Efficient Algorithms for PV and FV

- The PV of the cash flow $C_1, C_2, \ldots, C_n$ at times $1, 2, \ldots, n$ is

$$\frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_n}{(1+y)^n}.$$

- This formula and its variations are the engine behind most of financial calculations.
  - What is y?
  - What are C_i?
  - What is n?
- It can be computed by the algorithm on p. 28.

Algorithm for Evaluating PV

```
x := 0;
d := 1 + y;
for i = n, n - 1, ..., 1 do
    x := (x + C_i)/d;
end for
return x;
```


The Idea Behind p. 28: Horner's Rule

- This idea is

$$\left(\cdots \left(\left(\frac{C_n}{1+y} + C_{n-1} \right) \frac{1}{1+y} + C_{n-2} \right) \frac{1}{1+y} + \cdots \right) \frac{1}{1+y}.$$

  - Due to Horner (1786–1837) in 1819.
- The algorithm takes $O(n)$ time.
- It is the most efficient possible in terms of the absolute number of arithmetic operations.

Conversion between Compounding Methods

- Suppose r_1 is the annual rate with continuous compounding.
- Suppose r_2 is the equivalent rate compounded m times per annum.
- How are they related?

Conversion between Compounding Methods (concluded)

- Both interest rates must produce the same amount of money after one year.
- That is,

$$\left(1+\frac{r_2}{m}\right)^m = e^{r_1}.$$

- Therefore,

$$\begin{aligned} r_1 &= m \ln\left(1+\frac{r_2}{m}\right), \\ r_2 &= m\left(e^{r_1/m}-1\right). \end{aligned}$$

But Are They Really "Equivalent"?

- Recall r_1 and r_2 on the previous page.
- They are based on different cash flows.
- In what sense are they equivalent?

Annuities

- An annuity pays out the same C dollars at the end of each year for n years.
- With a rate of r, the FV at the end of the nth year is

$$\sum_{i=0}^{n-1} C(1+r)^i = C\,\frac{(1+r)^n-1}{r}. \qquad (2)$$

General Annuities

- If m payments of C dollars each are received per year (the general annuity), then Eq. (2) becomes

$$C\,\frac{\left(1+\frac{r}{m}\right)^{nm}-1}{\frac{r}{m}}.$$

- The PV of a general annuity is

$$\sum_{i=1}^{nm} C\left(1+\frac{r}{m}\right)^{-i} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-nm}}{\frac{r}{m}}. \qquad (3)$$

Amortization

- It is a method of repaying a loan through regular payments of interest *and* principal.
- The size of the loan (the original balance) is reduced by the principal part of each payment.
- The interest part of each payment pays the interest incurred on the remaining principal balance.
- As the principal gets paid down over the term of the loan, the interest part of the payment diminishes.

Example: Home Mortgage

- By paying down the principal consistently, the risk to the lender is lowered.
- When the borrower sells the house, the remaining principal is due the lender.
- Consider the equal-payment case, i.e., fixed-rate, level-payment, fully amortized mortgages.
  - They are called traditional mortgages in the U.S.

A Numerical Example

- Consider a 15-year, \$250,000 loan at 8.0% interest rate.
- Solving Eq. (3) on p. 34 with $\text{PV} = 250000$, $n = 15$, $m = 12$, and $r = 0.08$ gives a monthly payment of $C = 2389.13$.
- The amortization schedule is shown on p. 38.
- In every month (1) the principal and interest parts add up to \$2,389.13, (2) the remaining principal is reduced by the amount indicated under the Principal heading, and (3) the interest is computed by multiplying the remaining balance of the previous month by 0.08/12.

Month	Payment	Interest	Principal	Remaining principal
				250,000.000
1	2,389.13	1,666.667	722.464	249,277.536
2	2,389.13	1,661.850	727.280	248,550.256
3	2,389.13	1,657.002	732.129	247,818.128
		...		
178	2,389.13	47.153	2,341.980	4,730.899
179	2,389.13	31.539	2,357.591	2,373.308
180	2,389.13	15.822	2,373.308	0.000
Total	430,043.438	180,043.438	250,000.000	

Two Methods of Calculating the Remaining Principal

1. Go down the amortization schedule.
   - This method is relatively slow, however.
2. Right after the kth payment, the remaining principal is the PV of the future $nm - k$ cash flows,

$$\sum_{i=1}^{nm-k} C \left(1 + \frac{r}{m}\right)^{-i} = C \, \frac{1 - \left(1 + \frac{r}{m}\right)^{-nm+k}}{\frac{r}{m}}.$$

Yields

- The term yield denotes the return of investment.
- Two widely used yields are the bond equivalent yield (BEY) and the mortgage equivalent yield (MEY).
- BEY corresponds to the r in Eq. (1) on p. 22 that equates PV with FV when $m = 2$.
- MEY corresponds to the r in Eq. (1) on p. 22 that equates PV with FV when $m = 12$.

Internal Rate of Return (IRR)

- It is the interest rate which equates an investment's PV with its price P,

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_n}{(1+y)^n}.$$

- IRR assumes all cash flows are reinvested at the *same* rate as the internal rate of return.
- The above formula is the foundation upon which pricing methodologies are built.

Holding Period Return (HPR)

- Calculate the FV by whatever means and then find the yield y that satisfies $\text{PV} = \text{FV} \times e^{-yn}$.
- Explicit assumptions about the reinvestment rates must be made.
- If the reinvestment assumptions turn out to be wrong, the yield will not be realized.
  - This is the reinvestment risk.
- Financial instruments without intermediate cash flows do not have reinvestment risks.

Numerical Methods for Yields

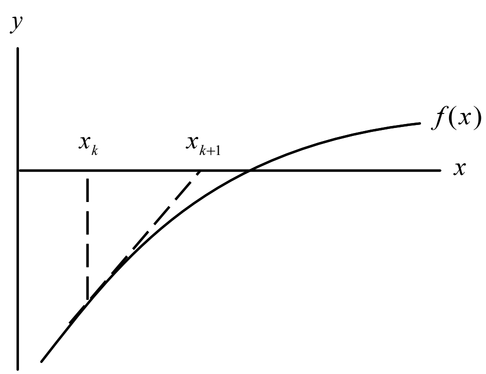
- Solve $f(y) = 0$ for $y \geq -1$, where
$$f(y) \equiv \sum_{t=1}^{n} \frac{C_t}{(1+y)^t} - P.$$
  - P is the market price.
- The function $f(y)$ is monotonic in y if $C_t > 0$ for all t.
- A unique solution exists for a monotonic $f(y)$.

The Bisection Method

- Start with a and b where $a < b$ and $f(a)\, f(b) < 0$.
- Then $f(\xi)$ must be zero for some $\xi \in [\,a, b\,]$.
- If we evaluate f at the midpoint $c \equiv (a+b)/2$, either (1) $f(c) = 0$, (2) $f(a)\, f(c) < 0$, or (3) $f(c)\, f(b) < 0$.
- In the first case we are done, in the second case we continue the process with the new bracket $[\,a, c\,]$, and in the third case we continue with $[\,c, b\,]$.
- The bracket is halved in the latter two cases.
- After n steps, we will have confined ξ within a bracket of length $(b-a)/2^n$.

The Newton-Raphson Method

- Converges faster than the bisection method.
- Start with a first approximation x_0 to a root of $f(x) = 0$.
- Then
$$x_{k+1} \equiv x_k - \frac{f(x_k)}{f'(x_k)}.$$
- When computing yields,
$$f'(x) = -\sum_{t=1}^{n} \frac{tC_t}{(1+x)^{t+1}}.$$



The Secant Method

- A variant of the Newton-Raphson method.
- Replace differentiation with difference.
- Start with two approximations x_0 and x_1.
- Then compute the $(k+1)$st approximation with

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

- Its convergence rate, 1.618, is slightly worse than the Newton-Raphson method's 2 but better than the bisection method's 1.

Solving Systems of Nonlinear Equations

- It is not easy to extend the bisection method to higher dimensions.
- But the Newton-Raphson method can be extended to higher dimensions.
- Let (x_k, y_k) be the kth approximation to the solution of the two simultaneous equations,

$$\begin{aligned} f(x,y) &= 0, \\ g(x,y) &= 0. \end{aligned}$$

Solving Systems of Nonlinear Equations (concluded)

- The $(k+1)$st approximation (x_{k+1}, y_{k+1}) satisfies the following linear equations,

$$\begin{bmatrix} \frac{\partial f(x_k,y_k)}{\partial x} & \frac{\partial f(x_k,y_k)}{\partial y} \\ \frac{\partial g(x_k,y_k)}{\partial x} & \frac{\partial g(x_k,y_k)}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{bmatrix} = - \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix},$$

  where $\Delta x_{k+1} \equiv x_{k+1} - x_k$ and $\Delta y_{k+1} \equiv y_{k+1} - y_k$.
- The above has a unique solution for $(\Delta x_{k+1}, \Delta y_{k+1})$ when the 2×2 matrix is invertible.
- Set $(x_{k+1}, y_{k+1}) = (x_k + \Delta x_{k+1}, y_k + \Delta y_{k+1})$.

Zero-Coupon Bonds (Pure Discount Bonds)

- The price of a zero-coupon bond that pays F dollars in n periods is

$$F/(1+r)^n,$$

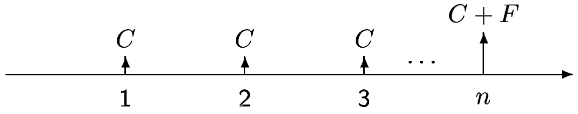
  where r is the interest rate per period.
- Can meet future obligations without reinvestment risk.
- Coupon bonds can be thought of as a matching package of zero-coupon bonds, at least theoretically.

Example

- The interest rate is 8% compounded semiannually.
- A zero-coupon bond that pays the par value 20 years from now will be priced at $1/(1.04)^{40}$, or 20.83%, of its par value.
- It will be quoted as 20.83.
- If the bond matures in 10 years instead of 20, its price would be 45.64.

Level-Coupon Bonds

- Coupon rate.
- Par value, paid at maturity.
- F denotes the par value and C denotes the coupon.
- Cash flow:



Pricing Formula

$$P = \sum_{i=1}^{n} \frac{C}{\left(1+\frac{r}{m}\right)^i} + \frac{F}{\left(1+\frac{r}{m}\right)^n}$$

$$= C\,\frac{1-\left(1+\frac{r}{m}\right)^{-n}}{\frac{r}{m}} + \frac{F}{\left(1+\frac{r}{m}\right)^n}. \tag{4}$$

- n: number of cash flows.
- m: number of payments per year.
- r: annual rate compounded m times per annum.
- $C = Fc/m$ when c is the annual coupon rate.

Yields to Maturity

- The r that satisfies Eq. (4) on p. 53 with P being the bond price.
- For a 15% BEY, a 10-year bond with a coupon rate of 10% paid semiannually sells for

$$5 \times \frac{1-[\,1+(0.15/2)\,]^{-2\times 10}}{0.15/2} + \frac{100}{[\,1+(0.15/2)\,]^{2\times 10}}$$
$$= 74.5138$$

percent of par.

Price Behavior (1)

- Bond prices fall when interest rates rise, and vice versa.
- "Only 24 percent answered the question correctly."[a]

[a] CNN, December 21, 2001.

Price Behavior (2)

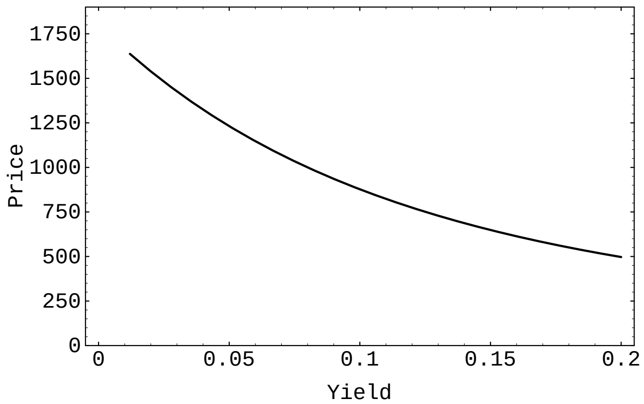
- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate is above the market interest rate;
  - at par (at its par value) when its coupon rate is equal to the market interest rate;
  - at a discount (below its par value) when its coupon rate is below the market interest rate.

Yield (%)	Price (% of par)
7.5	113.37
8.0	108.65
8.5	104.19
9.0	100.00
9.5	96.04
10.0	92.31
10.5	88.79

Terminology

- Bonds selling at par are called par bonds.
- Bonds selling at a premium are called premium bonds.
- Bonds selling at a discount are called discount bonds.

Price Behavior (3): Convexity



Day Count Conventions: Actual/Actual

- The first "actual" refers to the actual number of days in a month.
- The second refers to the actual number of days in a coupon period.
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ to date $D_2 \equiv (y_2, m_2, d_2)$ is

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1).$$

- Complications: 31, Feb 28, and Feb 29.

Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

$$\omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}. \quad (5)$$

- The price is now calculated by

$$\text{PV} = \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}. \quad (6)$$

Accrued Interest

- The buyer pays the quoted price plus the accrued interest

$$C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega).$$

- The yield to maturity is the r satisfying (6) when P is the invoice price, the sum of the quoted price and the accrued interest.
- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.

Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.

Example ("30/360") (concluded)

- The accrued interest is $(10/2) \times \frac{180-60}{180} = 3.3333$ per \$100 of par value.
- The yield to maturity is 3%.
- This can be verified by Eq. (6) with $\omega = 60/180$, $m = 2$, $C = 5$, PV$= 111.2891 + 3.3333$, and $r = 0.03$.

Price Behavior (2) Revisited

- Before: A bond selling at par if the yield to maturity equals the coupon rate.
- But it assumed that the settlement date is on a coupon payment date.
- Now suppose the settlement date for a bond selling at par (i.e., the *quoted price* is equal to the par value) falls between two coupon payment dates.
- Then its yield to maturity is less than the coupon rate.
  - The short reason: Exponential growth is replaced by linear growth, hence "overpaying" the coupon.