## Example

- Consider a call with strike $100 and an expiration date in September.
- The underlying asset is a forward contract with a delivery date in December.
- Suppose the forward price in July is $110.
- Upon exercise, the call holder receives a forward contract with a delivery price of $100.
- If an offsetting position is then taken in the forward market, a $10 profit in December will be assured.
- A call on the futures would realize the $10 profit in July.



## Some Pricing Relations

- Let delivery take place at time $T$, the current time be 0, and the option on the futures or forward contract have expiration date $t$ $(t \leq T)$.
- Assume a constant, positive interest rate.
- Although forward price equals futures price, a forward option does not have the same value as a futures option.
- The payoffs at time $t$ are

$$\text{futures option} = \max(F_t - X, 0), \quad (68)$$

$$\text{forward option} = \max(F_t - X, 0)\, e^{-r(T-t)}. \quad (69)$$



## Some Pricing Relations (concluded)

- A European futures option is worth the same as the corresponding European option on the underlying asset if the futures contract has the same maturity as the options.
  - Futures price equals spot price at maturity.
  - This conclusion is independent of the model for the spot price.



## Put-Call Parity

The put-call parity is slightly different from the one in Eq. (42) on p. 188.

**Theorem 15** *(1) For European options on futures contracts,* $C = P - (X - F)\, e^{-rt}$. *(2) For European options on forward contracts,* $C = P - (X - F)\, e^{-rT}$.

- Consider a portfolio of one short call, one long put, one long futures contract, and a loan of $(X - F)\, e^{-rt}$.

## The Proof (continued)

- Cash flow at time $t$:

| | $F_t \leq X$ | $F_t > X$ |
|---|---|---|
| A short call | 0 | $X - F_t$ |
| A long put | $X - F_t$ | 0 |
| A long futures | $F_t - F$ | $F_t - F$ |
| A loan of $(X - F)\, e^{-rt}$ | $F - X$ | $F - X$ |
| Total | 0 | 0 |

- Since the net future cash flow is zero in both cases, the portfolio must have zero value today.



## The Proof (concluded)

- This proves the theorem for futures option.
- The proof for forward options is identical except that the loan amount is $(X - F)\, e^{-rT}$ instead.



## Early Exercise and Forward Options

The early exercise feature is not valuable.

**Theorem 16** *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

- The proof is in the text.

Early exercise may be optimal for American futures options even if the underlying asset generates no payouts.

**Theorem 17** *American futures options may be exercised optimally before expiration.*



## Black Model (Black, 1976)

- Formulas for European futures options:

$$\begin{aligned} C &= Fe^{-rt} N(x) - Xe^{-rt} N(x - \sigma\sqrt{t}), \qquad (70) \\ P &= Xe^{-rt} N(-x + \sigma\sqrt{t}) - Fe^{-rt} N(-x), \end{aligned}$$

where $x \equiv \frac{\ln(F/X) + (\sigma^2/2)\, t}{\sigma\sqrt{t}}$.

- Formulas (70) are related to those for options on a stock paying a continuous dividend yield.
- In fact, they are exactly Eqs. (55) on p. 266 with the dividend yield $q$ set to the interest rate $r$ and the stock price $S$ replaced by the futures price $F$.

## Black Model (concluded)

- This observation incidentally proves Theorem 17 on p. 391.
- For European forward options, just multiply the above formulas by $e^{-r(T-t)}$.
  - Because forward options differ from futures options by a factor of $e^{-r(T-t)}$ based on Eqs. (68)–(69).



## Binomial Model for Forward and Futures Options

- Futures price behaves like a stock paying a continuous dividend yield of $r$.
- Under the BOPM, the risk-neutral probability for the futures price is

$$p_{\text{f}} \equiv (1-d)/(u-d)$$

by Eq. (56) on p. 267.
  - The futures price moves from $F$ to $Fu$ with probability $p_{\text{f}}$ and to $Fd$ with probability $1-p_{\text{f}}$.
- The binomial tree algorithm for forward options is identical except that Eq. (69) on p. 386 is the payoff.



## Spot and Futures Prices under BOPM

- The futures price is related to the spot price via $F = Se^{rT}$ if the underlying asset pays no dividends.
- The stock price moves from $S = Fe^{-rT}$ to $Fue^{-r(T-\Delta t)} = Sue^{r\Delta t}$ with probability $p_{\text{f}}$ per period.
- The stock price moves from $S = Fe^{-rT}$ to $Sde^{r\Delta t}$ with probability $1-p_{\text{f}}$ per period.



## Negative Probabilities Revisited

- As $0 < p_{\text{f}} < 1$, we have $0 < 1-p_{\text{f}} < 1$ as well.
- Solve the problem of negative risk-neutral probabilities:
  - Suppose the stock pays a continuous dividend yield of $q$.
  - Build the tree for the futures price $F$ of the futures contract expiring at the same time as the option.
  - Calculate $S$ from $F$ at each node via $S = Fe^{-(r-q)(T-t)}$.

## Swaps

- Swaps are agreements between two counterparties to exchange cash flows in the future according to a predetermined formula.
- There are two basic types of swaps: interest rate and currency.
- An interest rate swap occurs when two parties exchange interest payments periodically.
- Currency swaps are agreements to deliver one currency against another (our focus here).



## Currency Swaps

- A currency swap involves two parties to exchange cash flows in different currencies.
- Consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

| | Dollars | Yen |
|---|---|---|
| A | $D_{\rm A}\%$ | $Y_{\rm A}\%$ |
| B | $D_{\rm B}\%$ | $Y_{\rm B}\%$ |

- Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars.



## Currency Swaps (continued)

- A straightforward scenario is for A to borrow yen at $Y_{\rm A}\%$ and B to borrow dollars at $D_{\rm B}\%$.
- But suppose A is *relatively* more competitive in the dollar market than the yen market, and vice versa for B.
  - $Y_{\rm B} - Y_{\rm A} < D_{\rm B} - D_{\rm A}$.
- Consider this alternative arrangement:
  - A borrows dollars.
  - B borrows yen.
  - They enter into a currency swap with a bank as the intermediary.



## Currency Swaps (concluded)

- The counterparties exchange principal at the beginning and the end of the life of the swap.
- This act transforms A's loan into a yen loan and B's yen loan into a dollar loan.
- The total gain is $((D_{\rm B} - D_{\rm A}) - (Y_{\rm B} - Y_{\rm A}))\%$:
  - The total interest rate is originally $(Y_{\rm A} + D_{\rm B})\%$.
  - The new arrangement has a smaller total rate of $(D_{\rm A} + Y_{\rm B})\%$.
- Transactions will happen only if the gain is distributed so that the cost to each party is less than the original.

## Example

- A and B face the following borrowing rates:

| | Dollars | Yen |
|---|---|---|
| A | 9% | 10% |
| B | 12% | 11% |

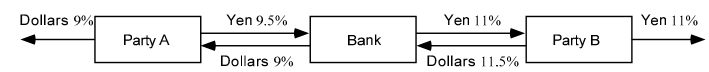
- A wants to borrow yen, and B wants to borrow dollars.
- A can borrow yen directly at 10%.
- B can borrow dollars directly at 12%.



## Example (concluded)

- As the rate differential in dollars (3%) is different from that in yen (1%), a currency swap with a total saving of $3 - 1 = 2\%$ is possible.
- A is relatively more competitive in the dollar market, and B the yen market.
- Figure next page shows an arrangement which is beneficial to all parties involved.
  - A effectively borrows yen at 9.5%. B borrows dollars at 11.5%.
  - The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.








## As a Package of Cash Market Instruments

- Assume no default risk.
- Take B on p. 403 as an example.
- The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest.
- The pricing formula is $SP_{\mathrm{Y}} - P_{\mathrm{D}}$.
  - $P_{\mathrm{D}}$ is the dollar bond's value in dollars.
  - $P_{\mathrm{Y}}$ is the yen bond's value in yen.
  - $S$ is the \$/yen spot exchange rate.

## As a Package of Cash Market Instruments (concluded)

- The value of a currency swap depends on the term structures of interest rates in the currencies involved and the spot exchange rate.
- It has zero value when $SP_{\text{Y}} = P_{\text{D}}$.



## Example

- Take a two-year swap on p. 403 with principal amounts of US$1 million and 100 million yen.
- The payments are made once a year.
- The spot exchange rate is 90 yen/$ and the term structures are flat in both nations—8% in the U.S. and 9% in Japan.
- For B, the value of the swap is (in millions of USD)

$$\frac{1}{90} \times (11 \times e^{-0.09} + 11 \times e^{-0.09\times 2} + 111 \times e^{-0.09\times 3}) - (0.115 \times e^{-0.08} + 0.115 \times e^{-0.08\times 2} + 1.115 \times e^{-0.08\times 3}) = 0.074.$$



## As a Package of Forward Contracts

- From Eq. (65) on p. 368, the forward contract maturing $i$ years from now has a dollar value of

$$f_i \equiv (SY_i)\, e^{-qi} - D_i e^{-ri}. \quad (71)$$

  - $Y_i$ is the yen inflow at year $i$.
  - $S$ is the $/yen spot exchange rate.
  - $q$ is the yen interest rate.
  - $D_i$ is the dollar outflow at year $i$.
  - $r$ is the dollar interest rate.



## As a Package of Forward Contracts (concluded)

- This formulation may be preferred to the cash market approach in cases involving costs of carry and convenience yields because forward prices already incorporate them.
- For simplicity, flat term structures were assumed.
- Generalization is straightforward.

## Example

- Take the swap in the example on p. 406.
- Every year, B receives 11 million yen and pays 0.115 million dollars.
- In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars.
- Each of these transactions represents a forward contract.
- $Y_1 = Y_2 = 11$, $Y_3 = 111$, $S = 1/90$, $D_1 = D_2 = 0.115$, $D_3 = 1.115$, $q = 0.09$, and $r = 0.08$.
- Plug in these numbers to get $f_1 + f_2 + f_3 = 0.074$ million dollars as before.



# *Stochastic Processes and Brownian Motion*



## Stochastic Processes

- A stochastic process $X = \{ X(t) \}$ is a time series of random variables.
- $X(t)$ (or $X_t$) is a random variable for each time $t$ and is usually called the state of the process at time $t$.
- A realization of $X$ is called a sample path.
- A sample path defines an ordinary function of $t$.



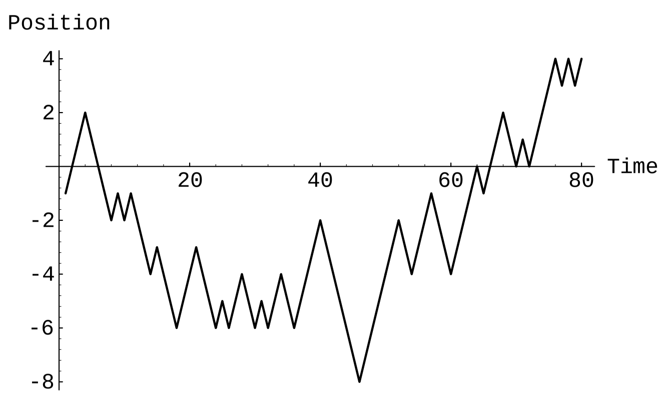
## Stochastic Processes (concluded)

- If the times $t$ form a countable set, $X$ is called a discrete-time stochastic process or a time series.
- In this case, subscripts rather than parentheses are usually employed, as in $X = \{ X_n \}$.
- If the times form a continuum, $X$ is called a continuous-time stochastic process.

## Random Walks

- The binomial model is a random walk in disguise.
- Consider a particle on the integer line, $0, \pm 1, \pm 2, \ldots$.
- In each time step, it can make one move to the right with probability $p$ or one move to the left with probability $1 - p$.
  - This random walk is symmetric when $p = 1/2$.
- Connection with the BOPM: The particle's position denotes the cumulative number of up moves minus that of down moves.








## Random Walk with Drift

$$X_n = \mu + X_{n-1} + \xi_n. \tag{72}$$

- $\xi_n$ are independent and identically distributed with zero mean.
- Drift $\mu$ is the expected change per period.



## Martingales

- $\{ X(t), t \geq 0 \}$ is a martingale if $E[\,|\,X(t)\,|\,] < \infty$ for $t \geq 0$ and

$$E[\, X(t) \,|\, X(u), 0 \leq u \leq s \,] = X(s). \tag{73}$$

- In the discrete-time setting, a martingale means

$$E[\, X_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] = X_n. \tag{74}$$

- $X_n$ can be interpreted as a gambler's fortune after the $n$th gamble.

## Martingales (concluded)

- Identity (74) says the expected fortune after the $(n+1)$th gamble equals the fortune after the $n$th gamble regardless of what may have occurred before.
- A martingale is therefore a notion of fair games.
- Apply the law of iterated conditional expectations to both sides of Eq. (74) to yield

$$E[\,X_n\,] = E[\,X_1\,] \tag{75}$$

for all $n$.

- $E[\,X(t)\,] = E[\,X(0)\,]$ in the continuous-time case.



## Example

- Consider the stochastic process $\{\,Z_n \equiv \sum_{i=1}^{n} X_i, n \geq 1\,\}$, where $X_i$ are independent random variables with zero mean.
- This process is a martingale because

$$\begin{aligned} & E[\,Z_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n\,] \\ = \; & E[\,Z_n + X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n\,] \\ = \; & E[\,Z_n \,|\, Z_1, Z_2, \ldots, Z_n\,] + E[\,X_{n+1} \,|\, Z_1, Z_2, \ldots, Z_n\,] \\ = \; & Z_n + E[\,X_{n+1}\,] = Z_n. \end{aligned}$$



## Probability Measure

- A martingale is defined with respect to a probability measure, under which the expectation is taken.
  - A probability measure assigns probabilities to states of the world.
- A martingale is also defined with respect to an information set.
  - In the characterizations (73)–(74), the information set contains the current and past values of $X$ by default.
  - But it needs not be so.



## Probability Measure (continued)

- A stochastic process $\{\,X(t), t \geq 0\,\}$ is a martingale with respect to information sets $\{\,I_t\,\}$ if, for all $t \geq 0$, $E[\,|\,X(t)\,|\,] < \infty$ and

$$E[\,X(u) \,|\, I_t\,] = X(t)$$

for all $u > t$.

- The discrete-time version: For all $n > 0$,

$$E[\,X_{n+1} \,|\, I_n\,] = X_n,$$

given the information sets $\{\,I_n\,\}$.

## Probability Measure (concluded)

- The above implies $E[\, X_{n+m} \,|\, I_n \,] = X_n$ for any $m > 0$ by Eq. (39) on p. 144.
  - A typical $I_n$ is the price information up to time $n$.
  - The above says the FVs of $X$ will not deviate systematically from today's value given the price history.



## Example

- Consider the stochastic process $\{\, Z_n - n\mu, n \geq 1 \,\}$.
  - $Z_n \equiv \sum_{i=1}^{n} X_i$.
  - $X_1, X_2, \ldots$ are independent random variables with mean $\mu$.
- Now,

$$\begin{aligned} & E[\, Z_{n+1} - (n+1)\,\mu \,|\, X_1, X_2, \ldots, X_n \,] \\ = \; & E[\, Z_{n+1} \,|\, X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\ = \; & Z_n + \mu - (n+1)\,\mu \\ = \; & Z_n - n\mu. \end{aligned}$$



## Example (concluded)

- Define

$$I_n \equiv \{\, X_1, X_2, \ldots, X_n \,\}.$$

- Then $\{\, Z_n - n\mu, n \geq 1 \,\}$ is a martingale with respect to $\{\, I_n \,\}$.



## Martingale Pricing

- The price of a European option is the expected discounted future payoff at expiration in a risk-neutral economy.
- This principle can be generalized using the concept of martingale.
- Recall the recursive valuation of European option via

$$C = [\, pC_u + (1-p)\, C_d \,]/R.$$

  - $p$ is the risk-neutral probability.
  - \$1 grows to \$$R$ in a period.

## Martingale Pricing (continued)

- Let $C(i)$ denote the value of the option at time $i$.
- Consider the discount process

$$\{ C(i)/R^i, i = 0, 1, \ldots, n \}.$$

- Then,

$$E\left[ \frac{C(i+1)}{R^{i+1}} \,\middle|\, C(i) = C \right] = \frac{pC_u + (1-p)\, C_d}{R^{i+1}} = \frac{C}{R^i}.$$



## Martingale Pricing (continued)

- In general,

$$E\left[ \frac{C(k)}{R^k} \,\middle|\, C(i) = C \right] = \frac{C}{R^i}, \quad i \leq k. \tag{76}$$

- The discount process is a martingale:

$$\frac{C(i)}{R^i} = E_i^{\pi}\left[ \frac{C(k)}{R^k} \right], \quad i \leq k. \tag{77}$$

  - $E_i^{\pi}$ is taken under the risk-neutral probability conditional on the price information up to time $i$.
- This risk-neutral probability is also called the equivalent martingale (probability) measure (EMM).



## Martingale Pricing (continued)

- In general, Eq. (77) holds for all assets, not just options.
- In the general case where interest rates are stochastic, the equation becomes

$$\frac{C(i)}{M(i)} = E_i^{\pi}\left[ \frac{C(k)}{M(k)} \right], \quad i \leq k. \tag{78}$$

  - $M(j)$ is the balance in the money market account at time $j$ using the rollover strategy with an initial investment of $1.
  - So it is called the bank account process.
- It says the discount process is a martingale under $\pi$.



## Martingale Pricing (concluded)

- If interest rates are stochastic, then $M(j)$ is a random variable.
  - $M(0) = 1$.
  - $M(j)$ is known at time $j - 1$.
- Identity (78) is the general formulation of risk-neutral valuation.

**Theorem 18** *A discrete-time model is arbitrage-free if and only if there exists a probability measure such that the discount process is a martingale. This probability measure is called the risk-neutral probability measure.*

## Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability.
  - The expected futures price in the next period is

$$p_f F u + (1 - p_f) F d = F \left( \frac{1-d}{u-d} u + \frac{u-1}{u-d} d \right) = F$$

(see p. 394).

- Can be generalized to

$$F_i = E_i^{\pi} [ F_k ], \quad i \le k, \tag{79}$$

where $F_i$ is the futures price at time $i$.

- It holds under stochastic interest rates.



## Martingale Pricing and Numeraire

- The martingale pricing formula (78) uses the money market account as numeraire.[a]
  - It expresses the price of any asset *relative to* the money market account.
- The money market account is not the only choice for numeraire.
- Suppose asset $S$'s value is positive at all times.

[a] Walras (1834–1910).



## Martingale Pricing and Numeraire (concluded)

- Choose $S$ as numeraire.
- Martingale pricing says there exists a risk-neutral probability $\pi$ under which the relative price of any asset $C$ is a martingale:

$$\frac{C(i)}{S(i)} = E_i^{\pi} \left[ \frac{C(k)}{S(k)} \right], \quad i \le k. \tag{80}$$

  - $S(j)$ denotes the price of $S$ at time $j$.
- So the discount process remains a martingale.



## Example

- Take the binomial model with two assets.
- In a period, asset one's price can go from $S$ to $S_1$ or $S_2$.
- In a period, asset two's price can go from $P$ to $P_1$ or $P_2$.
- Assume

$$(S_1/P_1) < (S/P) < (S_2/P_2)$$

for market completeness and to rule out arbitrage opportunities.

## Example (continued)

- For any derivative security, let $C_1$ be its price at time one if asset one's price moves to $S_1$.
- Let $C_2$ be its price at time one if asset one's price moves to $S_2$.
- Replicate the derivative by solving

$$\begin{aligned} \alpha S_1 + \beta P_1 &= C_1 \\ \alpha S_2 + \beta P_2 &= C_2 \end{aligned}$$

using $\alpha$ units of asset one and $\beta$ units of asset two.



## Example (continued)

- This yields

$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \text{ and } \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}.$$

- The derivative costs

$$\begin{aligned} C &= \alpha S + \beta P \\ &= \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2. \end{aligned}$$



## Example (concluded)

- It is easy to verify that

$$\frac{C}{P} = p \frac{C_1}{P_1} + (1-p) \frac{C_2}{P_2}. \tag{81}$$

  - $p \equiv \frac{(S/P)-(S_2/P_2)}{(S_1/P_1)-(S_2/P_2)}$.
- The derivative's price using asset two as numeraire is thus a martingale under the risk-neutral probability $p$.
- The expected returns of the two assets are irrelevant.



## Brownian Motion

- Brownian motion is a stochastic process $\{ X(t), t \geq 0 \}$ with the following properties.
  1. $X(0) = 0$, unless stated otherwise;
  2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X(t_k) - X(t_{k-1})$ for $1 \leq k \leq n$ are independent[a];
  3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where $\mu$ and $\sigma \neq 0$ are real numbers.

[a] So $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$.

## Brownian Motion (concluded)

- Such a process will be called a $(\mu, \sigma)$ Brownian motion with drift $\mu$ and variance $\sigma^2$.
- The existence and uniqueness of such a process is guaranteed by Wiener's theorem.
- Although Brownian motion is a continuous function of $t$ with probability one, it is almost nowhere differentiable.
- The $(0, 1)$ Brownian motion is also called the Wiener process.



## Example

- If $\{ X(t), t \geq 0 \}$ is the Wiener process, then $X(t) - X(s) \sim N(0, t - s)$.
- A $(\mu, \sigma)$ Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process:

$$Y(t) = \mu t + \sigma X(t). \tag{82}$$

- As $Y(t+s) - Y(t) \sim N(\mu s, \sigma^2 s)$, uncertainty about the future value of $Y$ grows as the square root of how far we look into the future.



## Brownian Motion as Limit of Random Walk

**Claim 1** *A* $(\mu, \sigma)$ *Brownian motion is the limiting case of random walk.*

- A particle moves $\Delta x$ to the left with probability $1 - p$.
- It moves to the right with probability $p$ after $\Delta t$ time.
- Assume $n \equiv t/\Delta t$ is an integer.
- Its position at time $t$ is

$$Y(t) \equiv \Delta x \,(X_1 + X_2 + \cdots + X_n).$$



## Brownian Motion as Limit of Random Walk (continued)

- (continued)
  - Here

$$X_i \equiv \begin{cases} +1 & \text{if the } i\text{th move is to the right} \\ -1 & \text{if the } i\text{th move is to the left} \end{cases}$$

  - $X_i$ are independent with $\text{Prob}[\, X_i = 1\,] = p = 1 - \text{Prob}[\, X_i = -1\,]$.
- Recall $E[\, X_i\,] = 2p - 1$ and $\text{Var}[\, X_i\,] = 1 - (2p-1)^2$.

## Brownian Motion as Limit of Random Walk (continued)

- Therefore,

$$E[\,Y(t)\,] = n(\Delta x)(2p-1),$$
$$\text{Var}[\,Y(t)\,] = n(\Delta x)^2\left(1-(2p-1)^2\right).$$

- With $\Delta x \equiv \sigma\sqrt{\Delta t}$ and $p \equiv (1+(\mu/\sigma)\sqrt{\Delta t})/2$,

$$\begin{aligned} E[\,Y(t)\,] &= n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t \\ \text{Var}[\,Y(t)\,] &= n\sigma^2\Delta t\left[\,1-(\mu/\sigma)^2\Delta t\,\right] \to \sigma^2 t \end{aligned}$$

as $\Delta t \to 0$.



## Brownian Motion as Limit of Random Walk (concluded)

- Thus, $\{\,Y(t), t \geq 0\,\}$ converges to a $(\mu,\sigma)$ Brownian motion by the central limit theorem.
- Brownian motion with zero drift is the limiting case of symmetric random walk by choosing $\mu = 0$.
-

$$\begin{aligned} &\text{Var}[\,Y(t+\Delta t)-Y(t)\,] \\ =&\text{Var}[\,\Delta x\,X_{n+1}\,] = (\Delta x)^2 \times \text{Var}[\,X_{n+1}\,] \to \sigma^2\Delta t. \end{aligned}$$

- Similarity to the the BOPM: The $p$ is identical to the probability in Eq. (54) on p. 248 and $\Delta x = \ln u$.



## Geometric Brownian Motion

- Let $X \equiv \{\,X(t), t \geq 0\,\}$ be a Brownian motion process.
- The process $\{\,Y(t) \equiv e^{X(t)}, t \geq 0\,\}$, is called geometric Brownian motion.
- Suppose further that $X$ is a $(\mu,\sigma)$ Brownian motion.
- $X(t) \sim N(\mu t, \sigma^2 t)$ with moment generating function

$$E\left[\,e^{sX(t)}\,\right] = E\,[\,Y(t)^s\,] = e^{\mu ts+(\sigma^2 ts^2/2)}$$

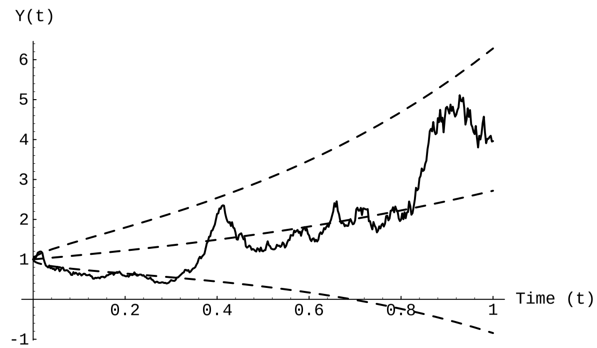
from Eq. (40) on p 146.



## Geometric Brownian Motion (continued)

- In particular,

$$E[\,Y(t)\,] = e^{\mu t+(\sigma^2 t/2)}, \tag{83}$$
$$\begin{aligned} \text{Var}[\,Y(t)\,] &= E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2 \\ &= e^{2\mu t+\sigma^2 t}(e^{\sigma^2 t}-1). \end{aligned} \tag{83'}$$






## Geometric Brownian Motion (concluded)

Useful for situations in which percentage changes are independent and identically distributed.

- Let $Y_n$ denote the stock price at time $n$ and $Y_0 = 1$.
- Assume relative returns $X_i \equiv Y_i / Y_{i-1}$ are independent and identically distributed.
- Then $\ln Y_n = \sum_{i=1}^{n} \ln X_i$ is a sum of independent, identically distributed random variables.
- Thus $\{ \ln Y_n, n \geq 0 \}$ is approximately Brownian motion.



# *Continuous-Time Financial Mathematics*



## Stochastic Integrals

- Use $W \equiv \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of $X$ from a class of stochastic processes,[a]

$$\mathrm{I}_t(X) \equiv \int_0^t X \, dW, \quad t \geq 0.$$

- $\mathrm{I}_t(X)$ is a random variable called the stochastic integral of $X$ with respect to $W$.
- The stochastic process $\{ \mathrm{I}_t(X), t \geq 0 \}$ will be denoted by $\int X \, dW$.

[a] Ito (1915–).

## Stochastic Integrals (concluded)

- Typical requirements for $X$ in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s)\, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)]\, ds < \infty$.
  - The information set at time $t$ includes the history of $X$ and $W$ up to that point in time.
  - But it contains nothing about the evolution of $X$ or $W$ after $t$ (nonanticipating, so to speak).
- The future cannot influence the present.
- $\{X(s), 0 \leq s \leq t\}$ is independent of $\{W(t+u) - W(t), u > 0\}$.



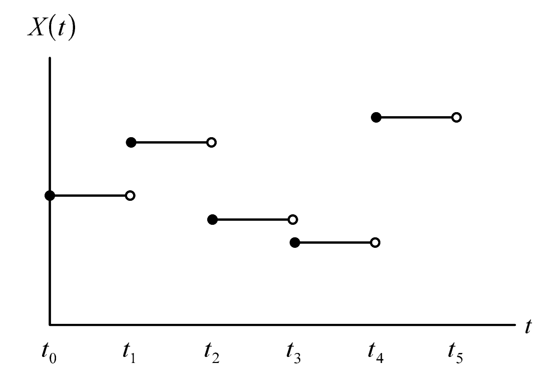
## Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{X(t)\}$ is simple if there exist $0 = t_0 < t_1 < \cdots$ such that

$$X(t) = X(t_{k-1}) \quad \text{for } t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots$$

for any realization (see figure next page).








## Ito Integral (continued)

- The Ito integral of a simple process is defined as

$$\text{I}_t(X) \equiv \sum_{k=0}^{n-1} X(t_k)[W(t_{k+1}) - W(t_k)], \qquad (84)$$

where $t_n = t$.

  - The integrand $X$ is evaluated at $t_k$, not $t_{k+1}$.
- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

## Ito Integral (continued)

- Let $X = \{ X(t), t \geq 0 \}$ be a general stochastic process.
- Then there exists a random variable $\mathrm{I}_t(X)$, unique almost certainly, such that $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that $X_n$ converges in probability to $X$.
- If $X$ is continuous with probability one, then $\mathrm{I}_t(X_n)$ converges in probability to $\mathrm{I}_t(X)$ as $\delta_n \equiv \max_{1 \leq k \leq n}(t_k - t_{k-1})$ goes to zero.



## Ito Integral (concluded)

- It is a fundamental fact that $\int X\, dW$ is continuous almost surely.
- The following theorem says the Ito integral is a martingale.
- A corollary is the mean value formula $E[\int_a^b X\, dW] = 0$.

**Theorem 19** *The Ito integral $\int X\, dW$ is a martingale.*



## Discrete Approximation

- Recall Eq. (84) on p. 452.
- The following simple stochastic process $\{ \widehat{X}(t) \}$ can be used in place of $X$ to approximate the stochastic integral $\int_0^t X\, dW$,

$$\widehat{X}(s) \equiv X(t_{k-1}) \text{ for } s \in [t_{k-1}, t_k),\ k = 1, 2, \ldots, n.$$

- Note the nonanticipating feature of $\widehat{X}$.
  - The information up to time $s$,

$$\{ \widehat{X}(t), W(t), 0 \leq t \leq s \},$$

  cannot determine the future evolution of $X$ or $W$.

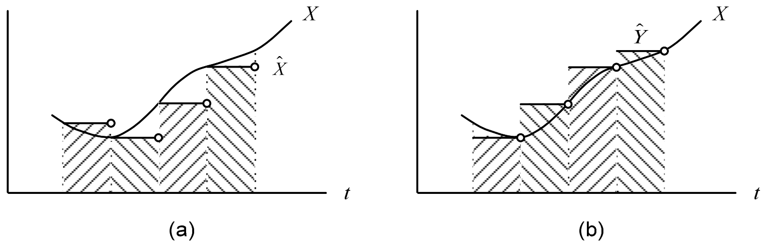


## Discrete Approximation (concluded)

- Suppose we defined the stochastic integral as $\sum_{k=0}^{n-1} X(t_{k+1})[ W(t_{k+1}) - W(t_k) ]$.
- Then we would be using the following different simple stochastic process in the approximation,

$$\widehat{Y}(s) \equiv X(t_k) \text{ for } s \in [t_{k-1}, t_k),\ k = 1, 2, \ldots, n.$$

- This clearly anticipates the future evolution of $X$.






## Ito Process

- The stochastic process $X = \{ X_t, t \geq 0 \}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s)\, ds + \int_0^t b(X_s, s)\, dW_s, \quad t \geq 0$$

is called an Ito process.

- Here, $X_0$ is a scalar starting point, and $\{ a(X_t, t) : t \geq 0 \}$ and $\{ b(X_t, t) : t \geq 0 \}$ are stochastic processes satisfying certain regularity conditions.
- The terms $a(X_t, t)$ and $b(X_t, t)$ are the drift and the diffusion, respectively.



## Ito Process (continued)

- A shorthand[a] is the following stochastic differential equation for the Ito differential $dX_t$,

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{85}$$

  – Or simply $dX_t = a_t\, dt + b_t\, dW_t$.
- This is Brownian motion with an instantaneous drift $a_t$ and an instantaneous variance $b_t^2$.
- $X$ is a martingale if the drift $a_t$ is zero by Theorem 19.

[a] Langevin, 1904.



## Ito Process (concluded)

- $dW$ is normally distributed with mean zero and variance $dt$.
- An equivalent form to Eq. (85) is

$$dX_t = a_t\, dt + b_t \sqrt{dt}\, \xi, \tag{86}$$

where $\xi \sim N(0, 1)$.

- This formulation makes it easy to derive Monte Carlo simulation algorithms.

## Euler Approximation

- The following approximation follows from Eq. (86),

$$\widehat{X}(t_{n+1}) = \widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\,\Delta W(t_n), \quad (87)$$

  where $t_n \equiv n\Delta t$.
- Called the Euler or Euler-Maruyama method.
- Under mild conditions, $\widehat{X}(t_n)$ converges to $X(t_n)$.
- Recall that $\Delta W(t_n)$ should be interpreted as $W(t_{n+1}) - W(t_n)$ instead of $W(t_n) - W(t_{n-1})$.



## More Discrete Approximations

- Under fairly loose regularity conditions, approximation (87) can be replaced by

$$\widehat{X}(t_{n+1}) = \widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,Y(t_n).$$

  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.



## More Discrete Approximations (concluded)

- A simpler discrete approximation scheme:

$$\widehat{X}(t_{n+1}) = \widehat{X}(t_n) + a(\widehat{X}(t_n), t_n)\,\Delta t + b(\widehat{X}(t_n), t_n)\sqrt{\Delta t}\,\xi. \quad (88)$$

  - $\text{Prob}[\,\xi = 1\,] = \text{Prob}[\,\xi = -1\,] = 1/2$.
  - Note that $E[\,\xi\,] = 0$ and $\text{Var}[\,\xi\,] = 1$.
- This clearly defines a binomial model.
- As $\Delta t$ goes to zero, $\widehat{X}$ converges to $X$.



## Trading and the Ito Integral

- Consider an Ito process $d\boldsymbol{S}_t = \mu_t\,dt + \sigma_t\,dW_t$.
  - $\boldsymbol{S}_t$ is the vector of security prices at time $t$.
- Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time $t$.
- The stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the value of the portfolio $\boldsymbol{\phi}_t$ at time $t$.
- $\boldsymbol{\phi}_t\,d\boldsymbol{S}_t \equiv \boldsymbol{\phi}_t(\mu_t\,dt + \sigma_t\,dW_t)$ represents the change in the value from security price changes occurring at time $t$.

## Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \equiv \int_0^T \boldsymbol{\phi}_t \, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t \, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t \, dW_t,$$

  measures the gains realized by the trading strategy over the period $[\,0, T\,]$.
- A strategy is self-financing if

$$\boldsymbol{\phi}_t \boldsymbol{S}_t = \boldsymbol{\phi}_0 \boldsymbol{S}_0 + G_t(\boldsymbol{\phi}) \tag{89}$$

  for all $0 \leq t < T$.
  - The investment at any time equals the initial investment plus the total capital gains.



## Ito's Lemma

**Theorem 20** *Suppose $f : R \to R$ is twice continuously differentiable and $dX = a_t \, dt + b_t \, dW$. Then $f(X)$ is the Ito process,*

$$\begin{aligned} & f(X_t) \\ = \; & f(X_0) + \int_0^t f'(X_s) \, a_s \, ds + \int_0^t f'(X_s) \, b_s \, dW + \frac{1}{2} \int_0^t f''(X_s) \, b_s^2 \, ds \end{aligned}$$

*for $t \geq 0$.*

- Basically says a smooth function of an Ito process is itself an Ito process.



## Ito's Lemma (continued)

- In differential form, Ito's lemma becomes

$$df(X) = f'(X) \, a \, dt + f'(X) \, b \, dW + \frac{1}{2} f''(X) \, b^2 \, dt. \tag{90}$$

- Compared with calculus, the interesting part is the third term on the right-hand side.
- A convenient formulation of Ito's lemma is

$$df(X) = f'(X) \, dX + \frac{1}{2} f''(X)(dX)^2. \tag{91}$$



## Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a \, dt + b \, dW)^2$ symbolically according to

| $\times$ | $dW$ | $dt$ |
|---|---|---|
| $dW$ | $dt$ | 0 |
| $dt$ | 0 | 0 |

  - The $(dW)^2 = dt$ entry is justified by a known result.
- This form is easy to remember because of its similarity to Taylor expansion.

## Ito's Lemma (continued)

**Theorem 21 (Higher-Dimensional Ito's Lemma)** *Let $W_1, W_2, \ldots, W_n$ be independent Wiener processes and $X \equiv (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : R^m \to R$ is twice continuously differentiable and $X_i$ is an Ito process with $dX_i = a_i\, dt + \sum_{j=1}^{n} b_{ij}\, dW_j$. Then $df(X)$ is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k,$$

*where $f_i \equiv \partial f / \partial x_i$ and $f_{ik} \equiv \partial^2 f / \partial x_i \partial x_k$.*



## Ito's Lemma (concluded)

- The multiplication table for Theorem 21 is

| $\times$ | $dW_i$ | $dt$ |
|---|---|---|
| $dW_k$ | $\delta_{ik}\, dt$ | 0 |
| $dt$ | 0 | 0 |

in which

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$



## Geometric Brownian Motion

- Consider the geometric Brownian motion process $Y(t) \equiv e^{X(t)}$
  - $X(t)$ is a $(\mu, \sigma)$ Brownian motion.
- Ito's formula (90) implies

$$\frac{dY}{Y} = \left(\mu + \sigma^2/2\right) dt + \sigma\, dW.$$

- The instantaneous rate of return is $\mu + \sigma^2/2$ not $\mu$.