates are nonstochastic![b]

[a] Cox, Ingersoll, and Ross (1981).

[b] This "justifies" treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.

Remarks

- When interest rates are stochastic, forward and futures prices are no longer theoretically identical.
  - Suppose interest rates are uncertain and futures prices move in the same direction as interest rates.
  - Then futures prices will exceed forward prices.
- For short-term contracts, the differences tend to be small.
- Unless stated otherwise, assume forward and futures prices are identical.

Futures Options

- The underlying of a futures option is a futures contract.
- Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price.
  - A call holder acquires a long futures position.
  - A put holder acquires a short futures position.
- The futures contract is then marked to market, and the futures position of the two parties will be at the prevailing futures price.

Futures Options (concluded)

- It works as if the call writer delivered a futures contract to the option holder and paid the holder the prevailing futures price minus the strike price.
- It works as if the put writer took delivery a futures contract from the option holder and paid the holder the strike price minus the prevailing futures price.
- The amount of money that changes hands upon exercise is the difference between the strike price and the prevailing futures price.

Forward Options

- Similar to futures options except that what is delivered is a forward contract with a delivery price equal to the option's strike price.
  - Exercising a call forward option results in a long position in a forward contract.
  - Exercising a put forward option results in a short position in a forward contract.
- Exercising a forward option incurs no immediate cash flows.

Example

- Consider a call with strike \$100 and an expiration date in September.
- The underlying asset is a forward contract with a delivery date in December.
- Suppose the forward price in July is \$110.
- Upon exercise, the call holder receives a forward contract with a delivery price of \$100.
- If an offsetting position is then taken in the forward market, a \$10 profit in December will be assured.
- A call on the futures would realize the \$10 profit in July.

Some Pricing Relations

- Let delivery take place at time T, the current time be 0, and the option on the futures or forward contract have expiration date t $(t \leq T)$.
- Assume a constant, positive interest rate.
- Although forward price equals futures price, a forward option does not have the same value as a futures option.
- The payoffs at time t are

$$\text{futures option} = \max(F_t - X, 0), \quad (35)$$
$$\text{forward option} = \max(F_t - X, 0)\, e^{-r(T-t)}. \quad (36)$$

Put-Call Parity

The put-call parity is slightly different from the one in Eq. (19) on p. 189.

Theorem 15 *(1) For European options on futures contracts,* $C = P - (X - F)\, e^{-rt}$*. (2) For European options on forward contracts,* $C = P - (X - F)\, e^{-rT}$*.*

- Consider a portfolio of one short call, one long put, one long futures contract, and a loan of $(X - F)\, e^{-rt}$.

The Proof (continued)

- Cash flow at time t:

	$F_t \leq X$	$F_t > X$
A short call	0	$X - F_t$
A long put	$X - F_t$	0
A long futures	$F_t - F$	$F_t - F$
A loan of $(X - F)\, e^{-rt}$	$F - X$	$F - X$
Total	0	0

- Since the net future cash flow is zero in both cases, the portfolio must have zero value today.

The Proof (concluded)

- This proves the theorem for futures option.
- The proof for forward options is identical except that the loan amount is $(X - F)\, e^{-rT}$ instead.

Early Exercise and Forward Options

The early exercise feature is not valuable.

Theorem 16 *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

- See text for proof.

Early exercise may be optimal for American futures options even if the underlying asset generates no payouts.

Theorem 17 *American futures options may be exercised optimally before expiration.*

Black Model[a]

- Formulas for European futures options:

$$C = Fe^{-rt}N(x) - Xe^{-rt}N(x - \sigma\sqrt{t}), \quad (37)$$
$$P = Xe^{-rt}N(-x + \sigma\sqrt{t}) - Fe^{-rt}N(-x),$$

where $x \equiv \frac{\ln(F/X) + (\sigma^2/2)\, t}{\sigma\sqrt{t}}$.

- Formulas (37) are related to those for options on a stock paying a continuous dividend yield.
- In fact, they are exactly Eqs. (26) on p. 268 with the dividend yield q set to the interest rate r and the stock price S replaced by the futures price F.

[a]Black (1976).

Black Model (concluded)

- This observation incidentally proves Theorem 17 on p. 391.
- For European forward options, just multiply the above formulas by $e^{-r(T-t)}$.
  - Forward options differ from futures options by a factor of $e^{-r(T-t)}$ based on Eqs. (35)–(36).

Binomial Model for Forward and Futures Options

- Futures price behaves like a stock paying a continuous dividend yield of r.
- Under the BOPM, the risk-neutral probability for the futures price is

$$p_{\mathrm{f}} \equiv (1-d)/(u-d)$$

by Eq. (27) on p. 269.
  - The futures price moves from F to Fu with probability p_{f} and to Fd with probability $1 - p_{\mathrm{f}}$.
- The binomial tree algorithm for forward options is identical except that Eq. (36) on p. 387 is the payoff.

Spot and Futures Prices under BOPM

- The futures price is related to the spot price via $F = Se^{rT}$ if the underlying asset pays no dividends.
- The stock price moves from $S = Fe^{-rT}$ to

$$Fue^{-r(T-\Delta t)} = Sue^{r\Delta t}$$

  with probability p_{f} per period.
- The stock price moves from $S = Fe^{-rT}$ to

$$Sde^{r\Delta t}$$

  with probability $1 - p_{\text{f}}$ per period.

Negative Probabilities Revisited

- As $0 < p_{\text{f}} < 1$, we have $0 < 1 - p_{\text{f}} < 1$ as well.
- Solve the problem of negative risk-neutral probabilities:
  - Suppose the stock pays a continuous dividend yield of q.
  - Build the tree for the futures price F of the futures contract expiring at the same time as the option.
  - Calculate S from F at each node via $S = Fe^{-(r-q)(T-t)}$.

Swaps

- Swaps are agreements between two counterparties to exchange cash flows in the future according to a predetermined formula.
- There are two basic types of swaps: interest rate and currency.
- An interest rate swap occurs when two parties exchange interest payments periodically.
- Currency swaps are agreements to deliver one currency against another (our focus here).

Currency Swaps

- A currency swap involves two parties to exchange cash flows in different currencies.
- Consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

	Dollars	Yen
A	$D_{\text{A}}\%$	$Y_{\text{A}}\%$
B	$D_{\text{B}}\%$	$Y_{\text{B}}\%$

- Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars.

Currency Swaps (continued)

- A straightforward scenario is for A to borrow yen at $Y_{\rm A}$% and B to borrow dollars at $D_{\rm B}$%.
- But suppose A is *relatively* more competitive in the dollar market than the yen market, and vice versa for B.
  - $Y_{\rm B} - Y_{\rm A} < D_{\rm B} - D_{\rm A}$.
- Consider this alternative arrangement:
  - A borrows dollars.
  - B borrows yen.
  - They enter into a currency swap with a bank as the intermediary.

Currency Swaps (concluded)

- The counterparties exchange principal at the beginning and the end of the life of the swap.
- This act transforms A's loan into a yen loan and B's yen loan into a dollar loan.
- The total gain is $((D_{\rm B} - D_{\rm A}) - (Y_{\rm B} - Y_{\rm A}))$%:
  - The total interest rate is originally $(Y_{\rm A} + D_{\rm B})$%.
  - The new arrangement has a smaller total rate of $(D_{\rm A} + Y_{\rm B})$%.
- Transactions will happen only if the gain is distributed so that the cost to each party is less than the original.

Example

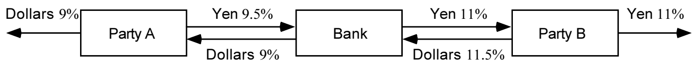
- A and B face the following borrowing rates:

	Dollars	Yen
A	9%	10%
B	12%	11%

- A wants to borrow yen, and B wants to borrow dollars.
- A can borrow yen directly at 10%.
- B can borrow dollars directly at 12%.

Example (concluded)

- As the rate differential in dollars (3%) is different from that in yen (1%), a currency swap with a total saving of $3 - 1 = 2$% is possible.
- A is relatively more competitive in the dollar market, and B the yen market.
- Figure next page shows an arrangement which is beneficial to all parties involved.
  - A effectively borrows yen at 9.5%. B borrows dollars at 11.5%.
  - The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.



As a Package of Cash Market Instruments

- Assume no default risk.
- Take B on p. 403 as an example.
- The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest.
- The pricing formula is $SP_{\mathrm{Y}} - P_{\mathrm{D}}$.
  - P_{D} is the dollar bond's value in dollars.
  - P_{Y} is the yen bond's value in yen.
  - S is the \$/yen spot exchange rate.

As a Package of Cash Market Instruments (concluded)

- The value of a currency swap depends on the term structures of interest rates in the currencies involved and the spot exchange rate.
- It has zero value when $SP_{\mathrm{Y}} = P_{\mathrm{D}}$.

Example

- Take a two-year swap on p. 403 with principal amounts of US$1 million and 100 million yen.
- The payments are made once a year.
- The spot exchange rate is 90 yen/$ and the term structures are flat in both nations—8% in the U.S. and 9% in Japan.
- For B, the value of the swap is (in millions of USD)

$$\frac{1}{90} \times \left(11 \times e^{-0.09} + 11 \times e^{-0.09 \times 2} + 111 \times e^{-0.09 \times 3}\right)$$
$$- \left(0.115 \times e^{-0.08} + 0.115 \times e^{-0.08 \times 2} + 1.115 \times e^{-0.08 \times 3}\right) \quad = \quad 0.074.$$

As a Package of Forward Contracts

- From Eq. (33) on p. 374, the forward contract maturing i years from now has a dollar value of

$$f_i \equiv (SY_i)\, e^{-qi} - D_i e^{-ri}. \tag{38}$$

  - Y_i is the yen inflow at year i.
  - S is the \$/yen spot exchange rate.
  - q is the yen interest rate.
  - D_i is the dollar outflow at year i.
  - r is the dollar interest rate.

As a Package of Forward Contracts (concluded)

- This formulation may be preferred to the cash market approach in cases involving costs of carry and convenience yields because forward prices already incorporate them.
- For simplicity, flat term structures were assumed.
- Generalization is straightforward.

Example

- Take the swap in the example on p. 406.
- Every year, B receives 11 million yen and pays 0.115 million dollars.
- In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars.
- Each of these transactions represents a forward contract.
- $Y_1 = Y_2 = 11$, $Y_3 = 111$, $S = 1/90$, $D_1 = D_2 = 0.115$, $D_3 = 1.115$, $q = 0.09$, and $r = 0.08$.
- Plug in these numbers to get $f_1 + f_2 + f_3 = 0.074$ million dollars as before.

Stochastic Processes and Brownian Motion

Stochastic Processes

- A stochastic process

$$X = \{ X(t) \}$$

is a time series of random variables.

- $X(t)$ (or X_t) is a random variable for each time t and is usually called the state of the process at time t.
- A realization of X is called a sample path.
- A sample path defines an ordinary function of t.

Stochastic Processes (concluded)

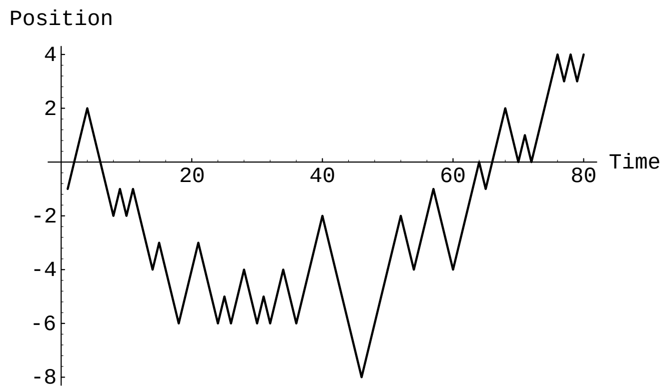
- If the times t form a countable set, X is called a discrete-time stochastic process or a time series.
- In this case, subscripts rather than parentheses are usually employed, as in

$$X = \{ X_n \}.$$

- If the times form a continuum, X is called a continuous-time stochastic process.

Random Walks

- The binomial model is a random walk in disguise.
- Consider a particle on the integer line, $0, \pm 1, \pm 2, \ldots$.
- In each time step, it can make one move to the right with probability p or one move to the left with probability $1 - p$.
  - This random walk is symmetric when $p = 1/2$.
- Connection with the BOPM: The particle's position denotes the cumulative number of up moves minus that of down moves.



Random Walk with Drift

$$X_n = \mu + X_{n-1} + \xi_n.$$

- ξ_n are independent and identically distributed with zero mean.
- Drift μ is the expected change per period.
- Note that this process is continuous in space.

Martingales

- $\{ X(t), t \geq 0 \}$ is a martingale if $E[\,|\,X(t)\,|\,] < \infty$ for $t \geq 0$ and

$$E[\,X(t)\,|\,X(u), 0 \leq u \leq s\,] = X(s). \tag{39}$$

- In the discrete-time setting, a martingale means

$$E[\,X_{n+1}\,|\,X_1, X_2, \ldots, X_n\,] = X_n. \tag{40}$$

- X_n can be interpreted as a gambler's fortune after the nth gamble.
- Identity (40) then says the expected fortune after the $(n+1)$th gamble equals the fortune after the nth gamble regardless of what may have occurred before.

Martingales (concluded)

- A martingale is therefore a notion of fair games.
- Apply the law of iterated conditional expectations to both sides of Eq. (40) on p. 416 to yield

$$E[\,X_n\,] = E[\,X_1\,] \tag{41}$$

for all n.

- Similarly, $E[\,X(t)\,] = E[\,X(0)\,]$ in the continuous-time case.

Example

- Consider the stochastic process

$$\{\, Z_n \equiv \sum_{i=1}^{n} X_i, n \geq 1 \,\},$$

where X_i are independent random variables with zero mean.

- This process is a martingale because

$$\begin{aligned}
& E[\,Z_{n+1}\,|\,Z_1, Z_2, \ldots, Z_n\,] \\
= \; & E[\,Z_n + X_{n+1}\,|\,Z_1, Z_2, \ldots, Z_n\,] \\
= \; & E[\,Z_n\,|\,Z_1, Z_2, \ldots, Z_n\,] + E[\,X_{n+1}\,|\,Z_1, Z_2, \ldots, Z_n\,] \\
= \; & Z_n + E[\,X_{n+1}\,] = Z_n.
\end{aligned}$$

Probability Measure

- A martingale is defined with respect to a probability measure, under which the expectation is taken.
  - A probability measure assigns probabilities to states of the world.
- A martingale is also defined with respect to an information set.
  - In the characterizations (39)–(40) on p. 416, the information set contains the current and past values of X by default.
  - But it needs not be so.

Probability Measure (continued)

- A stochastic process $\{ X(t), t \geq 0 \}$ is a martingale with respect to information sets $\{ I_t \}$ if, for all $t \geq 0$, $E[\,|\,X(t)\,|\,] < \infty$ and

$$E[\,X(u)\,|\,I_t\,] = X(t)$$

for all $u > t$.

- The discrete-time version: For all $n > 0$,

$$E[\,X_{n+1}\,|\,I_n\,] = X_n,$$

given the information sets $\{ I_n \}$.

Probability Measure (concluded)

- The above implies $E[\,X_{n+m}\,|\,I_n\,] = X_n$ for any $m > 0$ by Eq. (16) on p. 144.
  - A typical I_n is the price information up to time n.
  - Then the above identity says the FVs of X will not deviate systematically from today's value given the price history.

Example

- Consider the stochastic process $\{ Z_n - n\mu, n \geq 1 \}$.
  - $Z_n \equiv \sum_{i=1}^{n} X_i$.
  - $X_1, X_2, \ldots$ are independent random variables with mean μ.
- Now,

$$\begin{aligned} & E[\,Z_{n+1} - (n+1)\,\mu\,|\,X_1, X_2, \ldots, X_n\,] \\ = \; & E[\,Z_{n+1}\,|\,X_1, X_2, \ldots, X_n\,] - (n+1)\,\mu \\ = \; & Z_n + \mu - (n+1)\,\mu \\ = \; & Z_n - n\mu. \end{aligned}$$

Example (concluded)

- Define

$$I_n \equiv \{ X_1, X_2, \ldots, X_n \}.$$

- Then

$$\{ Z_n - n\mu, n \geq 1 \}$$

is a martingale with respect to $\{ I_n \}$.