Ornstein-Uhlenbeck Process

- The Ornstein-Uhlenbeck process:

$$dX = -\kappa X \, dt + \sigma \, dW, \tag{92}$$

where $\kappa, \sigma \geq 0$.

- It is known that

$$\begin{aligned} E[\,X(t)\,] &= e^{-\kappa(t-t_0)}\, E[\,x_0\,], \\ \text{Var}[\,X(t)\,] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-t_0)}\right) + e^{-2\kappa(t-t_0)}\, \text{Var}[\,x_0\,], \\ \text{Cov}[\,X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa(t-s)}\left[1 - e^{-2\kappa(s-t_0)}\right] \\ &\quad + e^{-\kappa(t+s-2t_0)}\, \text{Var}[\,x_0\,], \end{aligned}$$

for $t_0 \leq s \leq t$ and $X(t_0) = x_0$.

Ornstein-Uhlenbeck Process (continued)

- $X(t)$ is normally distributed if x_0 is a constant or normally distributed.
- X is said to be a normal process.
- $E[\,x_0\,] = x_0$ and $\text{Var}[\,x_0\,] = 0$ if x_0 is a constant.
- The Ornstein-Uhlenbeck process has the following mean reversion property.
  - When $X > 0$, X is pulled X toward zero.
  - When $X < 0$, it is pulled toward zero again.

Ornstein-Uhlenbeck Process (continued)

- Another version:

$$dX = \kappa(\mu - X)\, dt + \sigma\, dW, \tag{93}$$

where $\sigma \geq 0$.

- Given $X(t_0) = x_0$, a constant, it is known that

$$E[\,X(t)\,] = \mu + (x_0 - \mu)\, e^{-\kappa(t-t_0)} \tag{94}$$

$$\text{Var}[\,X(t)\,] = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa(t-t_0)}\right] \tag{95}$$

for $t_0 \leq t$.

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly μ and $\sigma/\sqrt{2\kappa}$, respectively.
- For large t, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$ (say $\mu > 4\sigma/\sqrt{2\kappa}$).
- The process is mean-reverting.
  - X tends to move toward μ.
  - Useful for modeling term structure, stock price volatility, and stock price return.

Interest Rate Models (Merton, 1970)

- Suppose the short rate r follows process $dr = \mu(r,t)\,dt + \sigma(r,t)\,dW$.
- Let $P(r,t,T)$ denote the price at time t of a zero-coupon bond that pays one dollar at time T.
- Write its dynamics as

$$\frac{dP}{P} = \mu_p\,dt + \sigma_p\,dW.$$

  - The expected instantaneous rate of return on a $(T-t)$-year zero-coupon bond is μ_p.
  - The instantaneous variance is σ_p^2.

Interest Rate Models (continued)

- Surely $P(r,T,T)=1$ for any T.
- By Ito's lemma (Theorem 21 on p. 469),

$$\begin{aligned} dP &= \frac{\partial P}{\partial T}\,dT + \frac{\partial P}{\partial r}\,dr + \frac{1}{2}\,\frac{\partial^2 P}{\partial r^2}\,(dr)^2 \\ &= -\frac{\partial P}{\partial T}\,dt + \frac{\partial P}{\partial r}\,(\mu(r,t)\,dt + \sigma(r,t)\,dW) \\ &\quad + \frac{1}{2}\,\frac{\partial^2 P}{\partial r^2}\,(\mu(r,t)\,dt + \sigma(r,t)\,dW)^2 \\ &= \left(-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2}\right) dt \\ &\quad + \sigma(r,t)\,\frac{\partial P}{\partial r}\,dW. \end{aligned}$$

Interest Rate Models (concluded)

- Hence,

$$-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2} = P\mu_p, \qquad (96)$$

$$\sigma(r,t)\,\frac{\partial P}{\partial r} = P\sigma_p.$$

- Models with the short rate as the only explanatory variable are called short rate models.

The Merton Model

- Assume the local expectations theory, which means μ_p equals the prevailing short rate $r(t)$ for all T.
- Assume further that μ and σ are constants.
- Then the partial differential equations (96) yield

$$P(r,t,T) = e^{-r(T-t) - \frac{\mu(T-t)^2}{2} + \frac{\sigma^2(T-t)^3}{6}}. \qquad (97)$$

- The dynamics of P is $dP/P = r\,dt - \sigma(T-t)\,dW$.
- Now, P has no upper limits as T becomes large, which does not square with the reality.

Duration under Parallel Shifts

- Consider duration with respect to parallel shifts in the spot rate curve.
- For convenience, assume $t = 0$.
- Parallel shift means $S(r + \Delta r, T) = S(r, T) + \Delta r$ for any Δr; so $\partial S(r, T)/\partial r = 1$.
- This implies $S(r, T) = r + g(T)$ for some function g with $g(0) = 0$ because $S(r, 0) = r$.
- Consequently, $P(r, T) = e^{-[\, r+g(T)\,]\, T}$.

Duration under Parallel Shifts (concluded)

- Substitute this identity into the left-hand part of Eq. (96) and assume the local expectations theory to obtain

$$g'(T) + \frac{g(T)}{T} = \mu(r) - \frac{\sigma(r)^2}{2}\, T.$$

- As the left-hand side is independent of r, so must the right-hand side.
- Since this holds for all T, both $\mu(r)$ and $\sigma(r)$ must be constants, i.e., the Merton model.
- As mentioned before, this model is flawed, so must duration as such.

Continuous-Time Derivatives Pricing

Towards the Black-Scholes Differential Equation

- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation.
- The key step is recognizing that the same random process drives both securities.
- As their prices are perfectly correlated, we figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio's return to be the riskless rate.

Assumptions

- The stock price follows $dS = \mu S\, dt + \sigma S\, dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at r.
- There is unlimited riskless borrowing and lending.
- t is the current time, T is the expiration time, and $\tau \equiv T - t$.

Black-Scholes Differential Equation

- Let C be the price of a derivative on S.
- From Ito's lemma (p. 469),

$$dC = (\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}) dt + \sigma S \frac{\partial C}{\partial S} dW.$$

  - The same W drives both C and S.
- Short one derivative and long $\partial C / \partial S$ shares of stock (call it Π).
- By construction,

$$\Pi = -C + S(\partial C / \partial S).$$

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time dt is

$$d\Pi = -dC + \frac{\partial C}{\partial S} dS.$$

- Substitute the formulas for dC and dS into the partial differential equation to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt.$$

- As this equation does not involve dW, the portfolio is riskless during dt time: $d\Pi = r \Pi\, dt$.

Black-Scholes Differential Equation (concluded)

- So,

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt = r \left(C - S \frac{\partial C}{\partial S} \right) dt.$$

- Equate the terms to finally obtain

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC. \qquad (98)$$

- When there is a dividend yield q,

$$\frac{\partial C}{\partial t} + (r - q) S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rC. \tag{99}$$

- Identity (99) leads to an alternative way of computing Θ numerically from Δ and Γ.
- When a portfolio is delta-neutral,

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rC.$$

  - A definite relation thus exists between Γ and Θ.

General Derivatives Pricing

- In general the underlying asset S may not be traded.
  - Interest rate, for instance, is not a traded security.
- Let S follow the Ito process $dS/S = \boldsymbol{\mu}\, dt + \boldsymbol{\sigma}\, dW$, where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ may depend only on S and t.
- Let $f_1(S,t)$ and $f_2(S,t)$ be the prices of two derivatives with dynamics $df_i/f_i = \mu_i\, dt + \sigma_i\, dW$, $i = 1, 2$.
  - They share the same Wiener process as S.

General Derivatives Pricing (continued)

- A portfolio consisting of $\sigma_2 f_2$ units of the first derivative and $-\sigma_1 f_1$ units of the second derivative is instantaneously riskless:

$$\begin{aligned} & \sigma_2 f_2\, df_1 - \sigma_1 f_1\, df_2 \\ = \;& \sigma_2 f_2 f_1 (\mu_1\, dt + \sigma_1\, dW) - \sigma_1 f_1 f_2 (\mu_2\, dt + \sigma_2\, dW) \\ = \;& (\sigma_2 f_2 f_1 \mu_1 - \sigma_1 f_1 f_2 \mu_2)\, dt. \end{aligned}$$

- Therefore,

$$(\sigma_2 f_2 f_1 \mu_1 - \sigma_1 f_1 f_2 \mu_2)\, dt = r(\sigma_2 f_2 f_1 - \sigma_1 f_1 f_2)\, dt,$$

or $\sigma_2 \mu_1 - \sigma_1 \mu_2 = r(\sigma_2 - \sigma_1)$.

General Derivatives Pricing (continued)

- After rearranging the terms,

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \equiv \lambda \text{ for some } \lambda.$$

- A derivative whose value depends only on S and t and which follows the Ito process $df/f = \mu\, dt + \sigma\, dW$ must thus satisfy

$$\frac{\mu - r}{\sigma} = \lambda \text{ or, alternatively, } \mu = r + \lambda\sigma. \tag{100}$$

- We call λ the market price of risk, which is independent of the specifics of the derivative.

General Derivatives Pricing (continued)

- Ito's lemma can be used to derive the formulas for μ and σ:

$$\mu = \frac{1}{f}\left(\frac{\partial f}{\partial t} + \boldsymbol{\mu} S\,\frac{\partial f}{\partial S} + \frac{1}{2}\,\boldsymbol{\sigma}^2 S^2\,\frac{\partial^2 f}{\partial S^2}\right),$$

$$\sigma = \frac{\boldsymbol{\sigma} S}{f}\,\frac{\partial f}{\partial S}.$$

- Substitute the above into Eq. (100) to obtain

$$\frac{\partial f}{\partial t} + (\boldsymbol{\mu} - \lambda\boldsymbol{\sigma})\,S\,\frac{\partial f}{\partial S} + \frac{1}{2}\,\boldsymbol{\sigma}^2 S^2\,\frac{\partial^2 f}{\partial S^2} = rf. \qquad (101)$$

General Derivatives Pricing (concluded)

- The presence of $\boldsymbol{\mu}$ shows that the investor's risk preference is relevant.
- The derivative may be dependent on the underlying asset's growth rate and the market price of risk.
- Only when the underlying variable is the price of a traded security can we assume $\boldsymbol{\mu} = r$ in pricing.

Hedging

Delta Hedge

- The delta (hedge ratio) of a derivative f is defined as $\Delta \equiv \partial f/\partial S$.
- Thus $\Delta f \approx \Delta \times \Delta S$ for relatively small changes in the stock price, ΔS.
- A delta-neutral portfolio is hedged in the sense that it is immunized against small changes in the stock price.
- A trading strategy that dynamically maintains a delta-neutral portfolio is called delta hedge.

Delta Hedge (concluded)

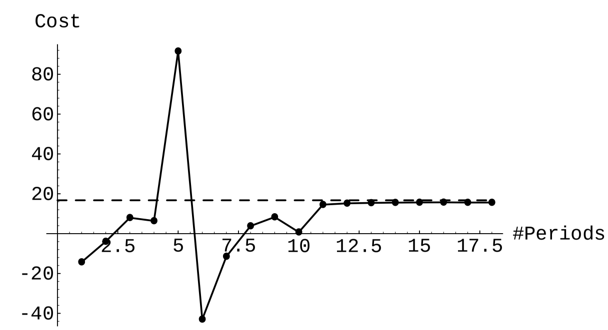
- Delta changes with the stock price.
- A delta hedge needs to be rebalanced periodically in order to maintain delta neutrality.
- In the limit where the portfolio is adjusted continuously, perfect hedge is achieved and the strategy becomes self-financing.
- This was the gist of the Black-Scholes-Merton argument.

Implementing Delta Hedge

- We want to hedge N short derivatives.
- Assume the stock pays no dividends.
- The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus B borrowed dollars such that

$$-N \times f + N \times \Delta \times S - B = 0.$$

- At next rebalancing point when the delta is Δ', buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares with a total borrowing of $B' = N \times \Delta' \times S' - N \times f'$.
- Delta hedge is the discrete-time analog of the continuous-time limit and will rarely be self-financing.



Example

- A hedger is short 10,000 European calls.
- $\sigma = 30\%$, and $r = 6\%$.
- This call's expiration is four weeks away, its strike price is \$50, and each call has a current value of $f = 1.76791$.
- As an option covers 100 shares of stock, $N = 1{,}000{,}000$.
- The trader adjusts the portfolio weekly.
- The calls are replicated well if the cumulative cost of trading stock is close to the call premium's FV.

Example (continued)

- As $\Delta = 0.538560$, $N \times \Delta = 538,560$ shares are purchased for a total cost of $538{,}560 \times 50 = 26{,}928{,}000$ dollars to make the portfolio delta-neutral.
- The trader finances the purchase by borrowing

$$B = N \times \Delta \times S - N \times f = 25{,}160{,}090$$

dollars net.

- The portfolio has zero net value now.

Example (continued)

- At 3 weeks to expiration, the stock price rises to \$51.
- The new call value is $f' = 2.10580$.
- So the portfolio is worth

$$-N \times f' + 538{,}560 \times 51 - Be^{0.06/52} = 171,622 \quad (102)$$

before rebalancing.

  – A delta hedge does not replicate the calls perfectly; it is not self-financing as \$171,622 can be withdrawn.

  – The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.

Example (continued)

- In fact, the tracking error is positive about 68% of the time even though its expected value is essentially zero (Boyle and Emanuel, 1980).
- It is furthermore proportional to vega.
- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.
- With a higher delta $\Delta' = 0.640355$, the trader buys $N \times (\Delta' - \Delta) = 101,795$ shares for \$5,191,545.
- The number of shares is increased to $N \times \Delta' = 640,355$.

Example (continued)

- The cumulative cost is

$$26{,}928{,}000 \times e^{0.06/52} + 5{,}191{,}545 = 32{,}150{,}634.$$

- The net borrowed amount is

$$B' = 640{,}355 \times 51 - N \times f' = 30{,}552{,}305.$$

  – Alternatively, the number could be arrived at via

$$Be^{0.06/52} + 5{,}191{,}545 + 171{,}622 = 30{,}552{,}305.$$

- The portfolio is again delta-neutral with zero value.

τ	S	Option value f	Delta Δ	Change in delta	No. shares bought $N \times (5)$	Cost of shares $(1) \times (6)$	Cumulative cost FV(8')+(7)
	(1)	(2)	(3)	(5)	(6)	(7)	(8)
4	50	1.7679	0.53856	—	538,560	26,928,000	26,928,000
3	51	2.1058	0.64036	0.10180	101,795	5,191,545	32,150,634
2	53	3.3509	0.85578	0.21542	215,425	11,417,525	43,605,277
1	52	2.2427	0.83983	−0.01595	−15,955	−829,660	42,825,960
0	54	4.0000	1.00000	0.16017	160,175	8,649,450	51,524,853

The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).

Example (concluded)

- At expiration, the trader has 1,000,000 shares.
- They are exercised against by the in-the-money calls for $50,000,000.
- The trader is left with an obligation of

$$51{,}524{,}853 - 50{,}000{,}000 = 1{,}524{,}853,$$

which represents the replication cost.

- Compared with the FV of the call premium,

$$1{,}767{,}910 \times e^{0.06 \times 4/52} = 1{,}776{,}088,$$

the net gain is $1{,}776{,}088 - 1{,}524{,}853 = 251{,}235$.

Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, Δf, due to changes in the stock price, ΔS.
- When ΔS is not small, the second-order term, gamma $\Gamma \equiv \partial^2 f / \partial S^2$, helps (theoretically).
- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.
- To meet this extra condition, one more security needs to be brought in.

Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call f_2 is brought in.
- To set up a delta-gamma hedge, we solve

$$\begin{aligned}
-N \times f + n_1 \times S + n_2 \times f_2 - B &= 0 \quad \text{(self-financing)},\\
-N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 &= 0 \quad \text{(delta neutrality)},\\
-N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 &= 0 \quad \text{(gamma neutrality)},
\end{aligned}$$

for n_1, n_2, and B.

- The gammas of the stock and bond are 0.

Other Hedges

- If volatility changes, delta-gamma hedge may not work well.
- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.
- To accomplish this, one more security has to be brought into the process.
- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.