Trees

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 241.
  - It cannot apply to American options.
- We will now apply it to price barrier options.

The Reflection Principle (André, 1887)

- Imagine a particle at position $(0, -\boldsymbol{a})$ on the integral lattice that is to reach $(n, -\boldsymbol{b})$.
- Without loss of generality, assume $\boldsymbol{a} > 0$ and $\boldsymbol{b} \geq 0$.
- This particle's movement:

$$(i, j) \to (i+1, j+1) \quad \text{up move } S \to Su$$

$$(i, j) \to (i+1, j-1) \quad \text{down move } S \to Sd$$

- How many paths touch the x axis?

The Reflection Principle (continued)

- For a path from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touches the x axis, let J denote the first point this happens.
- Reflect the portion of the path from $(0, -\boldsymbol{a})$ to J.
- A path from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$ is constructed.
- It also hits the x axis at J for the first time (see figure next page).
- The one-to-one mapping shows the number of paths from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touch the x axis equals the number of paths from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$.

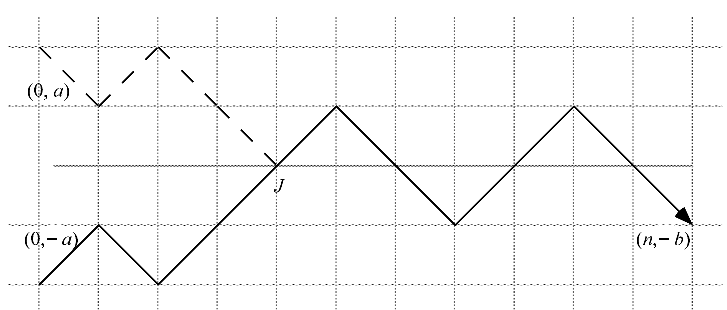
The Reflection Principle (concluded)

- Since a path of this kind has $(n + \boldsymbol{b} + \boldsymbol{a})/2$ down moves and $(n - \boldsymbol{b} - \boldsymbol{a})/2$ up moves, there are

$$\binom{n}{\frac{n+\boldsymbol{a}+\boldsymbol{b}}{2}} \qquad (103)$$

such paths for even $n + \boldsymbol{a} + \boldsymbol{b}$.

  – Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.



Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define

$$a \equiv \left\lceil \frac{\ln\left(X/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil, \qquad (104)$$

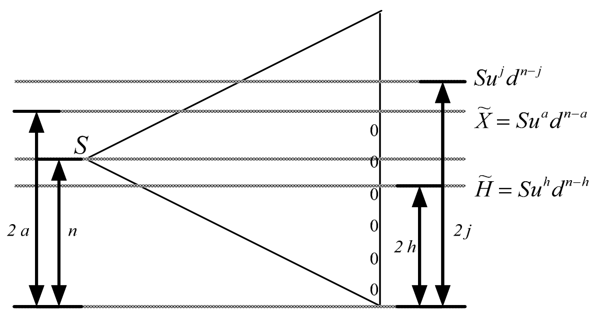
$$h \equiv \left\lfloor \frac{\ln\left(H/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor. \qquad (105)$$

  – h is such that $\tilde{H} \equiv Su^h d^{n-h}$ is the terminal price that is closest to, but does not exceed H.
  – a is such that $\tilde{X} \equiv Su^a d^{n-a}$ is the terminal price that is closest to, but is not exceeded by X.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with n moves hence ends up in the money if and only if the number of up moves is at least a.
- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price Sd^n on the binomial tree.
  –

$$Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-2k}. \qquad (106)$$



Pricing Barrier Options (continued)

- The number of paths from S to the terminal price Su^jd^{n-j} is $\binom{n}{j}$, each with probability $p^j(1-p)^{n-j}$.
- With reference to p. 517, the reflection principle can be applied with $\boldsymbol{a} = n - 2h$ and $\boldsymbol{b} = 2j - 2h$ in Eq. (103) by treating the S line as the x axis.
- Therefore,

$$\binom{n}{\frac{n+(n-2h)+(2j-2h)}{2}} = \binom{n}{n-2h+j}$$

paths hit $\tilde{H}$ in the process for $h \leq n/2$.

Pricing Barrier Options (concluded)

- The terminal price Su^jd^{n-j} is reached by a path that hits the effective barrier with probability

$$\binom{n}{n-2h+j} p^j(1-p)^{n-j}. \tag{107}$$

- The option value equals

$$R^{-n} \sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j(1-p)^{n-j} \left(Su^jd^{n-j} - X\right). \tag{108}$$

  - $R \equiv e^{r\tau/n}$ is the riskless return per period.
- It implies a linear-time algorithm.

Convergence of BOPM

- Equation (108) results in the sawtooth-like convergence shown on p. 320.
- The reasons are not hard to see.
- The true barrier most likely does not equal the effective barrier.
- The same holds for the strike price and the effective strike price.
- The issue of the strike price is less critical.
- The issue of the barrier is not negligible.

Convergence of BOPM (continued)

- Convergence is actually good if we limit n to certain values—191, for example.
- These values make the true barrier coincide with or occur just above one of the stock price levels, that is, $H \approx Sd^j = Se^{-j\sigma\sqrt{\tau/n}}$ for some integer j.
- The preferred n's are thus

$$n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

- There is only one minor technicality left.

Convergence of BOPM (continued)

- We picked the effective barrier to be one of the $n+1$ possible terminal stock prices.
- However, the effective barrier above, Sd^j, corresponds to a terminal stock price only when $n - j$ is even by Eq. (106).[a]
- To close this gap, we decrement n by one, if necessary, to make $n - j$ an even number.

[a] We could have adopted the form Sd^j $(-n \leq j \leq n)$ for the effective barrier.

Convergence of BOPM (concluded)

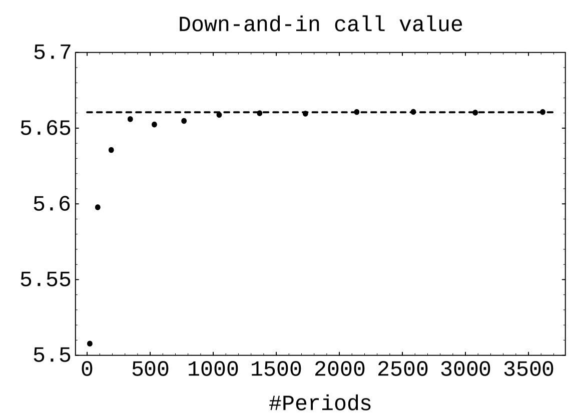
- The preferred n's are now

$$n = \begin{cases} \ell & \text{if } \ell - j \text{ is even} \\ \ell - 1 & \text{otherwise} \end{cases}, \qquad (109)$$

$j = 1, 2, 3, \ldots$, where

$$\ell \equiv \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.$$

- So evaluate pricing formula (108) only with the n's above.



Practical Implications

- Now that barrier options can be efficiently priced, we can afford to pick very large n's (see p. 526).
- This has profound consequences.
- For example, pricing is prohibitively time consuming when $S \approx H$ because $n \sim 1/\ln^2(S/H)$.
- This observation is indeed true of standard quadratic-time binomial tree algorithms.
- But it no longer applies to linear-time algorithms (see p. 527).

n	Combinatorial method		Trinomial tree algorithm	
	Value	Time	Value	Time
21	5.507548	0.30		
84	5.597597	0.90	5.634936	35.0
191	5.635415	2.00	5.655082	185.0
342	5.655812	3.60	5.658590	590.0
533	5.652253	5.60	5.659692	1440.0
768	5.654609	8.00	5.660137	3080.0
1047	5.658622	11.10	5.660338	5700.0
1368	5.659711	15.00	5.660432	9500.0
1731	5.659416	19.40	5.660474	15400.0
2138	5.660511	24.70	5.660491	23400.0
2587	5.660592	30.20	5.660493	34800.0
3078	5.660099	36.70	5.660488	48800.0
3613	5.660498	43.70	5.660478	67500.0
4190	5.660388	44.10	5.660466	92000.0
4809	5.659955	51.60	5.660454	130000.0
5472	5.660122	68.70		
6177	5.659981	76.70		

(All times in milliseconds.)

Barrier at 95.0			Barrier at 99.5			Barrier at 99.9		
n	Value	Time	n	Value	Time	n	Value	Time
	⋮		795	7.47761	8	19979	8.11304	253
2743	2.56095	31.1	3184	7.47626	38	79920	8.11297	1013
3040	2.56065	35.5	7163	7.47682	88	179819	8.11300	2200
3351	2.56098	40.1	12736	7.47661	166	319680	8.11299	4100
3678	2.56055	43.8	19899	7.47676	253	499499	8.11299	6300
4021	2.56152	48.1	28656	7.47667	368	719280	8.11299	8500
True	2.5615			7.4767			8.1130	

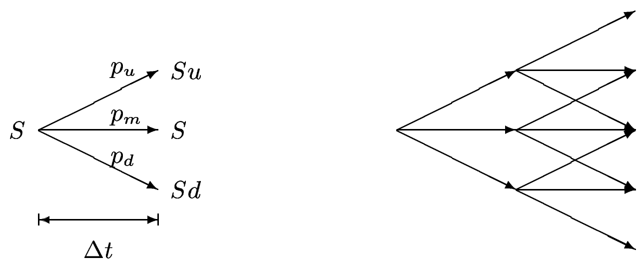
(All times in milliseconds.)

Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion $dS/S = r\,dt + \sigma\,dW$ (Boyle, 1988).
- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Impose the matching of mean and that of variance:

$$\begin{aligned} 1 &= p_u + p_m + p_d, \\ SM &\equiv (p_u u + p_m + (p_d/u))\,S, \\ S^2 V &\equiv p_u (Su - SM)^2 + p_m (S - SM)^2 + p_d (Sd - SM)^2, \end{aligned}$$

where $M \equiv e^{r\Delta t}$ and $V \equiv M^2(e^{\sigma^2 \Delta t} - 1)$ by Eqs. (41).



Trinomial Tree (continued)

- Use linear algebra to verify that

$$p_u = \frac{u\left(V + M^2 - M\right) - (M-1)}{(u-1)\,(u^2-1)},$$
$$p_d = \frac{u^2\left(V + M^2 - M\right) - u^3(M-1)}{(u-1)\,(u^2-1)}.$$

  - In practice, must make sure the probabilities lie between 0 and 1.
- Countless variations.

Trinomial Tree (concluded)

- Use $u = e^{\lambda\sigma\sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$p_u \to \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma},$$
$$p_d \to \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}.$$

- A nice choice for λ is $\sqrt{\pi/2}$.

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting λ so that the barrier is hit exactly (Ritchken, 1995).
- It takes

$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}$$

consecutive down moves to go from S to H if h is an integer, which is easy to achieve by adjusting λ.

Barrier Options Revisited (continued)

- Typically, we find the smallest $\lambda \geq 1$ such that h is an integer, that is,

$$\lambda = \max_{j=1,2,3,\ldots} \frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}}.$$

  – Such a λ may not exist for very small n's.

  – This is not hard to check.

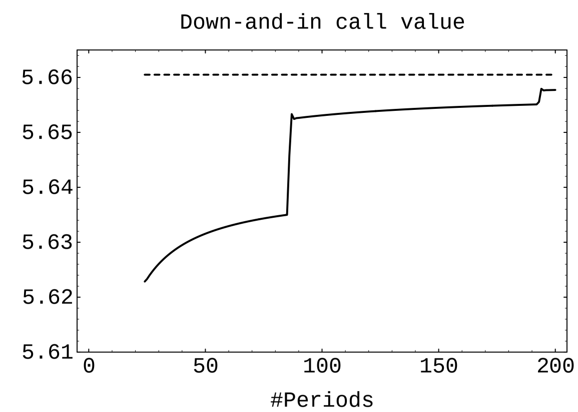
- This done, one of the layers of the trinomial tree coincides with the barrier.

Barrier Options Revisited (concluded)

- The following probabilities may be used,

$$\begin{aligned} p_u &= \frac{1}{2\lambda^2} + \frac{\mu'\sqrt{\Delta t}}{2\lambda\sigma}, \\ p_m &= 1 - \frac{1}{\lambda^2}, \\ p_d &= \frac{1}{2\lambda^2} - \frac{\mu'\sqrt{\Delta t}}{2\lambda\sigma}. \end{aligned}$$

  – $\mu' \equiv r - \sigma^2/2$.



Algorithms Comparison (Lyuu, 1998)

- So which algorithm is better?
- Algorithms are often compared based on the n value at which they converge.
  – The one with the smallest n wins.
- Giraffes are faster than cheetahs because they take fewer strides to travel the same distance.
- Performance must be based on actual running time.

Algorithms Comparison (concluded)

- Pages 320 and 535 show the trinomial model converges at a smaller n than BOPM.
- It is in this sense when people say trinomial models converge faster than binomial ones.
- But is trinomial model then better?
- The linear-time binomial tree algorithm actually performs better than the trinomial model (p. 526).

Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on m assets has the terminal payoff $\max(\sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0)$, where α_i is the percentage of asset i.
- Basket options are essentially options on a portfolio of stocks or index options.
- Option on the best of two risky assets and cash has a terminal payoff of $\max(S_1(\tau), S_2(\tau), X)$.

Correlated Trinomial Model

- Two risky assets S_1 and S_2 follow $dS_i/S_i = r\,dt + \sigma_i\,dW_i$ in a risk-neutral economy, $i = 1, 2$.
- Let

$$
\begin{aligned}
M_i &\equiv e^{r\Delta t},\\
V_i &\equiv M_i^2(e^{\sigma_i^2 \Delta t} - 1).
\end{aligned}
$$

  - $S_i M_i$ is the mean of S_i at time Δt.
  - $S_i^2 V_i$ the variance of S_i at time Δt.

Correlated Trinomial Model (continued)

- The value of $S_1 S_2$ at time Δt has a joint lognormal distribution with mean $S_1 S_2 M_1 M_2 e^{\rho\sigma_1\sigma_2\Delta t}$, where ρ is the correlation between dW_1 and dW_2.
- Next match the 1st and 2nd moments of the approximating discrete distribution to those of the continuous counterpart.
- At time Δt from now, there are five distinct outcomes.

Correlated Trinomial Model (continued)

- The five-point probability distribution of the asset prices is (as usual, we impose $u_i d_i = 1$)

Probability	Asset 1	Asset 2
p_1	$S_1 u_1$	$S_2 u_2$
p_2	$S_1 u_1$	$S_2 d_2$
p_3	$S_1 d_1$	$S_2 d_2$
p_4	$S_1 d_1$	$S_2 u_2$
p_5	S_1	S_2

Correlated Trinomial Model (continued)

- The probabilities must sum to one, and the means must be matched:

$$\begin{aligned} 1 &= p_1 + p_2 + p_3 + p_4 + p_5, \\ S_1 M_1 &= (p_1 + p_2)\, S_1 u_1 + p_5 S_1 + (p_3 + p_4)\, S_1 d_1, \\ S_2 M_2 &= (p_1 + p_4)\, S_2 u_2 + p_5 S_2 + (p_2 + p_3)\, S_2 d_2. \end{aligned}$$

Correlated Trinomial Model (continued)

- Let $R \equiv M_1 M_2 e^{\rho \sigma_1 \sigma_2 \Delta t}$.
- Match the variances and covariance:

$$\begin{aligned} S_1^2 V_1 &= (p_1 + p_2)((S_1 u_1)^2 - (S_1 M_1)^2) + p_5 (S_1^2 - (S_1 M_1)^2) \\ &\quad + (p_3 + p_4)((S_1 d_1)^2 - (S_1 M_1)^2), \\ S_2^2 V_2 &= (p_1 + p_4)((S_2 u_2)^2 - (S_2 M_2)^2) + p_5 (S_2^2 - (S_2 M_2)^2) \\ &\quad + (p_2 + p_3)((S_2 d_2)^2 - (S_2 M_2)^2), \\ S_1 S_2 R &= (p_1 u_1 u_2 + p_2 u_1 d_2 + p_3 d_1 d_2 + p_4 d_1 u_2 + p_5)\, S_1 S_2. \end{aligned}$$

Correlated Trinomial Model (continued)

- The solutions are

$$\begin{aligned} p_1 &= \frac{u_1 u_2 (R-1) - f_1 (u_1^2 - 1) - f_2 (u_2^2 - 1) + (f_2 + g_2)(u_1 u_2 - 1)}{(u_1^2 - 1)(u_2^2 - 1)}, \\ p_2 &= \frac{-u_1 u_2 (R-1) + f_1 (u_1^2 - 1) u_2^2 + f_2 (u_2^2 - 1) - (f_2 + g_2)(u_1 u_2 - 1)}{(u_1^2 - 1)(u_2^2 - 1)}, \\ p_3 &= \frac{u_1 u_2 (R-1) - f_1 (u_1^2 - 1) u_2^2 + g_2 (u_2^2 - 1) u_1^2 + (f_2 + g_2)(u_1 u_2 - u_2^2)}{(u_1^2 - 1)(u_2^2 - 1)}, \\ p_4 &= \frac{-u_1 u_2 (R-1) + f_1 (u_1^2 - 1) + f_2 (u_2^2 - 1) u_1^2 - (f_2 + g_2)(u_1 u_2 - 1)}{(u_1^2 - 1)(u_2^2 - 1)}. \end{aligned}$$

Correlated Trinomial Model (concluded)

- In the above,

$$
\begin{aligned}
f_1 &= p_1 + p_2 = \frac{u_1\left(V_1 + M_1^2 - M_1\right) - (M_1 - 1)}{(u_1 - 1)\,(u_1^2 - 1)},\\
f_2 &= p_1 + p_4 = \frac{u_2\left(V_2 + M_2^2 - M_2\right) - (M_2 - 1)}{(u_2 - 1)\,(u_2^2 - 1)},\\
g_1 &= p_3 + p_4 = \frac{u_1^2\left(V_1 + M_1^2 - M_1\right) - u_1^3(M_1 - 1)}{(u_1 - 1)\,(u_1^2 - 1)},\\
g_2 &= p_2 + p_3 = \frac{u_2^2\left(V_2 + M_2^2 - M_2\right) - u_2^3(M_2 - 1)}{(u_2 - 1)\,(u_2^2 - 1)}.
\end{aligned}
$$

- As $f_1 + g_1 = f_2 + g_2$, we can solve for u_2 given $u_1 = e^{\lambda\sigma_1\sqrt{\Delta t}}$ for an appropriate $\lambda > 1$.

Extrapolation

- Is a method to speed up numerical convergence.
- Say $f(n)$ converges to an unknown limit f at rate of $1/n$:

$$f(n) = f + \frac{c}{n} + o\left(\frac{1}{n}\right). \qquad (110)$$

- Assume that c is an unknown constant independent of n.
  - Convergence is basically monotonic and smooth.

Extrapolation (concluded)

- From two approximations $f(n_1)$ and $f(n_2)$ and by ignoring the smaller terms,

$$
\begin{aligned}
f(n_1) &= f + \frac{c}{n_1},\\
f(n_2) &= f + \frac{c}{n_2}.
\end{aligned}
$$

- A better approximation to the desired f is

$$f = \frac{n_1 f(n_1) - n_2 f(n_2)}{n_1 - n_2}. \qquad (111)$$

- This estimate should converge faster than $1/n$.
- The Richardson extrapolation uses $n_2 = 2n_1$.

Improving BOPM with Extrapolation

- Consider standard European options.
- Denote the option value under BOPM using n time periods by $f(n)$.
- It is known that BOPM convergences at the rate of $1/n$, consistent with Eq. (110).
- But the plots on p. 255 demonstrate that convergence to the true option value oscillates with n.
- Extrapolation is inapplicable at this stage.

Improving BOPM with Extrapolation (concluded)

- Take the at-the-money option in the left plot on p. 255.
- The sequence with odd n turns out to be monotonic and smooth (see the left plot on p. 550).
- Apply extrapolation (111) with $n_2 = n_1 + 2$, where n_1 is odd.
- Result is shown in the right plot on p. 550.
- The convergence rate is amazing.
- See Exercise 9.3.8 (p. 111) of the textbook for ideas in the general case.